Multi-station manufacturing system analysis: theoretical and simulation study

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Abstract: This paper deals with a flexible multi-station manufacturing system modelled by re-entrant queueing model. Our model incorporates classical queueing systems with exponential service times and controlled arrival process under a priority service discipline. The system is decomposed into N

fundamental multi-productive stations and $2N - 1$ classes, a part follows the route fixed by the system, where each one is processed by N stations requiring $2N - 1$ services. We assume that there is an infinite supply of work available, so that there are always parts ready for processing step 1. Our purpose in this paper is to present a detailed theoretical and simulation analysis of this priority multi-station manufacturing system.

Keywords: queues; manufacturing; priority scheduling policies; stability; modelling; virtual infinite buffers; simulation.

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1 Introduction

A manufacturing system can be defined as a combination of humans, machinery, and equipment that are bound by a common material and information flow. The materials input to a manufacturing system are raw materials and energy. Information is also input to a manufacturing system, in the form of customer demand for the system's products. The outputs of a system can likewise be divided into materials, such as finished goods and scrap, and information, such as measures of system performance (Chryssolouris, 2006).

These sort of systems have been widely studied, Tiwari and Tiwari (2014) gave a novel methodology for sensor placement for the multi-station manufacturing processes so that the dimensional variation in the manufactured product will be reduced, Sangwan (2013) presented a criteria catalogue and a multi-criteria decision model for the evaluation of manufacturing systems based on environmental aspects of the manufacturing system, Fazlollahtabar and Saidi-Mehrabad (2013) developed a mathematical model to assess the reliability of machines and automated guided vehicles in flexible manufacturing systems, Polotski et al. (in press) analysed a failure prone manufacturing system producing two part types and requiring a setup for switching from one part type to another.

In recent years, queueing theory constitutes a powerful tool in modelling and performance analysis of many complex systems, such as production/flexible manufacturing systems, computer networks, telecommunication systems, call centres, and service systems. Many researchers focused on analysing these different machining systems, let us cite for instance, Jain (2013), Jain et al. (2013, 2014) and references therein.

Stability and performance analysis of multi-class queueing networks is by nowadays an agreeably-researched field. Some preeminent papers in this research field are Harrison (1988), Chen and Mandelbaum (1991), and Kumar (1993). Some notorious contributions with respect to the stability analysis can be summarised in Rybko and Stolyar (1992), Baccelli and Foss (1994), Dai (1995), Bramson (2008), Chen and Yao (2001), Meyn (2008) and Gurvich (2014).

Queueing networks with product form have been greatly studied, Visschers et al. (2011) considered a memoryless single station service system with many servers, authors showed that there exist assignment probabilities under which the system has a product form stationary distribution, and obtained explicit expressions for it, the waiting time distributions in steady state have been derived. Mather et al. (2011) showed for some multi-class queueing networks that time-dependent distributions for the multi-class queue lengths can have a factorised form which reduces the problem of computing such distributions to a similar problem for related single-class queueing networks. Jung and Morrison (2010) gave closed form solutions for the equilibrium probability distribution in the closed Lu-Kumar network under two buffer priority policies, Kim and Morrison (2010) presented an equilibrium probabilities in a class of two station closed queueing network.

Re-entrant lines [described in Harrison (1988)] are a special case of queueing models related with systems composed of some machines/stations, in which customers are processed several times by the same server. These schemes are used to model a variety of real life systems including service centres, production/manufacturing systems, computer and communication networks…. Much attention has been devoted to obtain stability conditions for this kind of networks, Adan and Weiss (2005, 2006), Nazarathy and Weiss (2008), Weiss (2004, 2005) and Nazarathy (2008). A succinct study of these results is given in Guo et al. (2014), Kim and Morrison (2013), and Guo (2009).

Simulation technique is one of possible ways of modelling many complex systems. It can help to improve performance in terms of productivity, and most importantly it can help to identify and detect bottlenecks in production. Simulation has been used to study the behaviour of real systems in order to identify and understand problems associated with the systems. Therefore, in order to improve the performance in any manufacturing system, it is necessary to improve constraints also known as bottlenecks. Joseph and Sridharan (2014) focused on the evaluation of the routing flexibility of a flexible manufacturing system with the dynamic arrival of part types for processing in the system. A typical flexible manufacturing system configuration is chosen for detailed study and analysis. Ramaswami and Jeyakumar (2014) studied non-Markovian bulk queueing system with state dependent arrivals and multiple vacations using a simulation approach, Korytkowski and Wisniewski (2012) examined a multi-product production systems with in-line quality control, Rad et al. (2014) gave an analysis of a manufacturing system using simulation and multi-criteria decision-making tools were applied, Hasan et al. (2014) considered reconfigurable manufacturing systems to be one of the newer technologies which cannot only meet stochastic product demand but can also produce products having customised variety, Tajini et al. (2014) developed a flexible modelling environment for the simulation and analysis of different production systems, Boualem et al. (2015) focused on flexible production system modelled by re-entrant queueing network, where several performance measures have been investigated through expanded Monte Carlo simulations.

The main objective of this paper is to discuss the stability of a manufacturing system model under a specific service discipline, as an important source of the motivation of our research we mention Weiss (2004), where a stability analysis of a particular case 'a re-entrant line with two stations and three processing steps' of our general model is carried out. Thus, the main motivation of this research is to develop the stability study of general multi-station manufacturing system with an arbitrary N ($N \geq 2$) number of stations by using two different techniques including fluid approach and Foster's criterion. Note that it is not obvious how to extend the proofs beyond two machines queueing systems. In addition, in this paper the simulation is used to model our system. Using simulation technique as a means for improving existing manufacturing systems allows to evaluate the effect of local changes on the global system performance. The considered re-entrant model consists of N stations with infinite supply of work at the first one. In an infinite re-entrant line, we suppose that there are continually infinitely many class 1 customers available, which assures that the station serving class 1 will be always busy under non-idling service discipline.

Infinite supply of work expresses an ability to control the arrivals and is often a reasonable way to model a processing system. In some situations there may indeed be an infinite supply of work. In manufacturing systems, the supply of parts for processing at an expensive machine may be controlled and not allowed to exhaust. We refer to this as an infinite virtual queue: it acts like an infinite queue while in fact it only contains a few customers which are continually replenished. In standard queueing networks, one can regard the input stream as the output of a server which is fed by an infinite supply of work (Nazarathy and Weiss, 2010).

Our system is a generalised infinite re-entrant line initialised by Weiss (2004). By using Foster criterion, the author gave a sufficient condition for the stability of the system. In the present work, we study stability condition for our system using Foster criterion and fluid approach, then the effects of various parameters on the performance of the system have been examined numerically.

This paper is organised as follows: Section 2 describes the manufacturing system modelled by a re-entrant queueing model. In Section 3, the theoretical analysis is given, the stability via fluid model approach and Foster criterion approach is established. In Section 4, a detailed simulation study is carried out considering two different specific policies, then the obtained results are compared.

