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Approximate analysis of an $GI/M/\infty$ queue using the strong stability method

Aicha Bareche * **Mouloud Cherfaoui** ** **Djamil Aïssani** ***

* *Research Unit LaMOS (Modeling and Optimization of Systems), University of Bejaia, 06000 Bejaia, Algeria, (e-mail: aicha_bareche@yahoo.fr).*
** *Department of Mathematics, University of Biskra, 07000 Biskra, Algeria (e-mail: mouloudcherfaoui2013@gmail.com)*
*** *Research Unit LaMOS (Modeling and Optimization of Systems), University of Bejaia, 06000 Bejaia, Algeria, (e-mail: lamos_bejaia@hotmail.com)*

Abstract: In this work, we are interested in the approximation of the stationary characteristics of the $GI/M/\infty$ system by those of an $M/M/\infty$ system. In other words, we propose to study the strong stability of the $M/M/\infty$ system (ideal system) when the arrivals flow is subject to a small perturbation (the $GI/M/\infty$ is the resulting perturbed system).
For this purpose, we first determine the approximation conditions of the characteristics of the perturbed queuing system, and under these conditions we obtain the stability inequalities of the stationary distribution of the queue size.
To evaluate the performance of the proposed method, we develop an algorithm which allows us to compute the various theoretical results and which is executed on some systems ($Coxian_2/M/\infty$ and $E_2/M/\infty$) in order to compare its output results with those of simulation.

Keywords: Infinite-servers queue, Embedded Markov chain, Perturbation, Strong stability.

1. INTRODUCTION

Infinite servers queues (multi-servers queues) are common in real life. Indeed, in practice, many situations are modeled by an infinite-servers queue, for instance: the production systems, computer systems, computer communications and telecommunications, aerodrome management of an airport, ... However, the analytical results of such systems are generally expressed in terms of Laplace transforms or/and generating functions, which are not available in a closed form.

For this reason there exists, when a practical study is performed in queueing theory, a common technique for substituting the real but complicated elements governing a queueing system by simpler ones in some sense close to the real elements. The queueing model so constructed represents an idealization of the real queueing one, and hence the stability problem arises. The stability problem in queueing theory is concerned with the domain within which the ideal queueing model may be taken as a good approximation of the real queueing system under consideration. In other words, we clarify the conditions for which the proximity in one way or another of the parameters of the system involves the proximity of the studied characteristics.

In recent years, the practical needs have driven the research towards the determination of estimates and quantitative performance measurement methods of stability. It is why we will place more emphasis, in this work, on the strong stability method (see Aïssani and Kartashov (1983); Kartashov (1996)) which allows us to make both qualitative and quantitative analysis helpful in understanding complicated models by more simpler ones for which an evaluation can be made. This method, also called "method of operators" can be used to investigate the ergodicity and stability of the stationary and non-stationary characteristics of the imbedded Markov chains (see Kartashov (1996)). In contrast to other methods, it supposes that the perturbations of the transition kernel are small with respect to some norms in the operators space. This stringent condition gives better stability estimates and enables us to find precise asymptotic expansions of the characteristics of the perturbed system.

Besides, note that, in practice, all model parameters are imprecisely known because they are obtained by means of statistical methods. That is why the strong stability inequalities will allow us to numerically estimate the uncertainty shown during this analysis.

The applicability of the strong stability method is well proved and documented in various fields and for different purposes. In particular, it has been applied to several queueing models (see for example Bareche and Aïssani (2008); Benaouicha and Aïssani (2005); Berdjoudj et al. (2012); Bouallouche-Medjkoune and Aïssani (2006a,b)).

On the other hand, note that there are some classical works, in the literature, about the approximation of the infinite servers queues. A traditional trend is the use of the heavy traffic approximation based on a diffusion process (see Gross et al. (2013); Glynn and Whitt (1991) for

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general views). Some other particular cases can be found in Guillemin et al. (1996) for the $M/M/\infty$ queue, Hall (1985) for the $M/G/\infty$ queue, Halfin and Whitt (1981) for the $GI/M/s$ queue and Whitt (1982) for the $GI/G/\infty$ queue. Simulation is another approximation technique of own interest, see for example Ferreira (2013a,b). Ramalhoto (1999) gave a heuristic approximation of the infinite server queue by the multi-server queue with and without retrials. Helary and Pedrono (1983) used the theory of Markov chains to study the particular case of approximation of the $M/M/s$ queue by the $M/M/\infty$ one. The authors had produced two important results. The first one is summarized in the upper bound of the absolute difference (L_1 norm) between the stationary probabilities of the $M/M/s$ and $M/M/\infty$ systems, and in the second point the authors have proved that this difference tends to zero when the number of servers tends to infinity.

