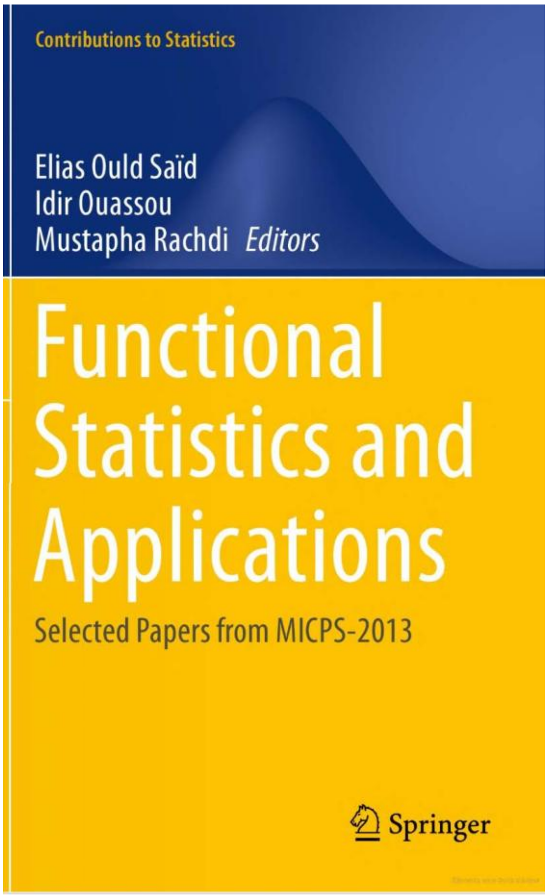

Stochastic Analysis of an M/G/1 Retrial Queue with FCFS

Mohamed Boualem, Mouloud Cherfaoui, Natalia Djellab, and Djamil Aïssani

Abstract The main goal of this paper is to investigate stochastic analysis of a single server retrial queue with a First-Come-First-Served (FCFS) orbit and non-exponential retrial times using the monotonicity and comparability methods. We establish various results for the comparison and monotonicity of the underlying embedded Markov chain when the parameters vary. Moreover, we prove stochastic inequalities for the stationary distribution and some simple bounds for the mean characteristics of the system. We validate stochastic comparison method by presenting some numerical results illustrating the interest of the approach.

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Elias Ould Saïd
Idir Ouassou
Mustapha Rachdi *Editors*

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Editors
Elias Ould Saïd
LMPA
Université du Littoral Côte d'Opale
Calais, France

Idir Ouassou
National School of Applied Sciences
Cadi Ayyad University
Marrakesh, Morocco

Mustapha Rachdi
Faculty of Science of Man and Society
Grenoble-Alpes University
Grenoble, France

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Stochastic Analysis of an M/G/1 Retrial Queue with FCFS

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Abstract The main goal of this paper is to investigate stochastic analysis of a single server retrial queue with a First-Come-First-Served (FCFS) orbit and non-exponential retrial times using the monotonicity and comparability methods. We establish various results for the comparison and monotonicity of the underlying embedded Markov chain when the parameters vary. Moreover, we prove stochastic inequalities for the stationary distribution and some simple bounds for the mean characteristics of the system. We validate stochastic comparison method by presenting some numerical results illustrating the interest of the approach.

1 Introduction

Queueing systems with repeated attempts have been widely used to model many problems in telecommunication and computer systems [1, 4, 19]. The essential feature of a retrial queue is that arriving customers who find all servers busy are obliged to abandon the service area and join a retrial group, called orbit, in order to try their luck again after some random time. For a detailed review of the main results

M. Boualem (✉)
Research Unit LaMOS (Modeling and Optimization of Systems), Faculty of Technology, University of Bejaia, Bejaia 06000, Algeria
e-mail: robertt15dz@yahoo.fr

M. Cherfaoui
Research Unit LaMOS (Modeling and Optimization of Systems), University of Bejaia, Bejaia 06000, Algeria

Department of Mathematics, University of Biskra, 07000 Biskra, Algeria
e-mail: mouloudcherfaoui2013@gmail.com

N. Djellab
Laboratory LaPS, Department of Mathematics, University of Annaba, Annaba 23000, Algeria
e-mail: djellab@yahoo.fr

D. Aïssani
Research Unit LaMOS (Modeling and Optimization of Systems), Faculty of Exact Sciences, University of Bejaia, Bejaia 06000, Algeria
e-mail: lamos_bejaia@hotmail.com

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and the literature on this topic the reader is referred to the monographs [2, 11]. In recent years, there has been an increasing interest in the investigation of the retrial phenomenon in cellular mobile network, see [3, 10, 15, 16, 24] and the references therein, and in many other telecommunication systems including star-like local area networks [14], wavelength-routed optical networks [26], circuit-switched systems with hybrid fiber-coax architecture [13], wireless sensor networks [25], etc.