2 The mathematical model

The multi-station manufacturing system considered in this paper consists of inputs, queues and servers as service centres (see Figure 1). Generally, it consists of N servers serving customers arriving in some manner and having some service requirements. The customers (the flow of entities) represent users, customers, transactions or programs. They arrive at the service facility for service, waiting for service, and leave the system after being served. The queueing system is described by distribution of inter-arrival times, distribution of service times, the number of servers, and the service discipline. More precisely we consider a multi-station manufacturing system model consisting of N stations, and a $2N - 1$ steps. Customers arrive to the system at rate α, and follow the route fixed by the system, each one is processed first by station 1 for the first step with rate μ_1, after that by station 2 for the second step with rate μ_2, then aligns all the steps of the network, until the $(2N - 1)^{\text{th}}$ one at which it will be processed again by the first station with rate μ_{2N-1} then leaves definitively the system. The processing times for each of the $2N - 1$ steps are independent sequences of independent identically distributed random variables, with means m_i and rates $\mu_i = \frac{1}{m_i}$, $i = \overline{1, N}$ and without loss of generality we scale time so that $\sum_{i=1}^{N} \mu_i = 1.$ It is well-known that the customers arrive at this system in a renewal stream, at rate α, and under the condition

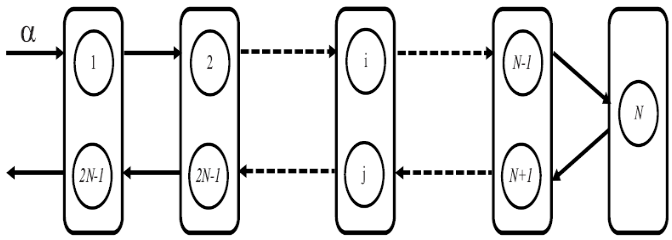
$$\begin{cases} \rho_i = \alpha\left(m_i + m_{2N-i}\right) < 1, i = \overline{1, N-1}, \\ \rho_N = \alpha m_N < 1, \end{cases} \tag{1}$$

the queues of customers waiting for each step are stable, and in fact the system is positive Harris recurrent, for any work conserving policy (see Dai and Weiss, 1996; Kumar and Kumar, 1994).

Assume that there is an infinite supply of work available, so that there are always customers ready for processing step 1. In this case, machine 1 will always be busy. In other words, the input and output rates at the first station are such that the offered load to all the resources is equal to $\rho_1 = 1$. We call buffer 1 a virtual infinite queue. The queue is virtual because in practice buffer 1 need not contain many customers, but it needs to be monitored so it will never be empty. The concept of infinite supply of work is quite natural in many practical situations, and in particular it is very relevant to manufacturing systems. The fact that there are always infinitely many class 1 customers available guarantees that the station serving class 1 will be always busy under non-idling service discipline.

Our purpose in this study is to discuss in details this system under last in first out policy (last buffer first server) (LIFO). In particular, we show that if $m_1 + m_{2N} > m_i + m_{2N-i}$, $i = \overline{2, N-1}$ and $m_1 + m_{2N} > m_N$ then under LIFO policy machine 1 will work all the time (that is we will have $\rho_1 = 1$), but the queues for steps 2, 3, …, $2N - 1$ will be stable. Suppose that in our system there are always customers available for processing of step 1. When these later finish processing step 1 by station 1, they queue in buffer 2 where they remain until they will be processed by machine 2 for step 2, then they move to machine 3 for step 3. The customers continue requiring services until they arrive to buffer N where they will be served by machine N for step N, after that they align the $(N + 1)^{\text{th}}$ queue requiring a service of mean m_{N+1} from the $(N - 1)^{\text{th}}$ station, and move to other stations for other services till the first one where they will be served with mean m_{2N-1}, and finally they leave the system. Each buffer is processed in FIFO order. Processing is non-idling, that is a machine will always process a customer when there is work. We assume that machine i, $i = \overline{1, N-1}$ gives preemptive priority to buffer j, $j = 2N - i$. Whenever there are customers in buffer j, machine i will work on the first of them. When buffer j empties, machine i will immediately resume processing of a customer in step i. If during the processing of step i a customer arrives from buffer $j - 1$ into buffer j, machine i will preempt its work at buffer i, and immediately start processing buffer j. Since the processing times are exponential, this system can be described as a discrete state continuous time Markov jump process, with the state given by the number of customers in buffers $\overline{2, 2N-1}$ denoted $n_2, \ldots, n_{2N-1}$.

Figure 1 Multi-station manufacturing model



Our result has an important practical applications in job-shop scheduling; the best realistic application of our model is job shop manufacturing system in which little batches of a variety of custom products are made. Job shops are usually businesses that perform custom parts manufacturing for other businesses. Examples of job shops include a large class of businesses a machine tool shop, a machining centre, a commercial printing shop, and many other manufacturers. Our model can be also found in many other realistic situations like communication network where each node transmits unlimited supply of materials and various classes of messages originating at this node, under some specific preemptive priority discipline. Metropolitan area networks is a particular computer communication case of our model, it is a network of ducting and fibre optic cable laid within a metropolitan area which can be used by a variety of businesses and organisations to provide services including telecom, internet access, television, etc.

3 Theoretical study

The main objective of this paper is to discuss the stability conditions of our multi-station manufacturing system (see Figure 1). A sufficient condition for the stability of two machines and a three step production process, with an infinite supply of work was established in Weiss (2004) using the Foster criterion. In the current work, we give the condition stability of large class of general re-entrant line queueing networks initialised by Weiss (2004) using two different approaches, namely, the Foster criterion approach, and the fluid model approach. The main result is given in the following theorem.

Theorem 1: The multi-station manufacturing system with N stations and $2N - 1$ processing step is stable if and only if

$$\begin{cases} m_1 + m_{2N} > m_i + m_{2N-i}, \quad i = \overline{2, N-1}, \\ m_1 + m_{2N} > m_N. \end{cases} \tag{2}$$

Proof: The proof of this theorem will be given into two ways, the first one is based on Foster criterion approach, the second one on fluid model approach, but at first, let us give a succinctly explanation of the suggestion of equation (2). By hypothesis, the first queue is infinite virtual, this assumption assures that station 1 is working all the time, which means that the traffic intensity of the station is $\rho_1 = 1$. So, every part which enter the system requires expected $m_1 + m_{2N}$ time units from it. Station 1 is always busy, thus the number of parts which processes in the system per time unit is $\frac{1}{m_1 + m_{2N}}$. Then for $j = \overline{2, N}$, the number of parts that are processed when the station is fully utilised is $m_i + m_j$, $i = \overline{2, N-1}$, $j = 2N - i$ and m_N per time unit. To this end, it is reasonable to say that the system will be stable if and only if $m_1 + m_{2N} > m_i + m_{2N-i}$, $i = \overline{2, N-1}$, and $m_1 + m_{2N} > m_N$.

The two following sub-sections establish the necessary and sufficient condition of our system, using two approaches.

3.1 First approach: the stability via foster criterion

In this part, we have to discuss the stability for $Q(t) = (Q_k)$, $k = \overline{2, 2N-1}$ by using the Foster criterion.

For our study, we need to employ the following result.

Lemma 2 (Meyn and Tweedie, 1993): Let h be a non-negative measurable function on state space of Markov chain Z_n (the set of all non-negative integer numbers, it is written as S). S_0 is a finite set of S.