The aim of our paper is the study of the strong stability of an $M/M/\infty$ system (ideal system) after a small perturbation of its arrivals flow (the $GI/M/\infty$ is the resulting perturbed system). We first clarify the conditions for such an approximation, then we provide an upper bound for the norm of the difference between the two stationary distributions of the considered systems. This work is motivated by the lack of results on approximation analysis of the particular case of the $GI/M/\infty$ queue. Unlike Helary and Pedrono (1983), we use here the general weight norm ($\|.\|_v$) instead of the norm (L_1) to approach the $GI/M/\infty$ system by the $M/M/\infty$ one. Moreover, we provide the domain within which this approximation is valid.

The paper is organized as follows: First, we give in Section 1 a brief description of the models: the $GI/M/\infty$ and the $M/M/\infty$ systems. In Section 2, we present briefly the basics of the strong stability method. The main results of this paper are given, respectively, in Section 3 where we provide the detailed study of the stability of the $M/M/\infty$ system, and in Section 4 where we give two illustrative numerical applications.

2. DESCRIPTION OF $GI/M/\infty$ AND $M/M/\infty$ MODELS

To analyze the $GI/M/\infty$ queue, we can use the embedded markov chain technique which consists to identify a set of renewal points and relate the state probabilities at successive renewal points to each other.

For this, we suppose that the customers arrive at epochs $T_1, T_2, ...$, and assume that the inter-arrival times $T_{k+1} - T_k$ $(k = 0, 1, ...;\ T_0 = 0)$ are random variables which are mutually independent and identically distributed (i.i.d) with common distribution function $H(t) = P\{T_{k+l} - T_k \leq t\}$ $(k = 0, 1, \ldots)$ and mean inter-arrival time $1/\lambda$. Let X_k be the number of customers in the system just prior to the arrival of the k^{th} customer; X_k is the number of customers present at $T_k - 0$. Since the input is recurrent, and since the service times are by assumption i.i.d exponential random variables with mean $1/\mu$ and are independent of the arrival epochs, the arrival epochs $T_1, T_2, ...$, are a renewal points. Hence, $X = (X_k; k = 0, 1, \ldots)$ is an homogeneous Markov chain with a state space $\mathbf{N} = \{0, 1, 2, \ldots\}$ and its transition probabilities:

$$P_{ij} = P\{X_{k+1} = j/X_k = i\} \quad (j = 0, 1, ...; i = 0, 1, ...) \tag{1}$$

are given as follows,

$$P_{ij} = \begin{cases} \int\limits_0^\infty C_{i+1}^j e^{-j\mu t} \left(1 - e^{-\mu t}\right)^{i+1-j} dH(t), & \text{if } i + 1 \geq j\ ; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It can be shown also using the theory of Markov chains that a unique stationary distribution

$$\pi_j = \lim_{k\to\infty} P\{X_k = j\} \quad (j = 0, 1, ...) \tag{3}$$

exists if and only if $\int\limits_0^\infty t dH(t) < \infty$, (see Cooper (1981)).

Let us consider the same situation as the previous one ($GI/M/\infty$ system) but, in this case we assume that the inter-arrival times $\tilde{T}_{k+1} - \tilde{T}_k (k = 0, 1, ...; T_0 = 0)$ are i.i.d exponential random variables with mean $1/\lambda$. Let $\tilde{X}_k$ be the number of customers in the system just prior to the arrival of the k^{th} customer. The arrival epochs $\tilde{T}_1, \tilde{T}_2, \ldots$, in this case are also renewal points.

Moreover, $(\tilde{T}_{k+1} - \tilde{T}_k; k = 0, 1, \ldots)$ is a sequence of i.i.d random variables with a same common distribution function

$$\tilde{E_\lambda}(t) = P\{\tilde{T}_l - \tilde{T}_0 \leq t\} = P\{T_l - T_0 \leq t\} = E_\lambda(t).$$

Therefore, $\tilde{X} = (\tilde{X}_k; k = 0, 1, \ldots)$ is an homogeneous Markov chain with a state space $\mathbf{N} = \{0, 1, 2, \ldots\}$. The transition probabilities

$$\tilde{P}_{ij} = P\{\tilde{X}_{k+1} = j/\tilde{X}_k = i\} \quad (j = 0, 1, ...; i = 0, 1, ...) \tag{4}$$

are given as follows,

$$\tilde{P}_{ij} = \begin{cases} \int\limits_0^\infty C_{i+1}^j e^{-j\mu t} \left(1 - e^{-\mu t}\right)^{i+1-j} dE_\lambda(t), & \text{if } i + 1 \geq j\ ; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

It can be shown also using the theory of Markov chains that the system has a unique stationary distribution

$$\tilde{\pi}_j = \lim_{k\to\infty} P\{\tilde{X}_k = j\} \quad (j = 0, 1, ...) \tag{6}$$

defined as follows:

$$\tilde{\pi}_j = \frac{(\lambda/\mu)^j}{j!} e^{-(\lambda/\mu)} \quad (j = 0, 1, ...). \tag{7}$$