It is well known that for the retrial queues we need to establish how the customers in orbit access to the server. The time between successive repeated attempts is important in telephony, where a call receiving a busy signal does not wait the termination of the busy condition. The most usual protocol described in the classical theory of retrial queues is the so-called classical retrial policy in which each source in orbit repeats its call after an exponentially distributed time with parameter θ. So, there is a probability $n\theta dt + o(dt)$ of a new retrial in the next interval $(t,\ t + dt)$ given that n customers are in orbit at time t. Such a policy has been motivated by applications in modeling subscriber's behavior in telephone networks since the 1940s. In past years, technology has considerably evolved. The literature on retrial queues describes several retrial protocols specific to some modern computer and communication networks in which the time between two successive repeated attempts is controlled by an electronic device and consequently, is independent of the number of units applying for service. In this case, the probability of a repeated attempt during $(t,\ t+dt)$, given the orbit is not empty, is $(1-\delta_{0,n})\alpha\ dt+o(dt)$ where $\delta_{0,n}$ denotes Kronecker's delta and n is the number of repeated customers. This type of retrial discipline is called the constant retrial policy.

An examination of the literature on the retrial queues reveals the remarkable fact that the non-homogeneity caused by the flow of repeated attempts is the key to understand most analytical difficulties arising in the study of retrial queues. Many efforts have been devoted to deriving performance measures such as queue length, waiting time, busy period distributions, and so on. However, these performance characteristics have been provided through transform methods which have made the expressions cumbersome and the obtained results cannot be put into practice. In the last decade there has been a trend towards the research of approximations and bounds. Qualitative properties of stochastic models constitute a basic theoretical basis for approximation methods. Some important approaches are monotonicity and comparability which can be investigated using the stochastic comparison method based on the general theory of stochastic orderings. Stochastic orders represent an important tool for many problems in probability and statistics [18, 20–23].

Stochastic comparison is a mathematical tool used in the performance study of systems modeled by continuous or discrete-time Markov chains. The general idea of this method is to bound a complex system by a new system, easier to solve and providing performance measures bounds. Many papers treat stochastic comparison methods of queueing systems with repeated attempts. Boualem et al. [6] investigate some monotonicity properties of an $M/G/1$ queue with constant retrial policy in which the server operates under a general exhaustive service and multiple vacation policy relative to strong stochastic ordering and convex ordering. These results imply in particular simple insensitive bounds for the stationary queue length distribution. Boualem et al. [7] use the tools of a qualitative analysis to investigate various monotonicity properties for an $M/G/1$ retrial queue with classical retrial policy and Bernoulli feedback. The obtained results allow to place in a prominent position the insensitive bounds for both the stationary distribution and the conditional distribution of the stationary queue of the considered model. Mokdad and Castel-Taleb [17] propose to use a mathematical method based on stochastic comparisons of Markov chains in order to derive bounds on performance indices of fixed and mobile networks. Their main objective consists in finding Markovian bounding models with reduced state spaces, which are easier to solve. They apply the methodology to performance evaluation of complex telecommunication systems modeled by large size Markov chains which cannot be solved by exact methods. They propose to define intuitively bounding systems in order to compute bounds on performance measures. Using stochastic comparison methods, they prove that the new systems represent bounds for the exact ones. To validate their approach and illustrate its interest, they present some numerical results. Bušić and Fourneau [9] illustrate through examples how monotonicity may help for performance evaluation of mobile networks, by considering two different applications. In the first application, they assume that a Markov chain of the model depends on a parameter that can be estimated only up to a certain level and they have only an interval that contains the exact value of the parameter. Instead of taking an approximated value for the unknown parameter, they show how monotonicity properties of the Markov chain can be used to take into account the error bound from the measurements. In the second application, they consider a well-known approximation method: the decomposition into Markovian submodels. They show that the monotonicity property may help to derive bounds for Markovian submodels and are sufficient conditions for convergence of iterative algorithms which are often designed to give approximations. More recently, Boualem et al. [8] investigate various monotonicity properties of a single server retrial queue with general retrial times using the mathematical method based on stochastic comparisons of Markov chains in order to derive bounds on performance indices. Bounds are derived for the mean characteristics of the busy period, number of customers served during a busy period, number of orbit busy periods, and waiting times. Boualem [5] addresses monotonicity properties of the single server retrial queue with no waiting room and server subject to active breakdowns, that is, the service station can fail only during the service period. The obtained results give insensitive bounds for the stationary distribution of the considered embedded Markov chain related to the model in the study. Numerical illustrations are provided to support the results.

In this paper we consider an $M/G/1$ retrial queue with non-exponential retrial times under the special assumption that only the customer at the head of the orbit queue is allowed to occupy the server. The performance characteristics of such a system are available in the literature (see [12]). The author obtains relevant performance characteristics expressed in terms of generating functions and Laplace transforms. However, there still remains the issue that numerical inversion is required for actually computing numbers and derive useable results. Indeed, it is sometimes possible to obtain the generating function and/or Laplace transforms of an unknown probability distribution but not to invert the generating function or the Laplace transforms to obtain an explicit form of the distribution. Moreover, the error for numerical inversion is difficult to control. For example, if we compare two systems which are "close" then it might be that due to the numerical error in the inversion, we may take the wrong system to perform better. Based on the relevant performance characteristics obtained by Gómez-Corral [12], we consider in our paper a qualitative analysis which is another field of own right to establish insensitive bounds on some performance measures by using the stochastic analysis approach relative to the theory of stochastic orderings . Finally, the effects of various parameters on the performance of the system have been examined numerically.