The chain Z_n is positive recurrent if there exist $\varepsilon > 0$ and $B < \infty$such that

$$\mathbb{E}_z\left(h(Z_1)\right) - h(z) < -\varepsilon \text{ for all } z \notin S_0. \tag{3}$$

$$\mathbb{E}_z\left(h(Z_1)\right) < B \text{ for all } z \in S_0. \tag{4}$$

Suppose that the non-negative function h satisfies

$$\mathbb{E}_z\left(h(Z_1)\right) \geq h(z) \text{ for all } z \in S \setminus S_0, \tag{5}$$

$$\sup_{z\in S} \mathbb{E}_z \left|h(Z_1) - h(z)\right| < \infty \text{ if for any } z_0 \in S \setminus S_0, \tag{6}$$

$$h(z_0) > h(z) \text{ for all } z \in S_0, \tag{7}$$

then we have $\mathbb{E}(\tau_{S_0}) = \infty$, where $\tau_{S_0} = \inf\{n \geq Z_n \in S_0\}$.

Now, we need to find a function $h(\cdot)$ defined on S such that inequalities (3) to (4) hold. Let $h(n) = n$, for all non-negative numbers.

Let us prove that

$$\begin{cases} m_1 + m_{2N-1} > m_i + m_{2N-i}, \text{ if } i = \overline{2, N-1}, \\ m_1 + m_{2N-1} > m_N, \end{cases}$$

is sufficient condition for the stability of process $Q(t)$, in other words the Markov chain Z_n is positive Harris recurrent.

We have, for $z_0 > L$, with L the random number of customers processed in the each busy period of an $M/M/1$ queue with arrival rate μ_j and service rate μ_{j+1}, $j = \overline{N, 2N-2}$;

$$\mathbb{E}_{z_0}\left(h(Z_1)\right) - h(z_0) \leq \mu_1 \frac{m_1}{m_1 + m_{2N-1}} - \mu_i \mathbb{E}(L').$$

L': number of customers served in a truncated busy period. Two cases are considered, either the busy period ends before all customers of class j are processed, and thus at that moment class $j + 1$ is empty, or class j empties the first.

We now choose $\xi > 0$ small enough, and define $S_0 = \{0, 1, \ldots, n_{2N-1}\}$, so that for any $z_0 \notin S_0$ we have:

$$\mathbb{E}_{z_0}\left(h(Z_1)\right) - h(z_0) \leq \mu_1 \frac{m_1}{m_1 + m_{2N-1}} - \mu_i \frac{m_i}{m_i + m_{2N-i}} + \xi < 0.$$

Thus, equation (3) holds and equation (4) follows directly from the definition of $h(\cdot)$, this yields that Z_s is positive recurrent.

Next, we need to find a function $h(\cdot)$ defined on S such that inequalities (5) to (7) hold.

To prove the necessity of the stability conditions, we suppose that

$$\begin{cases} m_1 + m_{2N-1} \leq m_i + m_{2N-i}, \text{ if } i = \overline{2, N-1}, \\ m_1 + m_{2N-1} \leq m_N, \end{cases}$$

and we have to demonstrate that the Markov chain Z_n is not positive Harris recurrent. So, for all $z_0 > n_{2N-1}$, and since $m_1 + m_{2N-1} \leq m_i + m_{2N-i}$, and $m_1 + m_{2N-1} \leq m_N$ we have

$$\mathbb{E}_{z_0}\left(h(Z_1)\right) - h(z_0) \geq \mu_1 \frac{m_1}{m_1 + m_{2N-1}} - \mu_i \frac{m_i}{m_i + m_{2N-i}} \geq 0.$$

Thus equation (5) holds.

Now, for all $z \in S$

$$\mathbb{E}_z \left|\left(h\left(Z_1\right)\right) - h(z)\right| = \mathbb{E}_z \left|Z_1 - z\right| \le \mu_1 \frac{m_1}{m_1 + m_{2N-1}} - \mu_i \frac{m_i}{m_{i+m_{2N-i}}}.$$

So, $\sup_{z \in S} \mathbb{E}_z \mid (h(Z_1)) - h(z) \mid < \infty$. Then (6) holds. Equation (7) follows directly from the definition of $h(\cdot)$, which completes the proof.

3.2 Second approach: the stability via fluid model

First, let us present some performance measures which are particularly interesting. Let $2N - 2$ dimensional queue length process $Q = (Q_k)$ with $Q_k = \{Q_k(t): t \ge 0\}$, where $Q_k(t)$ indicates the number of class k customers in the network at time t. The process $S = \{S_k(t), t \ge 0\}$, where $S_k(t)$ indicates the number of service completions for class k after station $\sigma(k)$ serves k for a cumulative of t units of time. $T = \{T_k(t): t \ge 0\}$, where $T_k(t)$ indicates the cumulative amount of processing time that the station $\sigma(k)$ has served class k customers during $[0, t]$. Thus, $S_k(T_k(t))$ is the total number of class k customer service completions by time t.

So, since there is a fixed route for all parts in the system, one can check that $S(\cdot)$, $T(\cdot)$, $Y(\cdot)$ and $Q(\cdot)$ satisfy the following queueing system:

$$Q_k(t) = Q_k(0) + S_{k-1}\left(T_{k-1}(t)\right) - S_k\left(T_k(t)\right) \ge 0, \quad k = \overline{2, 2N-1}. \tag{8}$$

$$T_k(t) = \int_0^t 1_{[Q_k(s)>0]} ds, \quad k = \overline{N, 2N-1}. \tag{9}$$

$$T_i(t) = t - T_{2N-i}(t) = Y_{2N-i}(t) = \int_0^t 1_{[Q_{2N-i}(s)=0]} ds, \quad i = \overline{1, N-1}. \tag{10}$$

$$Y_N(t) = t - T_N(t) = \int_0^t 1_{[Q_N(s)=0]} ds. \tag{11}$$

Then, referring to Chen (1995), and Dai and Weiss (1996), it is easy to verify that the fluid models corresponding to formulas (8)–(11) are given by:

$$\bar{Q}_k(t) = \bar{Q}_k(0) + \mu_{k-1}\left(\bar{T}_{k-1}(t)\right) - \mu_k\left(T_k(t)\right) \ge 0, \quad k = \overline{2, 2N-1}. \tag{12}$$

$$\bar{T}_k(t) = \int_0^t 1_{[\bar{Q}_k(s)>0]} ds, \quad k = \overline{N, 2N-1}. \tag{13}$$

$$\bar{T}_i(t) = t - \bar{T}_{2N-i}(t) = \bar{Y}_{2N-i}(t) = \int_0^t 1_{[\bar{Q}_{2N-i}(s)=0]} ds, \quad i = \overline{1, N-1}. \tag{14}$$

$$\bar{Y}_N(t) = t - \bar{T}_N(t) = \int_0^t 1_{[\bar{Q}_N(s)=0]} ds. \tag{15}$$

Using formulas (13) to (15), $Q_k(\cdot)$ and $T_k(\cdot), k = \overline{1, 2N-1}$, are Lipschitz continuous.

Now, it is well-known that the fluid model given by expressions (12) to (15) is strongly stable if there exists a time $\gamma > 0$ such that for any fluid solution $Q(\cdot)$, $T(\cdot)$ of the fluid model with the initial condition $\sum_{k=2}^{2N-1} \bar{Q}_k(0) = 1$, we have $\sum_{k=2}^{2N-1} \bar{Q}_k(t) = 0$, for $t \ge \gamma$.