3. THE STRONG STABILITY CRITERIA AND PRELIMINARY NOTATIONS

Let $\mathcal{M} = \{\nu_j\}$ be the space of finite measures on $\mathbf{N}$, and let $\mathcal{N} = \{f(j)\}$ be the space of bounded measurable functions on $\mathbf{N}$. We associate with each transition kernel P the linear mapping:

$$(\mu P)_k = \sum_{j\geq 0} \mu_j P_{ik}, \tag{8}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki}. \tag{9}$$

Introduce on $\mathcal{M}$ the υ-norm of the form:

$$\|\nu\|_\upsilon = \sum_{j\geq 0} \upsilon(j)|\nu_j|, \tag{10}$$

where $\upsilon(k) = \beta^k$, for all $k \in \mathbf{N}$ and $\beta > 1$ is a real parameter. This norm induces in the space $\mathcal{N}$ the norm

$$\|f\|_\upsilon = \sup_{k\geq 0} \frac{|f(k)|}{\upsilon(k)}. \quad (11)$$

Moreover, for all $\nu \in \mathcal{M}$ and $f \in \mathcal{N}$, the symbols νf and $f \circ \nu$ denote respectively the summation and the kernel defined as below

$$\nu f = \sum_{k=0}^{+\infty} f(k)\nu_k, \quad (12)$$

$$(f \circ \mu)(k,j) = f(k)\,\mu_j, \text{ for all } (k,j) \in \mathbf{N}\times\mathbf{N}. \quad (13)$$

Let us consider $\mathcal{B}$, the space of linear operators, with the norm

$$\|Q\|_\upsilon = \sup_{k\geq 0} \frac{1}{\upsilon(k)} \sum_{j\geq 0} \upsilon(j) Q_{kj}. \quad (14)$$

Let ν and $\tilde{\nu}$ be two measures and suppose that these measures have finite υ-norm. For all f such that $|f(k)| \leq \Lambda\beta^k$ for some finite positive number Λ, we have

$$|\nu f - \tilde{\nu} f| \leq \|\nu - \tilde{\nu}\|_\upsilon \|f\|_\upsilon \inf_{k\geq 0} \upsilon(k) = \|\nu - \tilde{\nu}\|_\upsilon \|f\|_\upsilon$$
$$= \|\nu - \tilde{\nu}\|_\upsilon \sup_{k\geq 0} \frac{|f(k)|}{\beta^k}. \quad (15)$$

Let us give the definition of the strong stability for an homogeneous Markov chain in the space $(\mathbf{N}, \mathcal{B}(\mathbf{N}))$ with respect to the υ-norm. Here $\mathcal{B}(\mathbf{N})$ is the σ-algebra generated by the singletons $\{j\}$, $j \in \mathbf{N}$.

Definition 1. (Aïssani and Kartashov (1983); Kartashov (1996)) The Markov chain X with a transition kernel P and an invariant measure π is said to be strongly υ-stable with respect to the norm $\|.\|_\upsilon$ if $\|P\|_\upsilon < \infty$ and each stochastic kernel Q on the space $(\mathbf{N}, \mathcal{B}(\mathbf{N}))$ in some neighborhood

$$\{Q : \|Q - P\| < \epsilon\}$$

has a unique invariant measure $\nu = \nu(Q)$ and $\|\pi - \nu\|_\upsilon \to 0$ as $\|Q - P\|_\upsilon \to 0$.

The following theorem gives necessary and sufficient conditions for the strong stability of a Markov chain.

Theorem 1. (Aïssani and Kartashov (1983)) A Markov chain X, with transition kernel P and stationary distribution π, is strongly υ-stable if and only if there exists a measure α and a nonnegative measurable function h on $\mathbf{E}$ (here $\mathbf{E} = \mathbf{N}$) such that

a) $\pi h > 0$, $\alpha\mathbf{1} = 1$, $\alpha h > 0$;
b) $\|P\|_\upsilon < \infty$;
c) $T = P - h \circ \alpha \geq 0$;
d) there exists $m \geq 1$ and $\rho < 1$ such that $T^m\upsilon(x) \leq \rho\upsilon(x)$ for all $x \in \mathbf{E}$;

where $\mathbf{1}$ is the identity function.

The quantitative estimates can be obtained by using the following results.

Theorem 2. (Kartashov (1986)) Let X be a strongly $\upsilon-$stable Markov chain, with an invariant measure π and satisfying the conditions of theorem 1. If ν is the invariant measure of a kernel Q, then for the norm $\|Q - P\|_\upsilon$ sufficiently small, we have

$$\nu = \pi\left[I - \Delta R_0 (I - \Pi)\right]^{-1} = \pi + \sum_{t\geq 1} \pi\left[\Delta R_0 (I - \Pi)\right]^t,$$

where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$, $\Pi = \mathbf{1} \circ \pi$ is the stationary projector of the kernel P and I the identity kernel on $\mathcal{M}$.