This paper is arranged as follows. In the next section, we describe the considered mathematical model. In Sect. 3, we introduce some pertinent definitions and notions of the three most important orderings. Section 4 focusses on monotonicity of the transition operator and gives comparability conditions of two transition operators. Stochastic inequalities for the stationary number of customers in the system are discussed in Sect. 5. The last section is devoted to the practical applications.

2 Mathematical Model

Primary customers arrive in a Poisson process with rate λ. If the server is free, the primary customer is served immediately and leaves the system after service completion. Otherwise, the customer leaves the service area and enters the retrial group in accordance with an FCFS discipline. We assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer joins the orbit again. Such a process is repeated until the customer finds the server idle and gets the requested service at the time of a retrial. Successive inter-retrial times of any customer follow an arbitrary law with common probability distribution function $A(x)$, Laplace-Stieltjes transform $\mathbb{L}_A(s)$ and first moment α_1. The service times are independently and identically distributed with probability distribution function $B(x)$, Laplace-Stieltjes transform $\mathbb{L}_B(s)$ and first two moments β_1, β_2. We suppose that inter-arrival times, retrial times, and service times are mutually independent.

The main characteristic of this queue is that, at any service completion, a competition between an exponential law and a general retrial time distribution determines the next customer who accesses the service facility. Thus, the retrial discipline does not depend on the orbit length.

Let τ_n be the time of the n-th departure and Z_n the number of customers in the orbit just after the time τ_n. We have the following fundamental recursive equation:

$$Z_{n+1} = Z_n + v^{n+1} - \delta_{Z_{n+1}},$$

where v^{n+1} is the number of primary customers arriving at the system during the service time which ends at τ_{n+1}. Its distribution is given by:

$$b_j = \mathbb{P}(v^{n+1} = j) = \int_0^{\infty} (\lambda x)^j (j!)^{-1} e^{-\lambda x} dB(x),\ j \geq 0,$$

with generating function $b(z) = \sum_{j\geq 0} b_j z^j = \mathbb{L}_B(\lambda(1-z))$.

The Bernoulli random variable $\delta_{Z_{n+1}}$ is equal to 1 or 0 depending on whether the customer who leaves the system at time τ_{n+1} proceeds from the orbit or otherwise.

The sequence of random variables $\{Z_n, n \geq 1\}$ forms an embedded Markov chain for our queueing system which is irreducible and aperiodic on the state-space $\mathbb{N}$. The stability condition is given in [12] as follows: $\rho < \mathbb{L}_A(\lambda)$, where $\rho = \lambda\beta_1$ is the load of the system.

3 Stochastic Orders

Stochastic orders are useful in comparing random variables measuring certain characteristics in many areas. Such areas include insurance, operations research, queueing theory, survival analysis, and reliability theory (see [22]). The simplest comparison is through comparing the expected value of the two comparable random variables. First, we define some notions on stochastic ordering which will be used in the context of the paper. For more details see [20–23].

Definition 1 Let $F(x)$ and $G(x)$ be two distribution functions of nonnegative random variables X and Y, respectively. Then:

(a) $F \leq_{\text{st}} G$ iff $F(x) \geq G(x)$ or $\overline{F}(x) = 1 - F(x) \leq \overline{G}(x)$, $\forall x \geq 0$.

(b) $F \leq_{icx} G$ iff $\int_x^{+\infty} \overline{F}(u)d(u) \leq \int_x^{+\infty} \overline{G}(u)d(u)$, $\forall x \geq 0$.

(c) $F \leq_L G$ iff $\int_0^{+\infty} \exp(-sx)dF(x) \geq \int_0^{+\infty} \exp(-sx)dG(x)$, $\forall s \geq 0$.

Definition 2 If the random variables of interest are of discrete type and $\alpha = (\alpha_n)_{n\geq 0}$, $\beta = (\beta_n)_{n\geq 0}$ are the corresponding distributions, then the above definitions can be given in the following form:

(a) $\alpha \leq_{\text{st}} \beta$ iff $\overline{\alpha}_m = \sum_{n\geq m} \alpha_n \leq \overline{\beta}_m = \sum_{n\geq m} \beta_n$, for all m.

(b) $\alpha \leq_{icx} \beta$ iff $\overline{\overline{\alpha}}_m = \sum_{n\geq m} \sum_{k\geq n} \alpha_k \leq \overline{\overline{\beta}}_m = \sum_{n\geq m} \sum_{k\geq n} \beta_k$, for all m.

(c) $\alpha \leq_L \beta$ iff $\sum_{n\geq 0} \alpha_n z^n \geq \sum_{n\geq 0} \beta_n z^n$, for all $z \in [0, 1]$.

Definition 3 Let X be a positive random variable with distribution function F:

1. F is *HNBUE* (Harmonically New Better than Used in Expectation) iff $F \leq_{icx} F^*$,
2. F is of class $\mathscr{L}$ iff $F \geq_L F^*$,

where F^* is the exponential distribution function with the same mean as F.