To prove that our multi-station queueing model is stable, it suffices to prove that its corresponding fluid is stable, to this end, we have to demonstrate that the fluid model given by (12) to (15) is stable if and only if condition (2) is satisfied.

First, let us suppose that

$$\begin{cases} m_1 + m_{2N} \le m_i + m_{2N-i}, & i = \overline{2, N-1}, \\ m_1 + m_{2N} \le m_N. \end{cases}$$

By assumption, the first station is always busy, there is an infinitely customers waiting for service all the time, so $\bar{T}_1(t) + \bar{T}_{2N-1}(t) = t,$ and $\dot{\bar{T}}_1(t) + \dot{\bar{T}}_{2N-1}(t) = 1.$

Thus, for station 1 we have $\frac{m_1 + m_{2N-1}}{m_1} \bar{T}_1(t) - t \ge 0.$

The processing rate of class 1 is $\mu_1 \dot{\bar{T}}_1(t),$ such that

$$\mu_1 \bar{T}_1(t) \ge \frac{1}{m_1 + m_{2N-1}} t, \text{ and } \mu_1 \dot{\bar{T}}_1(t) \ge \frac{1}{m_1 + m_{2N-1}},$$

Since for station $i = \overline{2, N},$ we have $\bar{T}_i(t) + \bar{T}_{2N-i}(t) \le t$ and $\bar{T}_N(t) \le t,$ this yields

$$\left(\frac{m_i + m_{2N-i}}{m_1 + m_{2N-1}} - 1 \right) t \ge 0, \quad \forall t > 0, \text{ and} \left(\frac{m_N}{m_1 + m_{2N-1}} - 1 \right) t \ge 0, \quad \forall t > 0.$$

So, if $m_i + m_{2N-i} \ge m_1 + m_{2N-1}$ and $m_N \ge m_1 + m_{2N-1},$ $|\bar{Q}(t)| \to \infty$ as $t \to \infty$, with $|\bar{Q}(t)| = \sum_{k=1}^{2N-1} Q_k(t).$ Thus, the necessity of the condition stability is proved.

Now, the necessary stability condition of the system turns out to be sufficient, it is simply to proceed by contra positive to get the sufficient result.

At first suppose that $\bar{Q}_{2N-1}(t) \ne 0,$ this yield $\dot{T}_1(t) = 0,$ and $\dot{T}_{2N-1}(t) = 1.$

Thus, $\sum_{k=1}^{2N-1} \dot{\bar{Q}}_k(t) = -\mu_{2N-1}.$

If $\bar{Q}_{2N-1}(t) = 0, \bar{Q}_j(t) \ne 0, j = \overline{2, 2N-1},$ and $\frac{m_i + m_{2N-i} - (m_1 + m_{2N-1})}{m_1 + m_{2N-1}} < 0.$

Since $\sum_{k=2}^{2N-1} \bar{Q}(0) = 1$ and Lipschitz continuity of $\sum_{k=2}^{2N-1} Q(\cdot),$ there exists a $\gamma > 0$ such that for $t \ge \gamma,$ $\sum_{k=2}^{2N-1} \bar{Q}(t) = 0.$

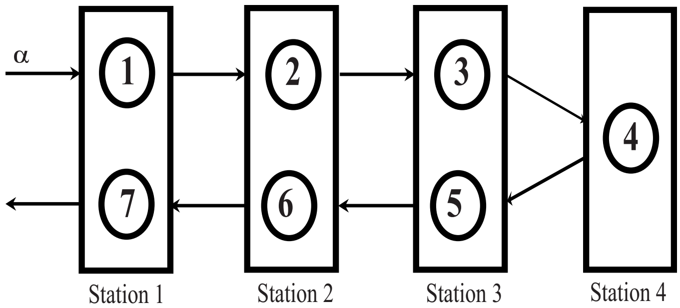
Finally, we have proved that the fluid model given by formulas (12) to (15) associated with our network given by expressions (8) to (11) is stable, thus our manufacturing system is stable. ▯

Next section is devoted to the simulation approach study, where a class of manufacturing system is modelled by a re-entrant queueing system to analyse its performance measures. In addition, we compare and verify the results obtained from the simulation techniques and the theoretical results given for this type of systems.

4 Manufacturing system modelling and performance analysis

Consider a multi-station re-entrant manufacturing system composed of four stations and seven classes (see Figure 2), a prototype of a general model given in Section 2 (see Figure 1), customers arrive from outside requiring services, when these later finish processing step 1 by station 1, they align the second queue where they remain until they will be processed for step 2, after that they move to station 3 for the third service, and then to station 4 for the fourth one. Later, customers align the fifth queue requiring service of mean m_5 from the third station, then continue requiring services from stations 2 and 1, afterwards they leave definitively the whole system. Stations 1, 2 and 3 give priority to buffer 7, 6 and 5 over buffer 1, 2, respectively.

Figure 2 Four station seven class manufacturing system



This system will be verified in order to ensure the theoretical result and finally analysis of the simulation model will be conducted. After the model is verified, there are different decisions that are made before the study proceeds any further. These include duration of the simulation, number of replication calculation, method of analysis used, etc. Finally, performance analysis and evaluation of the model and different operational procedures will be performed.

So, we analyse, evaluate and improve different performance measures of our system 'The mean number of customers in the whole system and in each class, the mean number of customers waiting in the global system and in each station, and the load in the whole system and in each station'. Subsequently, we analyse the influence of parameters of the considered system for two specific policies:

- First policy: The arriving customers follow a Poisson process with rate α, service priority is given to class i, class $2N - i$ is not interrupted since it begins to be served.
- Second policy: This latter has been already defined in Section 2 (the mathematical model).

Throughout the analysis, several conclusions will be drawn by comparing the results obtained of the two policies. The primary objective of this part of paper is to model a class of multi-station manufacturing line, to analyse, to evaluate and improve its performance using computer simulation techniques. Finally, conclusions are drawn from the analysis made and then recommendations are given based on those concluded points. Therefore, it is believed that the work will add some value to the existing knowledge. Analysis and evaluation of a multi-station manufacturing system usually uses performance indicators capable of assessing the adequacy of the model used with respect to the real system.

We first start by specifying performance measures which we consider interesting to study: In all what follows, we fixed $T_{\max}$ = 20,000 time units (duration of the simulation) and MC = 100 (number of replication of Monte Carlo).

The following notations are used throughout this paper.

- N_{i1}, N_{i2}: The mean number of customers in the i^{th} $(i = \overline{1, 7})$ class in the case of the first and the second policy respectively.
- Q_{i1}, Q_{i2}: The mean number of customers in the queue of the i^{th} $(i = \overline{1, 7})$ class in the case of the first and the second policy respectively.
- C_{i1}, C_{i2}: The load (%) of the i^{th} $(i = \overline{1, 7})$ class in the case of the first and the second policy respectively.
- N_{i1}, N_{i2} $(i = \overline{8, 10})$: The mean number of customers in the 1st, 2nd and 3d station in the case of the first and the second policy respectively.
- Q_{i1}, Q_{i2} $(i = \overline{8, 10})$: The mean number of customers waiting in the first (resp. in the second) station in the case of the first and the second policy respectively.
- C_{i1}, C_{i2} $(i = \overline{8, 10})$: The load (%) in the first (resp. in the second) station in the case of the first and the second policy respectively.
- $N_{11,1}$, $N_{11,2}$: The mean number of customers in the global system in the case of the first and the second policy respectively.
- $Q_{11,1}$, $Q_{11,2}$: The mean number of customers waiting in the global system in the case of the first and the second policy respectively.
- $C_{11,1}$, $C_{11,2}$: The load (%) of the global system in the case of the first and the second policy respectively.

