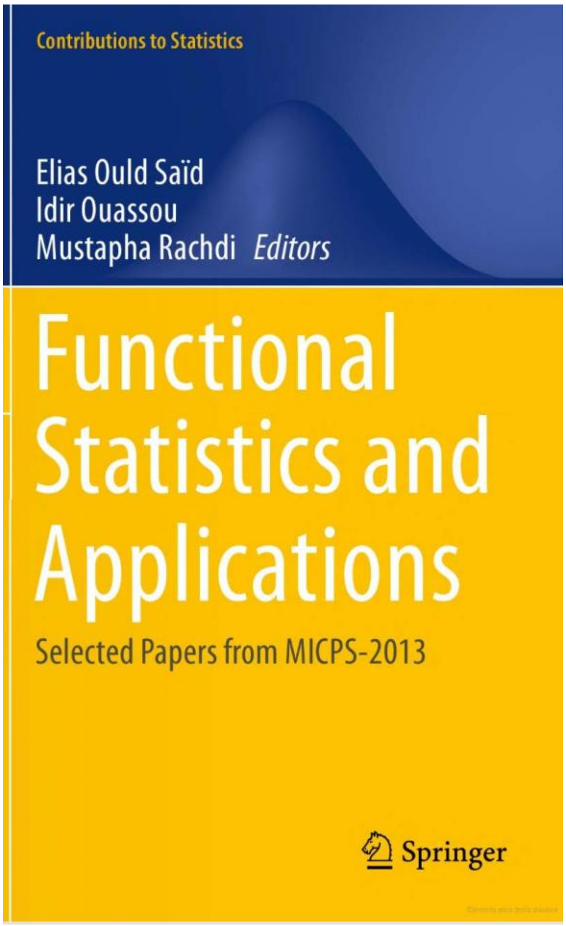

Impact of Nonparametric Density Estimation on the Approximation of the $G/G/1$ Queue by the $M/G/1$ One

Aïcha Bareche and Djamil Aïssani

Abstract In this paper, we show the interest of nonparametric boundary density estimation to evaluate a numerical approximation of $G/G/1$ and $M/G/1$ queueing systems using the strong stability approach when the general arrivals law G in the $G/G/1$ system is unknown. A numerical example is provided to support the results. We give a proximity error between the arrival distributions and an approximation error on the stationary distributions of the quoted systems.

Contributions to Statistics

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Functional Statistics and Applications

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ISSN 1431-1968
Contributions to Statistics
ISBN 978-3-319-22475-6 ISBN 978-3-319-22476-3 (eBook)
DOI 10.1007/978-3-319-22476-3

Library of Congress Control Number: 2015957212

Mathematics Subject Classification (2010): 60H30, 60H07, 62G08, 62G20, 92B15, 60H40

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

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1 Introduction

Because of the complexity of some queueing models, analytic results are generally difficult to obtain or are not very exploitable in practice. That is the case, for example, in the $G/G/1$ queueing system, where the Laplace transform or the generating function of the waiting time distribution is not available in a closed form [20]. Indeed, when a practical study is performed in queueing theory, one often replaces a real system by another one which is close to it in some sense but simpler in structure and/or components. The queueing model so constructed represents an idealization of the real queueing one, and hence the "stability" problem arises.

One of the stability methods is the strong stability approach [2, 19] which has been developed in the beginning of the 1980s. It can be used to investigate the ergodicity and stability of the stationary and non-stationary characteristics of Markov chains. In contrast to other methods, the strong stability approach supposes that the perturbation of the transition kernel is small with respect to a certain norm. Such a stringent condition allows us to obtain better estimates on the characteristics of the perturbed chain. Besides the ability to make qualitative analysis of some

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E. Ould Saïd et al. (eds.), *Functional Statistics and Applications*, Contributions to Statistics, DOI 10.1007/978-3-319-22476-3_7

complex systems, one importance particularity of the strong stability method is the possibility to obtain stability inequalities with an exact computation of the constants.

The applicability of this approach is well proved and documented in various situations and for different proposals. In particular, it has been applied to several queueing models [1, 10, 15, 17, 23] and inventory models [22].