The ageing classes are linked by the inclusion chain:

$$\text{NBU (New Better than Used)} \subset \text{NBUE (New Better than Used in Expectation)} \subset \text{HNBUE} \subset \mathscr{L}.$$

4 Monotonicity and Comparability of the Transition Operator

The one-step transition probabilities of $\{Z_n, n \geq 1\}$ are defined by

$$p_{nm} = \begin{cases} (1 - \mathbb{L}_A(\lambda))b_{m-n} + \mathbb{L}_A(\lambda)b_{m-n+1}, & \text{for } n \neq 0 \text{ and } m \geq 0, \\ b_m, & \text{for } n = 0 \text{ and } m \geq 0. \end{cases} \quad (1)$$

Let Θ be the transition operator of an embedded Markov chain which associates to every distribution $\alpha = \{\alpha_m\}_{m\geq 0}$ a distribution $\Theta\alpha = \{\beta_m\}_{m\geq 0}$ such that

$$\beta_m = \sum_{n\geq 0} \alpha_n p_{nm}.$$

Theorem 1 *The operator Θ is monotone with respect to the orders $\leq_{st}$ and $\leq_{icx}$.*

Proof The operator Θ is monotone with respect to $\leq_{\text{st}}$ if and only if $\overline{p}_{n-1m} \leq \overline{p}_{nm}$, and is monotone with respect to $\leq_{icx}$ if and only if $2\overline{\overline{p}}_{nm} \leq \overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m}$ for all n, m, where

$\overline{p}_{nm} = \sum_{l\geq m} p_{nl}$ and $\overline{\overline{p}}_{nm} = \sum_{k\geq m} \overline{p}_{nk} = \sum_{k\geq m}\sum_{l\geq k} p_{nl}$.

In our case:

$$\overline{p}_{nm} - \overline{p}_{n-1m} = (1 - \mathbb{L}_A(\lambda))b_{m-n} + \mathbb{L}_A(\lambda)b_{m-n+1} > 0.$$

$$\overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m} - 2\overline{\overline{p}}_{nm} = (1 - \mathbb{L}_A(\lambda))b_{m-n-1} + \mathbb{L}_A(\lambda)b_{m-n} > 0.$$

□

Theorems 2 till 4, we give comparability conditions of two transition operators. Consider two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(i)}, A^{(i)}, B^{(i)}$. Let Θ^i be the transition operator of the embedded Markov chain, in the i-th system, $i = 1, 2$.

Theorem 2 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\Theta^1 \leq_{st} \Theta^2$, i.e., for any distribution α, we have $\Theta^1\alpha \leq_{st} \Theta^2\alpha$.*

Proof From Stoyan [23], it is well known that to prove $\Theta^1 \leq_{\text{st}} \Theta^2$, we have to show the following inequality:

$$\overline{p}^{(1)}_{nm} \leq \overline{p}^{(2)}_{nm}, \;\; \forall \; n, m.$$

We have

$$\overline{p}^{(1)}_{nm} = (1 - \mathbb{L}_{A^{(1)}}(\lambda^{(1)}))b^{(1)}_{m-n} + \overline{b}^{(1)}_{m-n+1}.$$

Since $\lambda^{(1)} \leq \lambda^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then

$$\mathbb{L}_{A^{(1)}}(\lambda^{(1)}) \geq \mathbb{L}_{A^{(2)}}(\lambda^{(2)}),$$

and

$$\overline{p}^{(1)}_{nm} \leq (1 - \mathbb{L}_{A^{(2)}}(\lambda^{(2)}))b^{(1)}_{m-n} + \overline{b}^{(1)}_{m-n+1}.$$

But

$$(1 - \mathbb{L}_{A^{(2)}}(\lambda^{(2)}))b^{(1)}_{m-n} + \overline{b}^{(1)}_{m-n+1} = (1 - \mathbb{L}_{A^{(2)}}(\lambda^{(2)}))\overline{b}^{(1)}_{m-n} + \mathbb{L}_{A^{(2)}}(\lambda^{(2)})\overline{b}^{(1)}_{m-n+1}.$$

Using these inequalities we get:

$$\overline{p}^{(1)}_{nm} \leq (1 - \mathbb{L}_{A^{(2)}}(\lambda^{(2)}))\overline{b}^{(2)}_{m-n} + \mathbb{L}_{A^{(2)}}(\lambda^{(2)})\overline{b}^{(2)}_{m-n+1} = \overline{p}^{(2)}_{nm}.$$

□

Theorem 3 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_{icx} B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\Theta^1 \leq_{icx} \Theta^2$.*

Proof The proof is similar to that of Theorem 2. □

Theorem 4 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_L B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\Theta^1 \leq_L \Theta^2$.*

Proof Let α be a distribution and $\Theta_\alpha = \beta$, where

$$\beta_m = \sum_{n\geq 0} \alpha_n p_{nm} = \alpha_0 b_m + \sum_{n\geq 1} \alpha_n p_{nm}, \;\; \text{for all } m \geq 0.$$

The generating function of β is given by

$$G(z) = \sum_{m\geq 0} \beta_m z^m = \alpha_0 b(z) + \frac{1}{z} b(z)(\alpha(z) - \alpha_0)(z + (1 - z)\mathbb{L}_A(\lambda)).$$

If the conditions of Theorem 4 are fulfilled, then

$$b^{(1)}(z) \geq b^{(2)}(z) \text{ and } (1-z)\mathbb{L}_{A^{(1)}}(\lambda^{(1)}) \geq (1-z)\mathbb{L}_{A^{(2)}}(\lambda^{(2)}), \ \forall\, z \in [0,1].$$

Hence $G^{(1)}(z) \geq G^{(2)}(z)$. □

5 Stochastic Inequalities for the Stationary Distribution

Consider two $M/G/1$ retrial queues with non-exponential retrial times. Let $\pi_n^{(1)}$, $\pi_n^{(2)}$ be the corresponding stationary distributions of the number of customers in the system.