First, let us fixed the service rates μ_i, $(i = \overline{1, 7})$ and vary the arrival rates γ. Thereafter, we carry out inversely so as to obtain different states of the network.

4.1 First case: variation of the arrival rates α

For $\mu_i = 1/7$, $i = \overline{1, 7}$, we vary α with a pitch equal to 0.01 starting with $\alpha = 0.05$, for the two policies. The results are summarised in Tables 1 to 3.

Table 1 Variation of the mean number of customers in the system in terms of α

α	0.0300	0.0500	0.0700	0.0900
α_e	0.0738	0.0741	0.0739	0.0740
N_{11}	0.4004	1.4426	15.3908	163.6631
N_{12}	0.5180	0.5179	0.5176	0.5199
N_{21}	0.3916	1.5115	16.0990	36.7867
N_{22}	40.4849	35.9752	40.4291	37.5077
N_{31}	0.3942	1.5689	13.3126	24.9258
N_{32}	24.1205	27.5055	27.2439	26.9458
N_{41}	0.2665	0.5773	1.0858	1.1549
N_{42}	1.1492	1.1698	1.1564	1.1575
N_{51}	0.3076	0.6742	1.1712	1.2338
N_{52}	1.2411	1.2466	1.2465	1.2436
N_{61}	0.3152	0.6894	1.2447	1.3158
N_{62}	1.3300	1.3331	1.3326	1.3191
N_{71}	0.3159	0.6969	1.2772	1.3819
N_{72}	1.3881	1.3932	1.3913	1.3894
N_{81}	0.7163	2.1395	16.6680	165.0450
N_{82}	1.9062	1.9111	1.9089	1.9092
N_{91}	0.7068	2.2008	17.3437	38.1024
N_{92}	41.8150	37.3083	41.7617	38.8268
$N_{10,1}$	0.7018	2.2431	14.4837	26.1597
$N_{10,2}$	25.3616	28.7521	28.4904	28.1895
$N_{11,1}$	2.3914	7.1607	49.5812	230.4620
$N_{11,2}$	70.2319	69.1414	73.3173	70.0830

Table 2 Variation of the mean number of customers waiting in the system in terms of α

α	0.0300	0.0500	0.0700	0.0800
α_e	0.0738	0.0741	0.0739	0.0738
Q_{11}	0.1214	0.8799	14.4692	67.9457
Q_{12}	0.0000	0.0000	0.0000	0.0000
Q_{21}	0.1155	0.9466	15.1861	33.0355
Q_{22}	39.5183	35.0080	39.4592	38.3486
Q_{31}	0.1186	0.9998	12.4126	23.2096
Q_{32}	23.1791	26.5561	26.2948	24.5111
Q_{41}	0.0575	0.2298	0.6159	0.6535
Q_{42}	0.6693	0.6877	0.6747	0.6681
Q_{51}	0.0589	0.2199	0.5243	0.5575
Q_{52}	0.5721	0.5744	0.5743	0.5740
Q_{61}	0.0672	0.2487	0.6165	0.6607

Table 2 Variation of the mean number of customers waiting in the system in terms of α (continued)

Q_{62}	0.6794	0.6822	0.6807	0.6856
Q_{71}	0.0692	0.2620	0.6585	0.7229
Q_{72}	0.7414	0.7472	0.7433	0.7398
Q_{81}	0.2945	1.4425	15.7175	69.3034
Q_{82}	0.9062	0.9111	0.9089	0.9072
Q_{91}	0.2863	1.5038	16.3977	34.3246
Q_{92}	40.8358	36.3287	40.7805	39.6738
$Q_{10,1}$	0.2837	1.5457	13.5490	24.4103
$Q_{10,2}$	24.3998	27.7854	27.5236	25.7364
$Q_{11,1}$	1.5382	6.1744	48.5814	131.0905
$Q_{11,2}$	69.2319	68.1414	72.3173	69.4109

Table 3 Variation of the load of the system in terms of α

α	0.0300	0.0500	0.0700	0.0800
α_e	0.0738	0.0741	0.0739	0.0738
C_{11}	27.9034	56.2669	92.1597	98.3781
C_{12}	51.8036	51.7936	51.7620	51.8370
C_{21}	27.6098	56.4846	91.2942	95.5056
C_{22}	96.6659	96.7170	96.9885	96.8079
C_{31}	27.5605	56.9101	90.0016	93.6947
C_{32}	94.1472	94.9383	94.9073	94.5793
C_{41}	20.8983	34.7421	46.9887	47.9379
C_{42}	47.9867	48.2142	48.1625	48.0859
C_{51}	24.8660	45.4305	64.6858	66.4758
C_{52}	66.8971	67.2259	67.2148	66.9888
C_{61}	24.8001	44.0694	62.8204	64.5011
C_{62}	65.0587	65.0936	65.1888	65.1764
C_{71}	24.6708	43.4941	61.8679	64.0160
C_{72}	64.6784	64.5983	64.8022	64.8947
C_{81}	42.1755	69.7075	95.0519	98.9062
C_{82}	100.0000	100.0000	100.0000	100.0000
C_{91}	42.0528	69.7033	94.5976	97.1743
C_{92}	97.9106	97.9575	98.1130	98.0244
$C_{10,1}$	41.8036	69.7345	93.4719	95.8436
$C_{10,2}$	96.1802	96.6732	96.6763	96.4378
$C_{11,1}$	85.3177	98.6319	99.9780	99.9940
$C_{11,2}$	100.0000	100.0000	100.0000	100.0000

According to Tables 1–3, we constant that:

- In the case of the first policy:
  1. The mean number of customers (in the system and in the queues) is sensitive to the variation of α.
  2. By varying α, the mean number of customers and the load in the classes 1 and 2 as in stations 1 and 2 increases considerably compared to other classes, so the first station will be congested which causes a congestion 'bottleneck' of the second station.
- In the case of the second policy:
  1. There exists a considerable mean number of customers in the system and in the queue of each class.
  2. For any α, compared to the first policy, the mean number of customers is more important in the second station. The load in the classes and in the stations 2 and 3 are equilibrated.
  3. For $\alpha = 0.03$, the load in the second station and in class 2 is very high compared to other classes, this latter will causes a saturation of station 3, this bottleneck is due to the fact that $m_1 + m_7 = m_2 + m_6 = m_3 + m_5$, [equation (2) is not verified].

Graphical representations (see Figure 3) illustrate the details of some results in the case of the first policy.