Note that the first attempt to "measure" the performance of the strong stability method has been used in practice, and has been particularly applied to a simple system of queues [12, 13]. The approach proposed is based on the classical approximation method where the authors perform the numerical proximity of the stationary distribution of an $Hyp/M/1$ (respectively $M/Cox2/1$) system by the one of an $M/M/1$ system when applying the strong stability method. For the first time, Bareche and Aïssani [5] specify an approximation error on the stationary distributions of the $G/M/1$ (resp. $M/G/1$) and $M/M/1$ systems when the general law of arrivals (resp. service times) G is unknown and its density function is estimated by using the kernel density method. In [11], the authors use the discrete event simulation approach and the Student test to measure the performance of the strong stability method through simple numerical examples for a concrete case of queueing systems (the $G/M/1$ queue after perturbation of the service law [9], and the $M/G/1$ limit model for high retrial intensities (which is the classical $M/G/1$ system) after perturbation of the retrials parameter [10]). The same idea has been already investigated for an approximation analysis of the classical $G/G/1$ queue when the general law of service is unknown and must be estimated by different statistical methods, pointing out particularly to the impact of those taking into account the correction of boundary effects [6], see also the recent work of [7] and [8]. For example, in the latter work [8], besides of showing the interest of combining nonparametric methods with the strong stability principle for the study of the $M/G/1$ system, we also pointed out the importance of using the Student test to provide confidence intervals for the difference between the corresponding characteristics of the two considered queueing systems for the aim of comparing them (i.e., comparison of their characteristics).

Indeed, note that in practice all model parameters are imprecisely known because they are obtained by means of statistical methods. In this sense, our contribution concerns one aspect which is of some practical interest and has not been sufficiently studied in the literature; for instance, when a distribution governing a queueing system is unknown and we resort to nonparametric methods to estimate its density function. Besides, as the strong stability method assumes that the perturbation is small, then we suppose that the arrivals law of the $G/G/1$ system is close to the exponential one with parameter λ. This permits us to consider the problem of boundary bias correction [14, 16, 25] when performing nonparametric estimation of the unknown density of the law G, since the exponential law is defined on the positive real line.

It is why we use, in this paper, the tools of nonparametric density estimation to approximate the complex $G/G/1$ system by the simpler $M/G/1$ one, on the basis of the theoretical results addressed in [3] involving the strong stability of the $M/G/1$ system. When the distribution of arrivals is general but close to the exponential distribution, it is possible to approximate the characteristics of the $G/G/1$ system by those of the $M/G/1$ one, if we prove the fact of stability (see [2]). This substitution of characteristics is not justified without a prior estimation of the corresponding approximation error. This gives rise to the following question: Is it possible to precise the error of the proximity between the two systems?

Note that unlike [5] where kernel density estimation was used for the study of the strong stability of $M/M/1$ system, we consider here two new aspects. The first one concerns the model motivation: in queueing theory, there exist explicit formulas to determine some performance measures of the $M/G/1$ system. Unfortunately, for the $G/G/1$ system, these exact formulas are not known. So, if we suppose that the $G/G/1$ system is close to the $M/G/1$ one, then we can use the formulas obtained for the $M/G/1$ system to approximate the $G/G/1$ system characteristics. The second point deals with the use of a new class of nonparametric density estimation to remove boundary effects. This is the class of flexible estimators, for instance asymmetric kernels and smoothed histograms. Note also that unlike [6] where the perturbation concerns the service duration, we perturb the arrival flux.

This article is organized as follows: In Sect. 2, we describe the considered queueing models and we present briefly the strong stability of the $M/G/1$ system. In Sect. 3, we first provide a short review of boundary bias correction techniques in nonparametric density estimation, then we give the main results of this paper which are illustrated by a numerical case study based on simulation results.

2 Approximating $G/G/1$ Queue by the $M/G/1$ One Using Strong Stability Approach

2.1 *Description of the Models*

Consider a $G/G/1$ $(FIFO, \infty)$ queueing system with general service times distribution H and general inter-arrival times probability distribution G. The following notations are used: T_n (the arrival time of the nth customer), θ_n (the departure time of the nth customer), and γ_n (the time till the arrival of the following customer after θ_n). Let us designate by $\nu_n = \nu(\theta_n + 0)$ the number of customers in the system immediately after θ_n. ξ_n represents the service time of the nth customer arriving at the system. It is proved that $X_n = (\nu_n, \gamma_n)$ forms a homogeneous Markov chain with state space $\mathbb{N} \times \mathbb{R}^+$ and transition operator $Q = (Q_{ij})_{i,j\geq 0}$, where $Q_{ij}(x, dy) = P(\nu_{n+1} = j, \gamma_{n+1} \in dy/\nu_n = i, \gamma_n = x)$ is defined by (see [3]):