Theorem 5 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\{\pi_n^{(1)}\} \leq_s \{\pi_n^{(2)}\}$, where $\leq_s$ represents one of the symbols $\leq_{\text{st}}$ or $\leq_{icx}$.*

Proof Using Theorems 1–3 which state that Θ^i are monotone with respect to the order $\leq_s$ and $\Theta^1 \leq_s \Theta^2$, we have by induction $\Theta^{1,n}\alpha \leq_s \Theta^{2,n}\alpha$ for any distribution α, where $\Theta^{i,n} = \Theta^i(\Theta^{i,n-1}\alpha)$. Taking the limit, we obtain the stated result. Indeed, $\Theta^1\alpha_n^1 = \mathbb{P}[Z_k^1 = n] \leq_s \mathbb{P}[Z_k^2 = n] = \Theta^2\alpha_n^2$, when $k \to \infty$, we have $\{\pi_n^{(1)}\} \leq_s \{\pi_n^{(2)}\}$. □

Theorem 6 *If in the $M/G/1$ retrial queue with general retrial times the service time distribution $B(x)$ is HNBUE (Harmonically New Better than Used in Expectation) and the retrial time distribution is $\mathscr{L}$, then $\{\pi_n\} \leq_{icx} \{\pi_n^*\}$, where $\{\pi_n^*\}$ is the stationary distribution of the number of customers in the $M/M/1$ retrial queue with exponential retrial with the same parameters.*

Proof Consider an auxiliary $M/M/1$ retrial queue with exponentially distributed retrial time $A^*(x)$ and service time $B^*(x)$. If $B(x)$ is *HNBUE* and $A(x)$ is $\mathscr{L}$, then $B(x) \leq_{icx} B^*(x)$ and $A(x) \leq_L A^*(x)$. Therefore, by using Theorem 5, we deduce the statement of this theorem. □

6 Practical Aspect

Assume that we have two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$, respectively. Let $L^{(i)}$, $I^{(i)}$, $N_b^{(i)}$ and $W^{(i)}$ be the busy period length, the number of customers served during a busy period, the number of orbit busy periods which take place in $]0, L^{(i)}]$ and the waiting time , respectively, in the i-th system, $i = 1, 2$.

Theorem 7 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\mathbb{E}(L^{(1)}) \leq \mathbb{E}(L^{(2)})$ and $\mathbb{E}(I^{(1)}) \leq \mathbb{E}(I^{(2)})$, where $\leq_s$ is one of the symbols $\leq_{st}$, $\leq_{icx}$, $\leq_L$.*

Proof Gómez-Corral [12] shows that

$$\mathbb{E}(L) = \frac{\beta_1}{\mathbb{L}_A(\lambda) - \lambda\beta_1} \quad \text{and} \quad \mathbb{E}(I) = \frac{\mathbb{L}_A(\lambda)}{\mathbb{L}_A(\lambda) - \lambda\beta_1},$$

which are increasing with respect to λ and β_1, decreasing with respect to $\mathbb{L}_A(.)$. Under conditions of Theorem 7, we obtain the desired inequalities. □

Theorem 8 *For any $M/G/1$ retrial queue,*

$$\mathbb{E}(L) \leq \mathbb{E}(L)_{\text{Upper}} = \frac{\beta_1}{e^{-\lambda\alpha_1} - \lambda\beta_1},$$

$$\mathbb{E}(I) \leq \mathbb{E}(I)_{\text{Upper}} = \frac{e^{-\lambda\alpha_1}}{e^{-\lambda\alpha_1} - \lambda\beta_1}.$$

If A and B are of class $\mathscr{L}$, then

$$\mathbb{E}(L) \geq \mathbb{E}(L)_{\text{Lower}} = \frac{\beta_1(1+\lambda\alpha_1)}{1-\lambda\beta_1(1+\lambda\alpha_1)},$$

$$\mathbb{E}(I) \geq \mathbb{E}(I)_{\text{Lower}} = \frac{1}{1-\lambda\beta_1(1+\lambda\alpha_1)}.$$

Proof We consider auxiliary $M/D/1$ and $M/M/1$ retrial queues with the same arrival rates λ, mean service times β_1 and mean retrial times α_1. A represents Dirac distribution at α_1 for the $M/D/1$ system, and represents the exponential distribution for the $M/M/1$ system. Using the theorem above we obtain the stated results. □

Theorem 9 *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\mathbb{E}(N_b^{(1)}) \leq \mathbb{E}(N_b^{(2)})$ and $\mathbb{E}(W^{(1)}) \leq \mathbb{E}(W^{(2)})$.*