Figure 3 The state of the network when $\alpha = 0.08$ and $\mu = [1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7]$ (see online version for colours)

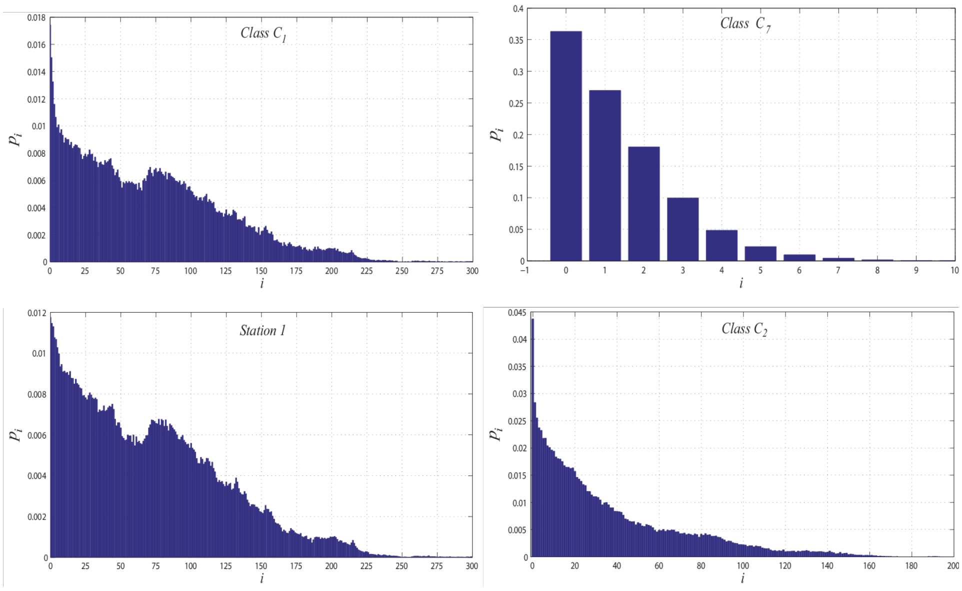
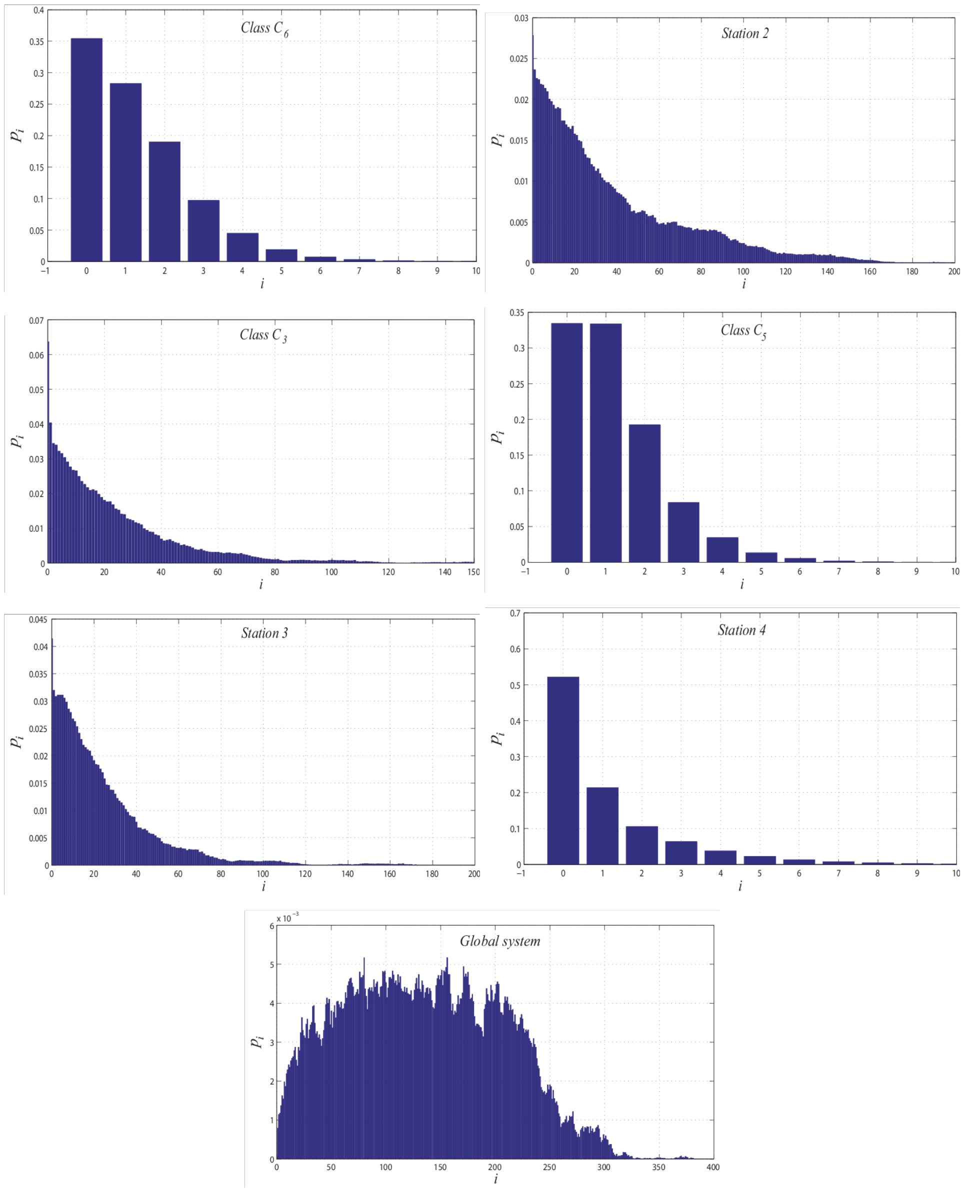


Figure 3 The state of the network when $\alpha = 0.08$ and $\mu = [1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7]$ (continued) (see online version for colours)



To conclude, we present a summary table on the results (see Table 4). This allows us to say that:

1 The network is unstable when one of the necessary conditions is not verified.

2 By varying α from 0.03 to 0.07, the system is stable in the case of the two policies ($\rho_i < 1,\ i = \overline{1, 4}$). On other hand, by varying α from 0.08 to 0.09, the system is unstable ($\rho_i > 1,\ i = \overline{1, 3}$).

Table 4 The state of the network in the case of the two policies

α	ρ_1	ρ_2	ρ_3	ρ_4	*Constatation*
0.03	0.42	0.42	0.42	0.21	Stable
0.04	0.56	0.56	0.56	0.28	
0.05	0.70	0.70	0.70	0.35	
0.06	0.84	0.84	0.84	0.42	
0.07	0.98	0.98	0.98	0.49	
0.08	1.12	1.12	1.12	0.56	Unstable
0.09	1.26	1.26	1.26	0.63	

4.2 Second case: variation of the service rates

4.2.1 Variation of the service rates of the first station

Let us vary the service rates μ_1 and μ_7, and fixed the rate α as the service rates of the second, the third and the fourth station. For $\alpha = 0.05$, $\mu_2 = 0.1$, $\mu_3 = 0.15$, $\mu_4 = 0.1$, $\mu_5 = 0.15$ and $\mu_6 = 0.2$. The results of the simulation in the case of the two policies are summarised in Tables 5 to 7.