Corollary 1. Under the conditions of theorem 1, for

$$\|\Delta\|_\upsilon < \frac{1-\rho}{c},$$

we have the estimation

$$\|\mu - \pi\|_\upsilon \leq \|\Delta\|_\upsilon c \|\pi\|_\upsilon \left(1 - \rho - c\|\pi\|_\upsilon\right)^{-1},$$

where

$$c = m\|P\|_\upsilon^{m-1}\left(1 + \|\mathbf{1}\|_\upsilon \|\pi\|_\upsilon\right)$$

and

$$\|\pi\|_\upsilon \leq (\alpha\upsilon)(1-\rho)^{-1}(\pi h) m \|P\|_\upsilon^{m-1}.$$

4. STRONG STABILITY IN THE $M/M/\infty$ SYSTEM

4.1 Strong stability conditions

The first step of the strong stability method is the determination of the $\upsilon-$stability conditions of the considered system, in other words it consists in delimiting the domain within the Markov chain $\tilde{X}_k$ associated to the analyzed system is strongly υ-stable after a small perturbation. In our case, the $\upsilon-$stability conditions of the $M/M/\infty$ system are given by Theorem 3.

Lemma 1. Suppose that in the $M/M/\infty$ system, the condition $\lambda < \mu$ is fulfilled, then there exists $\beta \in \left]1, 1 + \frac{(\mu-\lambda)(2\mu+\lambda)}{\lambda(\lambda+\mu)}\right[$ such that

$$\rho = \frac{1}{\beta}\left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < 1. \quad (16)$$

Proof. Let us consider relation (16).

$$(16) \Rightarrow \left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < \beta$$
$$\Rightarrow (\beta - 1) < \left(\frac{\mu-\lambda}{\mu+\lambda}\right)\left(\frac{2\mu+\lambda}{\lambda}\right)$$
$$\Rightarrow \beta < \left(\frac{\mu-\lambda}{\mu+\lambda}\right)\left(\frac{2\mu+\lambda}{\lambda}\right) + 1 = \phi(\lambda,\mu)$$
$$\Rightarrow \left(\frac{\mu-\lambda}{\mu+\lambda}\right)\left(\frac{2\mu+\lambda}{\lambda}\right) > 0, \quad \text{(since } \beta > 1)$$
$$\Rightarrow \mu - \lambda > 0, \quad \text{(because } \frac{2\mu+\lambda}{\lambda(\mu+\lambda)} > 0).$$

Hence, if $\mu > \lambda$ then, for all $\beta \in \left]1, 1 + \left(\frac{\mu-\lambda}{\mu+\lambda}\right)\left(\frac{2\mu+\lambda}{\lambda}\right)\right[$,

$$\rho = \frac{1}{\beta}\left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < 1.$$

Theorem 3. Suppose that in the $M/M/\infty$ system the condition of Lemma 1 holds. Then, for all β such that $1 < \beta < \beta_0$ the embedded Markov chain $\tilde{X}$ is $\upsilon-$strongly stable for the test function $\upsilon(k) = \beta^k$. Where $\beta_0 = \sup\{\beta/\beta > 1, \rho = \frac{1}{\beta}\left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < 1\}$.

Proof. To prove the strong υ-stability of the imbedded Markov chain $\tilde{X}$ for the test function $\upsilon(k) = \beta^k$, where $\beta > 1$, we use the strong stability criterion (Theorem 1). For this, we choose the measurable function:

$$h_i = \begin{cases} 0, \text{ if } i \geq 1; \\ 1, \text{ if } i = 0. \end{cases} \tag{17}$$

$$\alpha_j = P_{0j}(\infty) = \begin{cases} 0, & \text{if } j > 1; \\ \int_0^\infty (1 - e^{-\mu t})^{1-j} e^{-\mu j t} E_\lambda(t), & \text{if } j \leq 1. \end{cases} \tag{18}$$

Then, let us check conditions (a), (b), (c), and (d) of Theorem 1.

(1) We have $\tilde{\pi} h_i = \tilde{\pi}_0 > 0$ and $\alpha_j h_i = \alpha_0 = \tilde{P}_{00} > 0$. Therefore the conditions in a) are verified.

(2) We verify condition c). Indeed, we have

$$T_{ij} = \tilde{P}_{ij} - h_i \circ \alpha_j = \begin{cases} \tilde{P}_{ij} - \tilde{P}_{0j} \times 1, & \text{if } i = 0; \\ \tilde{P}_{ij} - \tilde{P}_{0j} \times 0 = \tilde{P}_{ij}, & \text{if } i \geq 1; \end{cases}$$

and from the definition of the kernel $(\tilde{P}_{ij})_{ij}$, it is obvious that $(T_{ij})_{ij} \geq 0$ for all $i, j \geq 0$.