$$Q_{ij} = \begin{cases} q_j(dy), & \text{if } i = 0; \\ q_{j-i}(x, dy), & \text{if } i \geq 1, \quad j \geq i; \\ p(x, dy), & \text{if } j = i - 1, \quad i \geq 1; \\ 0, & \text{otherwise}; \end{cases} \tag{1}$$

where

$$\begin{cases} q_j(dy) = \int P(T_j \leq u < T_{j+1}, T_{j+1} - u \in dy)dH(u); \\ q_j(x,dy) = \int\limits_x^\infty P(T_j \leq u - x < T_{j+1}, T_{j+1} - (u-x) \in dy)dH(u); \\ p(x,dy) = \int\limits_0^x P(x-u \in dy)dH(u). \end{cases}$$

Let us also consider an $M/G/1$ $(FIFO, \infty)$ system with exponential inter-arrivals distribution, E_λ, with parameter λ and take the same service times distribution than the $G/G/1$ one. We introduce the corresponding following notations: $\bar{T}_n$, $\bar{\theta}_n$, $\bar{\gamma}_n$, $\bar{\nu}_n = \bar{\nu}(\bar{\theta}_n - 0)$ and ξ_n defined as above. The transition operator $\bar{Q} = \left(\bar{Q}_{ij}\right)_{i,j>0}$ of the corresponding Markov chain $\bar{X}_n$ in the $M/G/1$ system has the same form as in (1), where

$$\begin{cases} \overline{q}_j(dy) = p_j E_\lambda(dy), \;\; \overline{q}_j(x,dy) = p_j(x)E_\lambda(dy), \;\; \overline{p}(x,dy) = p(x,dy); \\ p_j = \int \exp(-\lambda u)\frac{(\lambda u)^j}{j!}dH(u); \\ p_j(x) = \int_x^\infty \exp(-\lambda(u-x))\frac{(\lambda(u-x))^j}{j!}dH(u). \end{cases}$$

Let us suppose that the arrival flow of the $G/G/1$ system is close to the Poisson one. This proximity is then characterized by the metric:

$$w^* = w^*(G, E_\lambda) = \int \varphi^*(t)|G - E_\lambda|(dt), \tag{2}$$

where φ^* is a weight function and $|a|$ designates the variation of the measure a. We take $\varphi^*(t) = e^{\delta t}$, with $\delta > 0$. In addition, we use the following notations:

$$\begin{cases} E^* = \int \varphi^*(t)E_\lambda(dt), \\ G^* = \int \varphi^*(t)G(dt), \end{cases}$$

$$w_0 = w_0(G, E_\lambda) = \int |G - E_\lambda|(dt). \tag{3}$$

2.2 Strong Stability Criterion

For a general framework on the strong stability method, the reader is referred to [2, 19]. However, it is interesting to recall the following basic definition.

Definition 1 (See [2, 19]) The Markov chain X with transition kernel P and invariant measure π is said to be υ-strongly stable with respect to the norm $\|.\|_\upsilon$ (defined for each measure α as follows: $\|\alpha\|_\upsilon = \sum\limits_{j\geq 0} \upsilon(j)|\alpha_j|$), if $\|P\|_\upsilon < \infty$ and each stochastic kernel Q in some neighborhood $\{Q : \|Q - P\|_{\upsilon} < \epsilon\}$ has a unique invariant measure $\mu = \mu(Q)$ and $\|\pi - \mu\|_{\upsilon} \to 0$ as $\|Q - P\|_{\upsilon} \to 0$.

2.3 Strong Stability Bounds

The following theorem determines the υ-strong stability conditions of the $M/G/1$ system after a small perturbation of the arrivals law. It also gives the estimates of the deviations of both the transition kernels and the stationary distributions.

Theorem 1 ([3]) *Suppose that in the $M/G/1$ system, the following ergodicity condition holds:*

(a) $\lambda\ \mathbf{E}(\xi) < 1$; *(b)* $\exists a > 0 : \ \mathbf{E}(e^{a\xi}) = \int e^{au} dH(u) < \infty$.