Proof Gómez-Corral [12] shows that

$$\mathbb{E}(N_b) = \frac{1-\mathbb{L}_B(\lambda)}{\mathbb{L}_B(\lambda)} \text{ and } \mathbb{E}(W) = \frac{\lambda\beta_2 + 2\beta_1(1-\mathbb{L}_A(\lambda))}{2(\mathbb{L}_A(\lambda) - \lambda\beta_1)}.$$

These quantities are increasing with respect to λ, β_1 and β_2, decreasing with respect to $\mathbb{L}_B(.)$ and $\mathbb{L}_A(.)$. Under the conditions of Theorem 9, we obtain the desired inequalities. □

Theorem 10 *For any $M/G/1$ retrial queue,*

$$\mathbb{E}(N_b) \leq \mathbb{E}(N_b)_{\text{Upper}} = e^{\lambda\beta_1} - 1,$$

$$\mathbb{E}(W) \leq \mathbb{E}(W)_{\text{Upper}} = \frac{\lambda\beta_2 + 2\beta_1(1-e^{-\lambda\alpha_1})}{2(e^{-\lambda\alpha_1} - \lambda\beta_1)}.$$

If A and B are of class $\mathscr{L}$, then

$$\mathbb{E}(N_b) \geq \mathbb{E}(N_b)_{\text{Lower}} = \lambda\beta_1,$$
$$\mathbb{E}(W) \geq \mathbb{E}(W)_{\text{Lower}} = \frac{\lambda\beta_2(1+\lambda\alpha_1) + 2\lambda\beta_1\alpha_1}{2(1-\lambda\beta_1(1+\lambda\alpha_1))}.$$

Proof The proof is similar to that of Theorem 8. □

6.1 Numerical Application

We give a numerical illustration concerning the mean busy period length $\mathbb{E}(L)_{A(x)}$ and the mean waiting time $\mathbb{E}(W)_{A(x)}$ in the $M/M/1$ retrial queue with general retrial times given respectively in Theorems 8 and 10. To this end, for the retrial time distributions $A(x)$, we have considered the most representative distributions which are:

1. Exponential (exp): $A(x) = 1 - e^{-\alpha_1 x}$.
2. Two-Stage Erlang (E_2): $A(x) = 1 - (1 - 2\gamma x)e^{-2\gamma x}$.
3. Gamma (Γ): $A(x) = \frac{1}{b^a \Gamma(a)} \int_0^x t^{a-1} e^{-t/b} dt$.
4. Two-Stage Hyper-Exponential (H_2): $A(x) = 1 - pe^{-\gamma_1 x} - (1-p)e^{-\gamma_2 x}$.

In Table 1 we present the values of the system parameters according the above cases.

The obtained results are presented in Figs. 1 and 2. From these results, we note that:

- The lower bound $\mathbb{E}(L)_{\text{Lower}}$ (respectively, $\mathbb{E}(W)_{\text{Lower}}$) is nothing else than the mean length of the busy period $\mathbb{E}(L)$ (respectively, the mean waiting time $\mathbb{E}(W)$) in the $M/M/1$ retrial queue with exponential retrial times.
- The inequality $\mathbb{E}(L)_{A(x)} \leq \mathbb{E}(L)_{\text{Upper}}$ (respectively, $\mathbb{E}(W)_{A(x)} \leq \mathbb{E}(W)_{\text{Upper}}$) always holds. In addition, if the law $A \in \mathscr{L}$, then the inequality $\mathbb{E}(L)_{\text{Lower}} \leq \mathbb{E}(L)_{A(x)}$ (respectively, $\mathbb{E}(W)_{\text{Lower}} \leq \mathbb{E}(W)_{A(x)}$) holds.
- If α_1 and ρ are small enough then the mean length of the busy period (respectively, the mean waiting time) in the system is closer to the $\mathbb{E}(L)_{A(x)}$ (respectively, $\mathbb{E}(W)$), in other words, closer to the $\mathbb{E}(L)_{\text{Lower}}$ (respectively, $\mathbb{E}(W)_{\text{Lower}}$).

Table 1 Different values of the system parameters

ρ	λ	β_1	α_1	γ	(a, b)	(p, γ_1, γ_2)
0.3		0.3	[0.500, 0.400, 0.333, 0.286, 0.250]		$a = 3.5$	$p = 0.3$
0.6	1	0.6	[0.125, 0.143, 0.167, 0.200, 0.250]	$\frac{2}{\alpha_1}$		$\gamma_1 = 4$
0.8		0.8	[0.083, 0.091, 0.100, 0.111, 0.125]		$b = \frac{\alpha_1}{3.5}$	$\gamma_2 = \frac{(1-p)\alpha_1\gamma_1}{(\gamma_1 - \alpha p)}$