(3) Let us verify condition d). From (5) and (9), we get

$$\begin{aligned}
(T\upsilon)(k) &= \sum_{j \geq 0} \beta^j T_{kj} = \sum_{j=0}^{k+1} \beta^j \tilde{P}_{kj} \\
&= \sum_{j=0}^{k+1} \beta^j \int_0^\infty C_{k+1}^j \left(1 - e^{-\mu t}\right)^{k+1-j} e^{-j\mu t} dE_\lambda(t) \\
&= \beta^{k+1} \int_0^\infty \left[\frac{1 - e^{-\mu t}}{\beta} + e^{-\mu t}\right]^{k+1} dE_\lambda(t) \\
&\leq \beta^{k+1} \int_0^\infty \left[\frac{1 - e^{-\mu t}}{\beta} + e^{-\mu t}\right]^{2} dE_\lambda(t) \\
&\leq \beta^k \, \frac{\lambda}{\beta} \left[\frac{1}{\lambda} + \frac{2(\beta - 1)}{\mu + \lambda} + \frac{(\beta - 1)^2}{2\mu + \lambda}\right].
\end{aligned} \tag{19}$$

It suffices to take

$$\rho = \frac{1}{\beta}\left(\frac{\lambda(\beta - 1)^2}{2\mu + \lambda} + \frac{2\lambda(\beta - 1)}{\mu + \lambda} + 1\right), \tag{20}$$

and so we obtain for all $k \geq 0$

$$T\upsilon(k) \leq \rho \upsilon(k). \tag{21}$$

According to Lemma 1, for $\beta \in]1, \beta_0[$ we have $\rho < 1$, then the condition d) of Theorem 3 is verified.

(4) Let us check condition b). First, according to equations (8)-(10), we obtain

$$\|T\|_\upsilon = \sup_{k \geq 0} \frac{1}{\beta^k} \sum_{j \geq 0} \beta^j T_{kj} \leq \rho(\beta) = \rho < 1,$$

$$\|\alpha\|_\upsilon = 1,$$

$$\|h\|_\upsilon = \sum_{j \geq 0} \beta^j \alpha_j = 1.$$

Hence,

$$T = \tilde{P} - h \circ \alpha \Rightarrow \|\tilde{P}\|_\upsilon \leq \|T\|_\upsilon + \|h\|_\upsilon \|\alpha\|_\upsilon \leq \rho + 1 < \infty.$$

4.2 Estimation of the strong stability

Before estimating the deviation between the stationary distributions of the imbedded Markov chains $\tilde{X}$ and X using the strong stability method, we must first estimate the deviation norm of the transition operators. This deviation is given by the following theorem.

Theorem 4. Let $\tilde{P}$ (respectively P) be the transition operator of the embedded Markov chain in the $M/M/\infty$ system (respectively in the $GI/M/\infty$ system). Then, for all $1 < \beta < \beta_0$, we have:

$$\|P - \widetilde{P}\|_\upsilon \leq \beta w, \tag{22}$$

where $w = \int_0^\infty |H(t) - E_\lambda(t)| \, dt$.

Proof.

$$\begin{aligned}
\|P - \widetilde{P}\|_\upsilon &= \sup_{k \geq 1} \frac{1}{\beta^k} \sum_{j \geq 0} \beta^j |P_{kj} - \widetilde{P}_{kj}| \\
&\leq \beta \int_0^\infty \left(e^{-\mu t} + \frac{1 - e^{-\mu t}}{\beta}\right)^2 |H(t) - E_\lambda(t)| \, dt \\
&\leq \frac{1}{\beta} \int_0^\infty \left(1 + (\beta - 1)e^{-\mu t}\right)^2 |H(t) - E_\lambda(t)| \, dt \\
&\leq \frac{1}{\beta}\left(w + 2(\beta - 1)w + (\beta - 1)^2 w\right) \\
&\leq \beta w.
\end{aligned}$$

After elaborating the stability conditions, it remains to determine the deviation between the stationary distributions of the imbedded Markov chains $\tilde{X}$ and X which can be done by using Theorem 2 and Corollary 1.

The following theorem allows us to obtain the stability inequalities with exactly computing of the constants.

Theorem 5. Let $\tilde{\pi}$ and π be the stationary distributions of the embedded Markov chains $\tilde{X}$ and X respectively. Then, for all $1 < \beta < \beta_0$, and under the condition:

$$w < \frac{\left(2\mu^2 - \lambda\mu\beta - \lambda^2\beta\right)(\beta - 1)}{\beta^2 (2\mu + \lambda)(\mu + \lambda)\left(1 + e^{\lambda/\mu(\beta - 1)}\right)}, \tag{23}$$

we have

$$\|\pi - \widetilde{\pi}\|_\upsilon \leq \tag{24}$$

$$\frac{w\beta^2 (2\mu + \lambda)(\mu + \lambda)\left(e^{\lambda/\mu(\beta-1)}\right)\left(1 + e^{\lambda/\mu(\beta-1)}\right)}{\left(2\mu^2 - \lambda\mu\beta - \lambda^2\beta\right)(\beta - 1) - w\beta^2 (2\mu + \lambda)(\mu + \lambda)\left(1 + e^{\lambda/\mu(\beta-1)}\right)} = E_\beta.$$

Proof. First, let us estimate $\|\widetilde{\pi}\|_\upsilon$ and $\|1\|_\upsilon$.