Suppose also that $E^* < \infty$ and $\beta_0 = \sup(\beta : H^*(\lambda - \lambda\beta) < \beta)$, where H^* is the Laplace transform of the probability density of the service times. Then, for all β such that $1 < \beta < \beta_0$, the Markov chain $\overline{X}_n$ is υ-strongly stable for the function $\upsilon(n, t) = \beta^n[\exp(-\alpha t) + c^{-1}\varphi^*(t)]$, where:

$$\alpha > 0, \ \ c = \frac{\beta E^*}{1 - \rho}, \ \ \textit{and} \ \ \rho = \frac{H^*(\lambda - \lambda\beta) + \beta}{2\beta} < 1.$$

In addition, if $G^* < \infty$, and $w_0 \le \frac{(\beta_0 - \beta)}{\beta_0^2}$, then we have the margin between the transition operators:

$$\|Q - \overline{Q}\|_{\upsilon} \le w^*(1 + \beta) + w_0 G^*(1 + \lambda\beta)\frac{\beta_0^4}{(\beta_0 - \beta)^2}.$$

Moreover, if the general distribution of arrivals G is such that:

$$w^*(G, E_\lambda) \le \frac{1 - \rho}{2c_0(1 + c)}(1 + \beta + c_1)^{-1},$$

$$w_0(G, E_\lambda) \le \frac{(\beta_0 - \beta)}{\beta_0^2},$$

we obtain the deviation between the stationary distributions π and $\bar{\pi}$ associated, respectively, to the Markov chains X_n and $\bar{X}_n$, given by:

$$Er := \|\pi - \overline{\pi}\| \le 2[(1 + \beta)w^* + c_1 w_0]c_0 c_2(1 + c), \tag{4}$$

where $c_0, \ c_1, \ c_2$ are defined as follows:

$$c_0^{'} \le c_0,$$

where

$$c_0^{'} = 1 + \frac{(1-\lambda m)(\beta-1)(2-\rho)E^*}{2(1-\rho)^2} \quad \text{and} \quad m = \mathbf{E}(\xi),$$

$$c_1 = G^*(1+\lambda\beta)\frac{\beta_0^4}{(\beta_0-\beta)^2},$$

$$c_2 = \frac{(1-\lambda m)(\beta-1)(2-\rho)}{2(1-\rho)\beta}.$$

Note that the bound in formula (4) of Theorem 1 involves the computation of w^* and w_0 and methods to do so will be discussed in the following.

3 Nonparametric Estimation for Approximating the $G/G/1$ System by the $M/G/1$ One

We want to apply nonparametric density estimation methods to determine the variation distances w_0 and w^* defined, respectively, in (2) and (3), together with the proximity error Er defined in (4) between the stationary distributions of the $G/G/1$ and $M/G/1$ systems. We first give an overview of nonparametric estimation methods which are required to compute w_0 and w^* measures, then we perform a simulation study.

3.1 Nonparametric Density Estimation Methods

The most known and used nonparametric estimation method is the kernel density estimation. If $X_1, \ldots, X_n$ is a sample coming from a random variable X with probability density function f and distribution F, then the Parzen–Rosenblatt kernel estimator [21, 24] of the density $f(x)$ for each point $x \in \mathbb{R}$ is given by:

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h_n}\right), \tag{5}$$

where K is a symmetric density function called the kernel and h_n is the bandwidth.

The classical symmetric kernel estimate works well when estimating densities with unbounded support. However, when these latter are defined on the positive real line $[0, \infty[$, without correction, the kernel estimates suffer from boundary effects since they have a boundary bias (the expected value of the standard kernel estimate at $x = 0$ converges to the half value of the underlying density when f is twice continuously differentiable on its support $[0, +\infty)$ [14, 25]). In fact, using a fixed symmetric kernel is not appropriate for fitting densities with bounded supports as a weight is given outside the support.

Several approaches for handling the boundary effects in nonparametric density estimation have been introduced. They propose the use of estimators based on flexible kernels (asymmetric kernels [14, 16] and smoothed histograms [14]). They are very simple in implementation, free of boundary bias, always nonnegative, their support matches the support of the probability density function to be estimated, and their rate of convergence for the mean integrated squared error is $O(n^{-4/5})$.

Below, are briefly discussed the estimators which we will use in the context of this paper.