Table 5 Variation of the mean number of customers in the system in terms of μ_1 and μ_7

μ_1	0.2500	0.2000	0.1500	0.1000
μ_7	0.0500	0.1000	0.1500	0.2000
α_e	0.0426	0.0699	0.0845	0.0676
N_{11}	83.1417	2.1207	1.2689	2.2128
N_{12}	0.1699	0.3499	0.5611	0.6770
N_{21}	5.6732	2.8849	2.2431	2.1444
N_{22}	6.3445	60.0884	185.8040	30.6144
N_{31}	1.5271	1.3659	1.2280	1.2364
N_{32}	1.6018	5.0560	5.9585	4.8238
N_{41}	1.0883	1.1272	1.0609	1.0667
N_{42}	1.0861	2.3074	2.4475	2.2486
N_{51}	0.5737	0.6414	0.6228	0.6297
N_{52}	0.5886	1.0453	1.0846	1.0397
N_{61}	0.5921	0.6708	0.6600	0.6498
N_{62}	0.6087	1.1124	1.1585	1.0985
N_{71}	4.4914	1.1130	0.6775	0.6825
N_{72}	4.6337	2.1769	1.2861	1.1767
N_{81}	87.6331	3.2337	1.9464	2.8953
N_{82}	4.8036	2.5268	1.8472	1.8538

Table 5 Variation of the mean number of customers in the system in terms of μ_1 and μ_7 (continued)

N_{91}	6.2654	3.5557	2.9031	2.7942
N_{92}	6.9532	61.2008	186.9625	31.7129
$N_{10,1}$	2.1008	2.0073	1.8508	1.8660
$N_{10,2}$	2.1903	6.1014	7.0431	5.8636
$N_{11,1}$	97.0875	9.9239	7.7613	8.6223
$N_{11,2}$	15.0332	72.1364	198.3003	41.6789

Table 6 Variation of the mean number of customers waiting in the system in terms of μ_1 and μ_7

μ_1	0.2500	0.2000	0.1500	0.1000
μ_7	0.0500	0.1000	0.1500	0.2000
α_e	0.0426	0.0699	0.0845	0.0676
Q_{11}	82.1622	1.5175	0.7397	1.5407
Q_{12}	0.0000	0.0000	0.0000	0.0000
Q_{21}	5.1120	2.2175	1.5788	1.4795
Q_{22}	5.7668	59.1165	184.8068	29.6484
Q_{31}	1.0693	0.8333	0.7021	0.7067
Q_{32}	1.1339	4.2576	5.1360	4.0287
Q_{41}	0.6742	0.6270	0.5626	0.5652
Q_{42}	0.6685	1.6571	1.7868	1.5972
$Q_5 1$	0.2186	0.2194	0.2032	0.2056
Q_{52}	0.2251	0.4676	0.4932	0.4635
Q_{61}	0.2432	0.2521	0.2410	0.2303
Q_{62}	0.2506	0.5274	0.5612	0.5176
Q_{71}	3.6509	0.5764	0.2690	0.2735
Q_{72}	3.7829	1.4479	0.6945	0.6115
Q_{81}	86.6452	2.4889	1.2818	2.1414
Q_{82}	3.8036	1.5268	0.8472	0.8538
Q_{91}	5.6468	2.8097	2.1556	2.0453
Q_{92}	6.3207	60.2214	185.9647	30.7368
$Q_{10,1}$	1.5521	1.3458	1.1883	1.1983
$Q_{10,2}$	1.6324	5.2340	6.1570	4.9968
$Q_{11,1}$	96.0878	8.9296	6.7729	7.6326
$Q_{11,2}$	14.0332	71.1364	197.3003	40.6789

Table 7 Variation of the load of the system in terms of μ_1 and μ_7

μ_1	0.2500	0.2000	0.1500	0.1000
μ_7	0.0500	0.1000	0.1500	0.2000
α_e	0.0426	0.0699	0.0845	0.0676
C_{11}	97.9498	60.3274	52.9176	67.2133
C_{12}	16.9894	34.9891	56.1078	67.7047
C_{21}	56.1236	66.7418	66.4305	66.4924
C_{22}	57.7720	97.1954	99.7163	96.6012
C_{31}	45.7777	53.2558	52.5813	52.9699
C_{32}	46.7848	79.8424	82.2512	79.5141
C_{41}	41.4088	50.0126	49.8366	50.1474
C_{42}	41.7577	65.0351	66.0729	65.1352
C_{51}	35.5053	42.1999	41.9689	42.4062
C_{52}	36.3444	57.7753	59.1420	57.6242
C_{61}	34.8975	41.8650	41.9068	41.9525
C_{62}	35.8115	58.5049	59.7296	58.0978
C_{71}	84.0495	53.6568	40.8493	40.9059
C_{72}	85.0794	72.9019	59.1633	56.5210
C_{81}	98.7844	74.4812	66.4573	75.3980
C_{82}	100.0000	100.0000	100.0000	100.0000
C_{91}	61.8624	74.5914	74.7526	74.8883
C_{92}	63.2502	97.9446	99.7810	97.6081
$C_{10,1}$	54.8687	66.1481	66.2551	66.7718
$C_{10,2}$	55.7938	86.7312	88.6102	86.6811
$C_{11,1}$	99.9684	99.4264	98.8406	98.9624
$C_{11,2}$	100.0000	100.0000	100.0000	100.0000

Following the numerical results given in Tables 5–7, we constat that:

- In the case of the first policy:
  1. For $\mu = [0.25, 0.1, 0.15, 0.1, 0.15, 0.2, 0.05]$, the mean number of customers as well as the load in the first class and in station 1, also in the overall system is very high compared to other classes, therefore the network is unstable. The instability is caused by the saturation of the first station ($\rho_1 > 1$).
  2. For (μ_1, μ_7) varying from (0.2, 0.1) to (0.15, 0.15), the mean number of customers decreases in the class i, $i = \overline{2, 7}$ as in stations 1, 2, 3 and in the global system.
- In the case of the second policy:
  1. For $\mu = [0.15, 0.1, 0.15, 0.1, 0.15, 0.2, 0.15]$, the mean number of customers and the load in the second class and in the second station is very high, this later is due to the fact that equation (2) is not verified, $m_1 + m_7 < m_2 + m_6$.

2 For (μ_1, μ_7) varying from (0.25, 0.05) to (0.1, 0.2), the load of the first station and the global system is stable.

3 For (μ_1, μ_7) varying from (0.2, 0.1) to (0.15, 0.15), the mean number of customers increases.

For some results of Tables 5 to 7, graphical representations ‘in the case of the first policy’ are illustrated in Figures 4 and 5.