By definition, we have:

$$\|\widetilde{\pi}\|_\upsilon = \sum_{j \geq 0} \beta^j \pi_j = \sum_{j \geq 0} \beta^j \frac{(\lambda/\mu)^j}{j!} e^{-(\lambda/\mu)} = e^{\frac{\lambda}{\mu}(\beta - 1)}. \tag{25}$$

Thus, $\|\widetilde{\pi}\|_\upsilon = c_0 = e^{\frac{\lambda}{\mu}(\beta - 1)}$.

According to formula (11), we have:

$$\|1\|_\upsilon = \sup_{k \geq 0} \frac{1}{\beta^k} = 1. \tag{26}$$

We have therefore:

$$c = 1 + \|\widetilde{\pi}\|_v = 1 + e^{\frac{\lambda}{\mu}(\beta-1)}. \quad (27)$$

If we replace ρ, $\|\Delta\|_v$ and c by their formulas, given in (20), (22) and (27) respectively, in the condition of Corollary 1, we obtain:

$$\beta w < \frac{\left(2\mu^2 - \lambda\mu\beta - \lambda^2\beta\right)(\beta - 1)}{\beta\,(2\mu+\lambda)\,(\mu+\lambda)\left(1 + e^{\lambda/\mu(\beta-1)}\right)},$$

from where

$$w < \frac{\left(2\mu^2 - \lambda\mu\beta - \lambda^2\beta\right)(\beta - 1)}{\beta^2\,(2\mu+\lambda)\,(\mu+\lambda)\left(1 + e^{\lambda/\mu(\beta-1)}\right)}.$$

Hence, for all $w < \frac{\left(2\mu^2-\lambda\mu\beta-\lambda^2\beta\right)(\beta-1)}{\beta^2(2\mu+\lambda)(\mu+\lambda)\left(1+e^{\lambda/\mu(\beta-1)}\right)}$, we obtain the result.

5. NUMERICAL APPLICATION

The primary objective of this section is to compare the bound put forward in Theorem 5 against the one given by simulation. This allows us to obtain an idea about the performance and the validity of the strong stability approximation. For this, we consider two examples: $Coxian_2/M/\infty$ and $E_2/M/\infty$ systems.

To realize this work, we have developed an algorithm that allows us to check the different conditions of strong stability and calculates the various required quantities. The steps of this algorithm are the same as those proposed by Bouallouche-Medjkoune and Aïssani Bouallouche-Medjkoune and Aïssani (2006a,b) adapted for our case study, then we obtain the following algorithm:

Step 1. Introduce the input parameters: Inter-arrivals density $h(t)$ and service mean rate μ,
Step 2. Determine the arrivals mean rate $\lambda := 1/\int_0^\infty th(t)dt$.
Step 3. Verify the existence of β_0
if $\mu > \lambda$ **then**
the system is stable and **goto** step 4
else ***disp*** *'the system is not stable'* and **goto** step 8.
Step 4. Determine the constant $\beta_0 := \arg\max_\beta \rho < 1$.
Step 5. Determine the constant $\beta_{min} := \arg\min_\beta\{1 \le \beta \le \beta_0/$ the lowest value of β satisfying the condition 23$\}$.
Step 6. Determine the constant $\beta_{max} := \arg\max_\beta\{1 \le \beta \le \beta_0/$ the highest value of β satisfying the condition 23$\}$.
Step 7. Determine the constant β_{opt}:= the value of β minimizing the bound 24 where $\beta_{min} \le \beta \le \beta_{max}$.
Step 8. end.

5.1 *Example 1*

In this example, we consider the $GI/M/\infty$ queuing system, where we set the service rate $\mu = 2$ and we assume that the inter-arrival times distribution $H(t)$ is a Coxian law with order two (a mixture of two exponential law), having the parameters $\lambda_1 = 1.25$, $\lambda_2 = 1.5$ and α, and defined by its probability density $h(t)$ written as follows:

$$h(t) = \alpha\lambda_1 e^{-\lambda_1 t} + (1-\alpha)\lambda_2 e^{-\lambda_2 t}, \quad (t \ge 0,\ 0 < \alpha < 1). (28)$$

The obtained results in this case, for different values of α (0.1,0.2,..., 0.9), are presented in Figure 1 (at the top) and listed in Table 1 below:

Discussion of results

Table 1. Numerical results: Case $Coxian_2/M/\infty$ with $\lambda = (1.25, 1.5)$ and $\mu = 2$