Reflection Method

Schuster [25] suggests creating the mirror image of the data on the other side of the boundary and then applying the estimator (5) for the set of the initial data and their reflection. $f(x)$ is then estimated, for $x \geq 0$, as follows:

$$\tilde{f}_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} \left[K\left(\frac{x - X_j}{h_n}\right) + K\left(\frac{x + X_j}{h_n}\right) \right]. \tag{6}$$

Asymmetric Gamma Kernel Estimator

Asymmetric kernels [14, 16] are defined by the form

$$\hat{f}_b(x) = \frac{1}{n} \sum_{i=1}^{n} K(x, b)(X_i), \tag{7}$$

where b is the bandwidth and the asymmetric kernel K can be taken as a Gamma density K_G with the parameters $(x/b + 1, b)$ given by

$$K_G\left(\frac{x}{b} + 1, b\right)(t) = \frac{t^{x/b} e^{-t/b}}{b^{x/b+1} \Gamma(x/b + 1)}. \tag{8}$$

Smoothed Histograms

Smoothed histograms [14] are defined by the form

$$\hat{f}_k(x) = k \sum_{i=0}^{+\infty} \omega_{i,k} p_{ki}(x), \tag{9}$$

where the random weights $\omega_{i,k}$ are given by

$$\omega_{i,k} = F_n\left(\frac{i+1}{k}\right) - F_n\left(\frac{i}{k}\right),$$

where F_n is the empiric distribution, k is the smoothing parameter, and $p_{ki}(.)$ can be taken as a Poisson distribution with parameter kx,

$$p_{ki}(x) = e^{-kx}\frac{(kx)^i}{i!}, \qquad i = 0, 1, \ldots \tag{10}$$

3.2 Algorithm

To realize this work, we use the discrete event simulation approach [4] to simulate the according systems and we elaborate an algorithm which follows the following steps:

(1) Generation of a sample of size n of general arrivals distribution G with theoretical density $g(x)$.
(2) Use of a nonparametric estimation method to estimate the theoretical density function $g(x)$ by a function denoted in general $g_n^*(x)$.
(3) Calculation of the mean arrival rate given by:
$\lambda = 1/\int xdG(x) = 1/\int xg(x)dx = 1/\int xg_n^*(x)dx$.
(4) Verification, in this case, of the strong stability conditions given in Sect. 2.3. For calculation considerations, the variation distances w_0 and w^* are given, respectively, by: $w_0 = \int |G - E_\lambda|(dx) = \int |g_n^* - e_\lambda|(x)dx$ and $w^* = \int e^{\delta x}|G - E_\lambda|(dx) = \int e^{\delta x}|g_n^* - e_\lambda|(x)dx$, where $\delta > 0$.
(5) Computation of the minimal error on the stationary distributions of the considered systems according to (4).

Simulation studies were performed under Matlab 7.1 environment. The Epanechnikov kernel [26] is used throughout for estimators involving symmetric kernels. The bandwidth h_n is chosen to minimize the criterion of the “least squares cross-validation” [18]. The smoothing parameters b and k are chosen according to a bandwidth selection method which leads to an asymptotically optimal window in the sense of minimizing L_1 distance [14].

3.3 Numerical Example

We consider a $G/G/1$ system such that the general inter-arrivals distribution G is assumed to be a Gamma distribution with parameters $\alpha = 0.7, \beta = 2$, denoted $\Gamma(0.7, 2)$, with a theoretical density $g(x)$ and the service times distribution is Cox2 with parameters: $\mu_1 = 3$, $\mu_2 = 10$, $a = 0.005$.

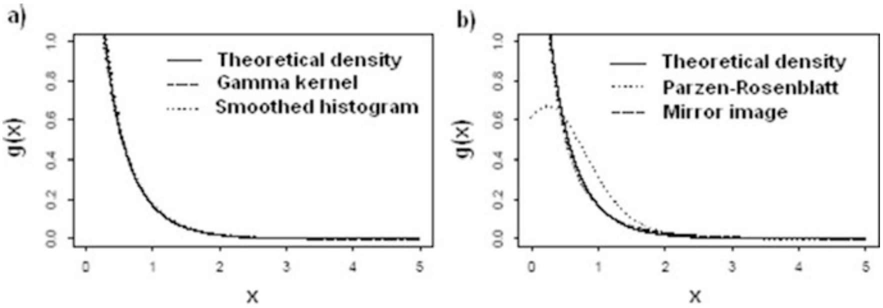

Fig. 1 Theoretical density $g(x) = \gamma(0.7, 2)(x)$ and estimated densities. (**a**) Gamma kernel and smoothed histogram estimates; (**b**) Parzen-Rosenblatt and Mirror image estimates. Taken from [8]. Published with the kind permission of ©SCITEPRESS 2014. All rights reserved