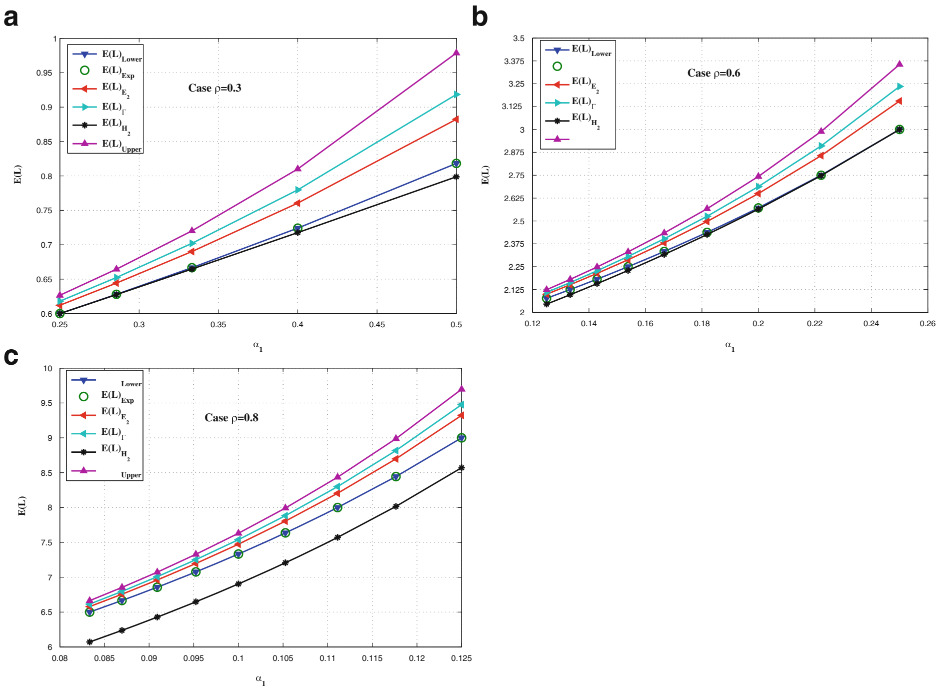

Fig. 1 Comparison of the $\mathbb{E}(L)$ in $M/M/1$ queue with general retrial times versus α_1

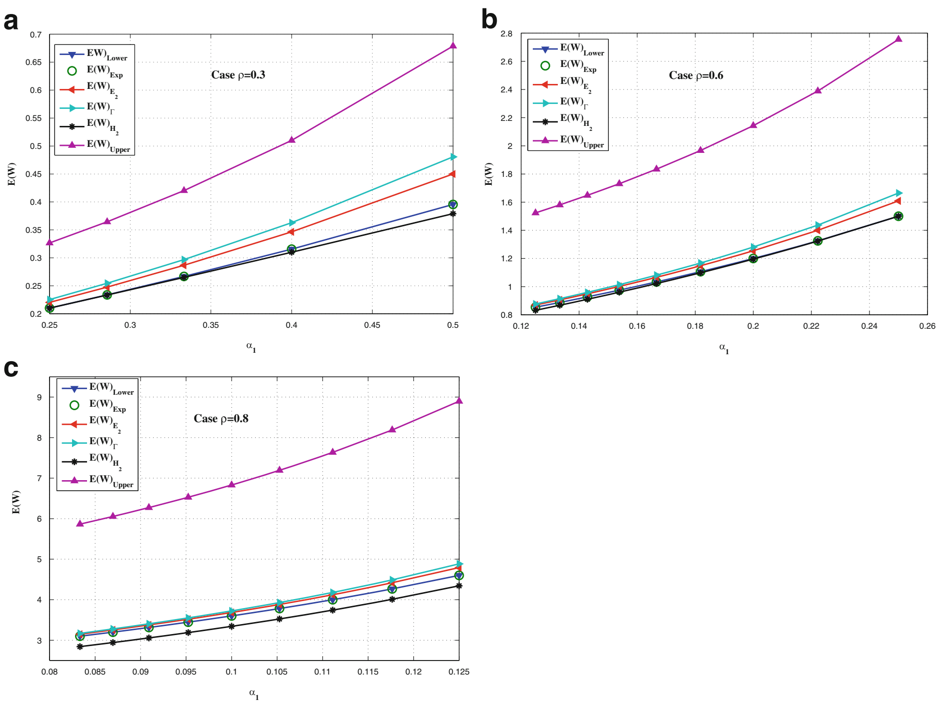

Fig. 2 Comparison of the $\mathbb{E}(W)$ in $M/M/1$ queue with general retrial times versus α_1

- If the distribution of the retrial time is close to the exponential distribution in the Laplace transform, then the exact value $\mathbb{E}(L)_{A(x)}$ (respectively, $\mathbb{E}(W)_{A(x)}$) is closer to the lower bound $\mathbb{E}(L)_{\text{Lower}}$ (respectively, $\mathbb{E}(W)_{\text{Lower}}$) (see the case of $E(L)_{E_2}$ and $\mathbb{E}(W)_{E_2}$).
- Both considered characteristics depend closely on the inter-retrial times distribution and its first moment α_1. In addition, this dependence appears clearly in the case of heavy traffic , i.e., when $\rho \to 1$.

7 Conclusion

The main result of this paper consists to give insensitive bounds for the stationary distribution and some performance measures of the considered embedded Markov chain by using the theory of stochastic orderings. The result is confirmed by numerical illustrations.

In conclusion, the monotonicity approach holds promise for the solution of several systems with repeated attempts. Hence, it is worth noting that our approach can be further extended to more complex systems.