α	λ	β_0	β_{min}	β_{max}	β_{opt}	algor. error	Simu. error
0.1	1.4706	1.5675	1.0279	1.5155	1.2169	0.3923	0.1714
0.2	1.4423	1.6113	1.0489	1.5196	1.2339	0.7611	0.1844
0.3	1.4151	1.6554	1.0618	1.5372	1.2490	0.9987	0.3741
0.4	1.3889	1.6997	1.0668	1.5682	1.2624	1.0476	0.9365
0.5	1.3636	1.7441	1.0647	1.6117	1.2740	0.9387	0.7251
0.6	1.3393	1.7888	1.0572	1.6660	1.2840	0.7461	0.5208
0.7	1.3158	1.8337	1.0458	1.7297	1.2926	0.5317	0.3885
0.8	1.2931	1.8787	1.0320	1.8018	1.2997	0.3292	0.2423
0.9	1.2712	1.9239	1.0165	1.8818	1.3056	0.1513	0.1512

We note that for small values of α, the deviation $\|\pi - \tilde{\pi}\|_v$ is small too, which is also valid for large enough values of α. This can be explained by the fact that:

- For small values of α, the law $h(t) = \lambda_2 e^{-\lambda_2 t} + \epsilon$ is very close to an exponential law with parameter λ_2.
- For large values of α, the law $h(t) = \lambda_1 e^{-\lambda_1 t} + \epsilon$ is very close to an exponential law with parameter λ_1.

That is to say, when the value of α is large enough or small enough, the Coxian law tends to become an exponential law, hence the characteristics of the $Coxian_2/M/\infty$ and the $M/M/\infty$ systems will be very close.

If we compare the two curves of Figure 1 (at the top) (or the numerical errors stored in the two last columns of Table 1), we see that the simulation results are always lower than the algorithmic results. This warrants and confirms that the bound E_β is an upper bound for the deviation $\|\pi - \tilde{\pi}\|_v$.

5.2 *Example 2*

In this example we change, according to the first example, the inter-arrival law $H(t)$ into an Erlang's law of order two (the sum of two exponential laws) having a probability density $h(t)$ given by:

$$h(t) = \lambda^2 t e^{-\lambda t}, \quad t \ge 0. \quad (29)$$

The obtained results in this case, for different values of λ (0.200,0.225,...), are presented in Figure 1 (at the bottom) and listed in Table 2 below:

Table 2. Numerical results: Case $E_2/M/\infty$

λ/n	β_0	β_{min}	β_{max}	β_{opt}	algor. error	Simu. error
0.1000	78.0488	1.9009	53.8383	9.9781	0.4396	0.3375
0.1125	69.1658	1.9252	46.0763	9.0923	0.5166	0.3722
0.1250	62.0606	1.9506	39.9691	8.3852	0.6007	0.4008
0.1375	56.2483	1.9772	35.0521	7.8081	0.6931	0.4086
0.1500	51.4056	2.0051	31.0175	7.3285	0.7948	0.5435
0.1625	47.3088	2.0343	27.6542	6.9241	0.9075	0.5482
0.1750	43.7981	2.0651	24.8119	6.5787	1.0329	0.6868
0.1875	40.7562	2.0975	22.3817	6.2806	1.1733	0.7029
0.2000	38.0952	2.1319	20.2822	6.0210	1.3316	0.7824
> 0.2	-	-	-	-	-	-

Discussion of results

We note that, the difference $\|\pi - \tilde{\pi}\|_v$ increases with the increase of the value of λ (arrivals rate λ/n), and this until $\lambda = 0.4$ ($\lambda/n = 0.2$). This can be explained by the fact that the Erlang's law departs from the exponential law for large enough values of λ (λ/n).

We also note that, for $\lambda \geq 0.425$ ($\lambda/n \geq 0.2125$), the stability conditions are not satisfied. This means that the $M/M\infty$ system is not $\nu-$strongly stable for the test function β^k, for this perturbation.

Comparing the two curves in Figure 1 (at the bottom) (or the numerical errors stored in the two last columns of Table 2), we see that the bound E_β is an upper bound for the deviation $\|\pi - \tilde{\pi}\|_v$ for all the errors obtained by simulation (which are below this bound).

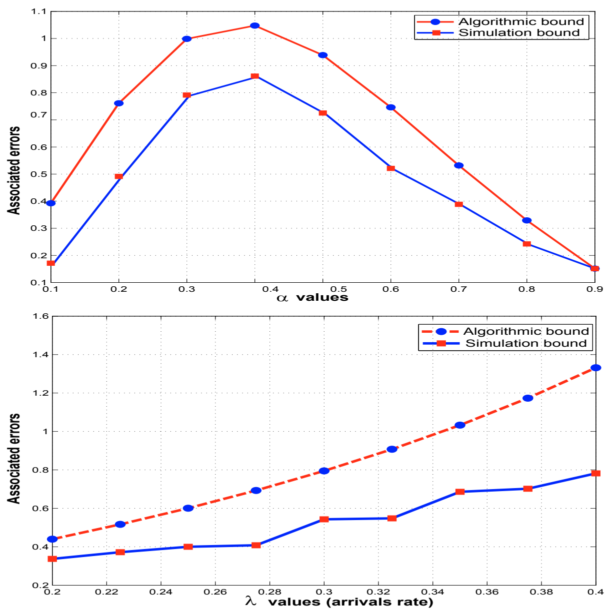

Fig. 1. Top: $Coxian_2$ Errors , Bottom: $Erlang$ errors

6. CONCLUSION

In this work, we applied for the first time the strong stability method for the study of the $M/M/\infty$ queue after a small perturbation of the arrivals rate.