Table 1 Performance measures with different estimators

	$g(x)$	$g_n(x)$	$\tilde{g}_n(x)$	$\hat{g}_b(x)$	$\hat{g}_k(x)$
Mean arrival rate λ	1.6874	1.5392	1.6503	1.6851	1.6840
Traffic intensity of the system $\frac{\lambda}{\mu}$	0.1562	0.1578	0.1570	0.1564	0.1567
Variation distance w_0	0.0096	0.1287	0.0114	0.0102	0.0105
Variation distance w^*	0.0183	0.2536	0.0311	0.0206	0.0224
Error on stationary distributions Er	0.0356		0.0452	0.0378	0.0377

Taken from [8]. Published with the kind permission of ©SCITEPRESS 2014. All rights reserved

By generating a sample coming from the $\Gamma(0.7, 2)$ distribution, we use the different nonparametric estimators given, respectively, in (5)–(10) to estimate the theoretical density $g(x)$.

For these estimators, we take the sample size $n = 200$ and the number of simulations $R = 100$.

Curves of the theoretical and estimated densities are illustrated in Fig. 1 (taken from [8]). Different performance measures are listed in Table 1 (taken from [8]).

Interpretation of Results

Figure 1 shows that the use of nonparametric density estimation methods taking into account the correction of boundary effects improves the quality of the estimation (compared to the curve of the Parzen–Rosenblatt estimator, those of mirror image, asymmetric Gamma kernel, and smoothed histogram estimators are closer to the curve of the theoretical density).

We note in Table 1 that the approximation error on the stationary distributions of the $G/G/1$ and $M/G/1$ systems was given when applying nonparametric density estimation methods by considering the correction of boundary effects such in the cases of using the mirror image estimator ($Er = 0.0452$), asymmetric Gamma kernel estimator ($Er = 0.0378$), and smoothed histogram ($Er = 0.0377$). In addition, these two last errors are close to the one given when using the theoretical density $g(x)$ ($Er = 0.0356$). But, when applying the Parzen–Rosenblatt estimator which does not take into account the correction of boundary effects, the approximation error Er on the stationary distributions of the quoted systems could not be given. This shows the importance of the smallness of the proximity error of the two corresponding arrival distributions of the considered systems, characterized by the variation distances w_0 and w^*.

4 Conclusion

We use statistical techniques, for instance, nonparametric density estimation with boundary effects considerations to measure the performance of the strong stability method in a $M/G/1$ queueing system after perturbation of the arrival flow.

The obtained results show particularly the interest of nonparametric estimation methods and the techniques of correction of boundary effects to determine the approximation error of the stationary distributions between two queueing systems when applying the strong stability method in order to substitute the characteristics of a complex real system by another simpler ideal one.

Note that, in practice, all model parameters are imprecisely known because they are obtained by means of statistical methods. That is why the strong stability inequalities will allow us to numerically estimate the uncertainty shown during this analysis. In our case, if one had real data, then one could apply the kernel density method to estimate the density function. By combining the techniques of correction of boundary effects with the calculation of the variation distance characterizing the proximity of the quoted systems, one will be able to check if this density is sufficiently close to that of the Poisson law (or that of the exponential law), and apply then the strong stability method to approximate the characteristics of the real system by those of a classical one.

A close field of some practical interest is networks of queues. Indeed, for modeling some complex physical systems, a simple queue is not sufficient, so we may resort to networks of queues. However, few among them have simple analytic solutions. This is mainly due to the difficulty of studying the properties of inter-stations fluxes. In fact, the only known exact results are those of networks having the product form property, such as the Jackson networks. There comes the interest of analyzing such networks by combining the strong stability aspect and the boundary correction techniques.

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Impact of Nonparametric Density Estimation on the Approximation of the $G/G/1$ Queue by the $M/G/1$ One

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Abstract In this paper, we show the interest of nonparametric boundary density estimation to evaluate a numerical approximation of $G/G/1$ and $M/G/1$ queueing systems using the strong stability approach when the general arrivals law G in the $G/G/1$ system is unknown. A numerical example is provided to support the results. We give a proximity error between the arrival distributions and an approximation error on the stationary distributions of the quoted systems.