References

1. Artalejo, J.R., Gómez-Corral, A.: Channel idle periods in computer and telecommunication systems with customer retrials. Telecommun. Syst. **24**, 29–46 (2003)
2. Artalejo, J.R., Gómez-Corral, A.: Retrial Queueing Systems: A Computational Approach. Springer, Berlin (2008)
3. Artalejo, J.R., López-Herrero, M. J.: Cellular mobile networks with repeated calls operating in random environment. Comput. Oper. Res. **37**, 1158–1166 (2010)
4. Artalejo, J.R., Pla, V.: On the impact of customer balking, impatience and retrials in telecommunication systems. Comput. Math. Appl. **57**, 217–229 (2009)
5. Boualem, M.: Insensitive bounds for the stationary distribution of a single server retrial queue with server subject to active breakdowns. Adv. Oper. Res. **2014**, Article ID 985453, 12 pp. (2014)
6. Boualem, M., Djellab, N., Aïssani, D.: Stochastic inequalities for $M/G/1$ retrial queues with vacations and constant retrial policy. Math. Comput. Model. **50**, 207–212 (2009)
7. Boualem, M., Djellab, N., Aïssani, D.: Stochastic approximations and monotonicity of a single server feedback retrial queue. Math. Probl. Eng. **2012**, Article ID 536982, 13 pp. (2012)
8. Boualem, M., Djellab, N., Aïssani, D.: Stochastic bounds for a single server queue with general retrial times. Bull. Iranian Math. Soc. **40**, 183–198 (2014)
9. Bušić, A., Fourneau, J.M.: Monotonicity and performance evaluation: applications to high speed and mobile networks. Cluster Comput. **15**, 401–414 (2012)
10. Economou, A., López-Herrero, M.J.: Performance analysis of a cellular mobile network with retrials and guard channels using waiting and first passage time measures. Eur. Trans. Telecommun. **20**, 389–401 (2009)
11. Falin, G.I., Templeton, J.G.C.: Retrial Queues. Chapman and Hall, London (1997)
12. Gómez-Corral, A.: Stochastic analysis of a single server retrial queue with the general retrial times. Nav. Res. Logist. **46**, 561–581 (1999)

13. Houck D.J., Lai, W.S.: Traffic modeling and analysis of hybrid fiber-coax systems. Comput. Netw. ISDN Syst. **30**, 821–834 (1998)
14. Janssens, G.K.: The quasi-random input queueing system with repeated attempts as a model for a collision-avoidance local area network. IEEE Trans. Commun. **45**, 360–364 (1997)
15. Liu, X., Fapojuwo, A.O.: Performance analysis of hierarchical cellular networks with queueing and user retrials. Int. J. Commun. Syst. **19**, 699–721 (2006)
16. Machihara, F., Saitoh, M.: Mobile customers model with retrials. Eur. J. Oper. Res. **189**, 1073–1087 (2008)
17. Mokdad, L., Castel-Taleb, H.: Stochastic comparisons: a methodology for the performance evaluation of fixed and mobile networks. Comput. Commun. **31**, 3894–3904 (2008)
18. Müller, A., Stoyan, D.: Comparison Methods for Stochastic Models and Risk. Wiley, New York (2002)
19. Pustova, S.V.: Investigation of call centers as retrial queuing systems. Cybern. Syst. Anal. **46**, 494–499 (2010)
20. Ross, S.M.: Stochastic Processes. Wiley, New York (1983)
21. Shaked, M., Shanthikumar, J.G.: Stochastic Orders and their Applications. Academic, San Diego (1994)
22. Shaked, M., Shanthikumar, J.G.: Stochastic Orders. Academic, New York (2006)
23. Stoyan, D.: Comparison Methods for Queues and Other Stochastic Models. Wiley, New York (1983)
24. Van Do, T.: A new computational algorithm for retrial queues to cellular mobile systems with guard channels. Comput. Ind. Eng. **59**, 865–872 (2010)
25. Wüchner, P., Sztrik, J., de Meer, H.: Modeling wireless sensor networks using finite-source retrial queues with unreliable orbit. In: Hummel, K.A. et al. (eds.) PERFORM 2010 (Haring Festschrift), LNCS 6821, pp. 73–86 (2011)
26. Xue, F., Yoo, S.J.B., Yokoyama, H., Horiuchi, Y.: Performance analysis of wavelength-routed optical networks with connection request retrials. In: IEEE International Conference on Communication ICC 2005, vol. 3, pp. 1813–1818. IEEE, New York (2005)

Stochastic Analysis of an M/G/1 Retrial Queue with FCFS

Mohamed Boualem, Mouloud Cherfaoui, Natalia Djellab, and Djamil Aïssani

Abstract The main goal of this paper is to investigate stochastic analysis of a single server retrial queue with a First-Come-First-Served (FCFS) orbit and non-exponential retrial times using the monotonicity and comparability methods. We establish various results for the comparison and monotonicity of the underlying embedded Markov chain when the parameters vary. Moreover, we prove stochastic inequalities for the stationary distribution and some simple bounds for the mean characteristics of the system. We validate stochastic comparison method by presenting some numerical results illustrating the interest of the approach.