This method allows us to determine the stability conditions of the $M/M/\infty$ system and to obtain the stability inequalities between the stationary characteristics of the $GI/M/\infty$ system and those of the $M/M/\infty$ one.

To evaluate the performance of the proposed method, and to validate the obtained theoretical results, we developed an algorithm which was executed on two real examples (the $Coxian_2/M/\infty$ system and the $E_2/M/\infty$ system). These results were compared to those obtained by applying the discrete event simulation on the same systems.

We can adopt the same scenario used in this paper to study the strong stability of multi-servers queues with finite number of servers. Indeed, in practice, many systems are modeled by a multi-servers queue (finite number of servers), but for which we seek to have the average number of the waiting customers equals to zero (tends to zero). Such a situation arises, for example when we seek to avoid to wait for landing for an aircraft of a free aerodrome, or to avoid to wait for liberating a buffer to make a call in a telecommunications center, etc. This situation coincides with an infinite-servers queue. The main question here is: at what number of servers we may consider a multi-servers queue to behave approximately as an infinite-servers queue?

REFERENCES

Aïssani, D., Kartashov, N.V. (1983). Ergodicity and Stability of Markov Chains with Respect to Operator Topology in the Space of Transition Kernels. *Compte Rendu Academy of Sciences U.S.S.R, ser.A*, 11: 3–5.

Bareche, A., Aïssani, D. (2008). Kernel density in the study of the strong stability of the $M/M/1$ queueing system. *Operations Research Letters*, 36: 535–538.

Benaouicha, M., Aïssani, D. (2005). Strong stability in a $G/M/1$ queueing system. *Theor. Probab. Math. Statist.*, 71: 25–36.

Berdjoudj, L., Benaouicha, M., Aïssani, D. (2012). Measure of performances of the strong stability method. *Mathematical and Computer Modelling*, 56: 241–246.

Bouallouche-Medjkoune, L., Aïssani, D. (2006a). Measurement and performance of the strong stability method. *Theor. Probab. Math. Statist.*, 72: 1–9.

Bouallouche-Medjkoune, L., Aïssani, D. (2006b). Performance analysis approximation in a queueing system of type $M/G/1$. *Math. Method Oper. Res.*, 63: 341–356.

Cooper R. B. (1981). *Introduction to queueing theory.* Second Edition. North Holland, Amsterdam.

Ferreira, M.A.M. (2013a). An infinite servers queue systems with Poisson and non Poisson arrivals busy period simulation. *Emerging Issues in the Natural and Applied Sciences*, 3(1): 38–58. DOI: 10.7813/einas.2013/3-1/4

Ferreira, M.A.M. (2013b). Infinite servers queue systems computational simulation. *Proceedings: 12th Conference on Applied Mathematics APLIMAT 2013, At Bratislava*, 1: 1–16.

Glynnn, P.W., Whitt, W. (1991). A new view of the heavy-traffic limit theorem for infinite-server queues. *Adv. Appl. Prob.*, 23(1): 188–209.

Gross, D., Shortie, J.F., Thompson, J.M., Harris, C.M. (2013). *Bounds and Approximations. Fundamentals of Queueing Theory.* John Wiley and Sons, Fourth edition.

Guillemin, F.M., Mazumdar, R.R., Simonian, A.D. (1996). On heavy traffic approximations for transient characteristics of $M/M/\infty$ queues. *Journal of Applied Probability (Applied Probability Trust)*, 33(2): 490–506.

Halfin, S., Whitt, W. (1981). Heavy traffic limits for queues with many exponenetial servers. *Operation Research*, 29(3): 567–588.

Hall, P. (1985). Heavy traffic approximations for busy period in an $M/G/\infty$ queue. *Stochastic Processes and their Applications*, 19: 259–269.

Helary, J.M., Pedrono, R. (1983). *Recherche opérationnelle.* Hermann.

Kartashov, N.V. (1986). Strong stability of Markov chains. *Journal of Soviet Mathematics*, 34: 1493–1498.

Kartashov, N.V. (1996). *Strong Stable Markov Chains. VSP Utrech.* TbiMC Scientific Publishers, Utrecht, Kiev.

Ramalhoto, M.F. (1999). The infinite server queue and heuristic approximations to the multi-server queue with and without retrials. *Sociedad de Estadlstica e Investigacidn Operativa*, 7: 333–350.

Whitt, W. (1982). On the heavy-traffic limit theorem for $GI/G/\infty$ queues. *Adv. Appl. Prob.*, 14: 171–190.