Figure 4 The state of the network when μ = [0.2, 0.1, 0.15, 0.1, 0.15, 0.2, 0.1] and α = 0.05 (see online version for colours)

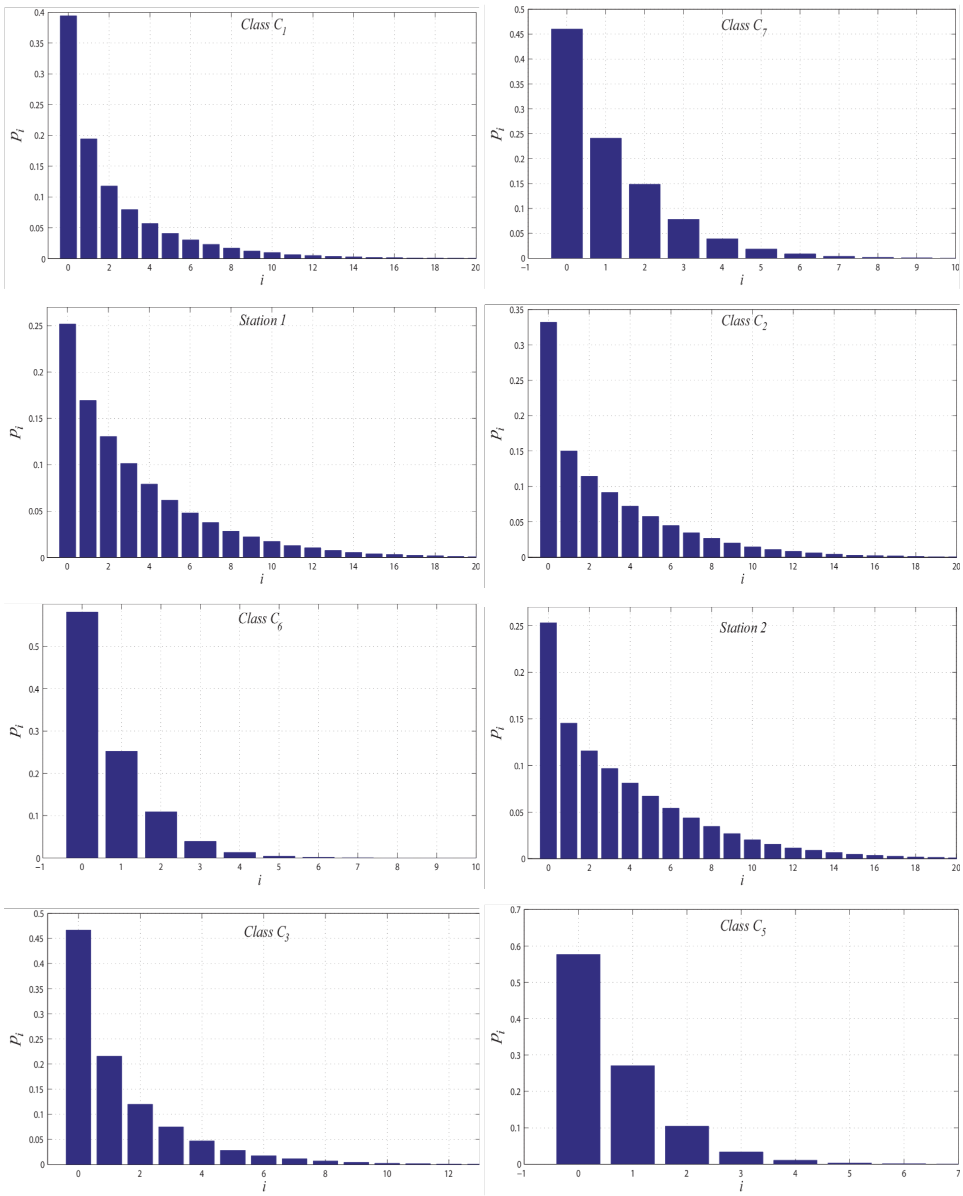


Figure 4 The state of the network when μ = [0.2, 0.1, 0.15, 0.1, 0.15, 0.2, 0.1] and α = 0.05 (continued) (see online version for colours)

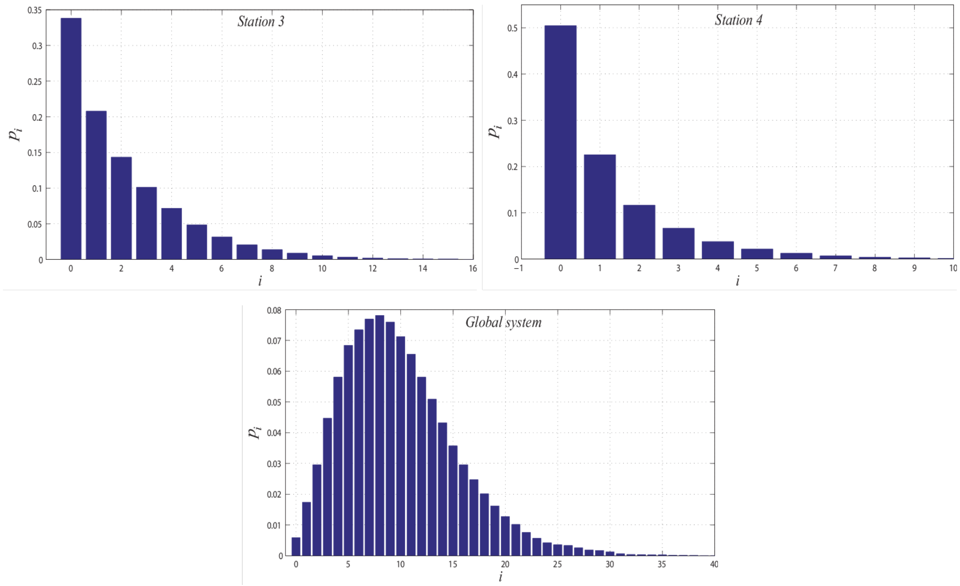


Figure 5 The state of the network when μ = [0.25, 0.1, 0.15, 0.1, 0.15, 0.2, 0.05] and α = 0.05 (see online version for colours)

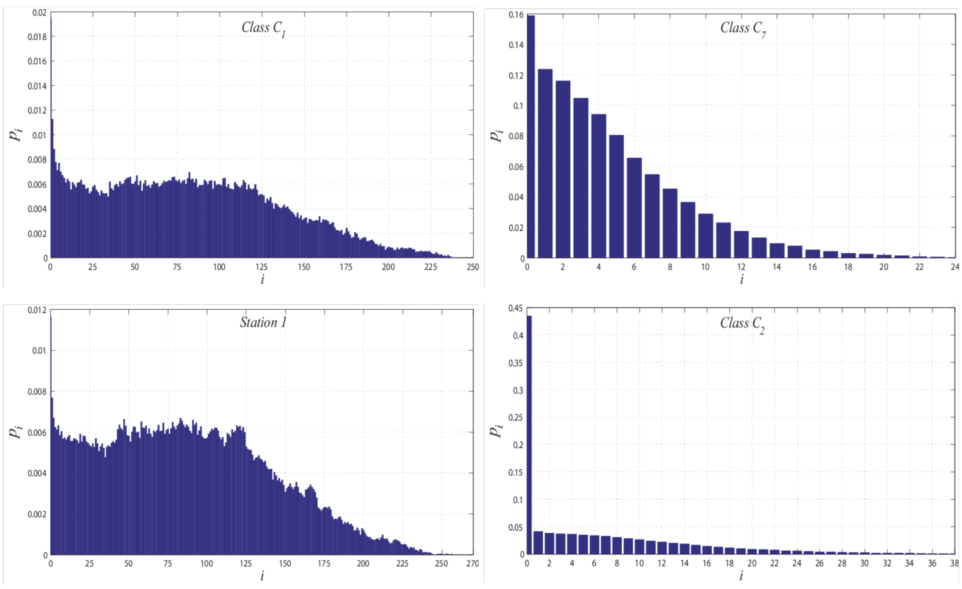
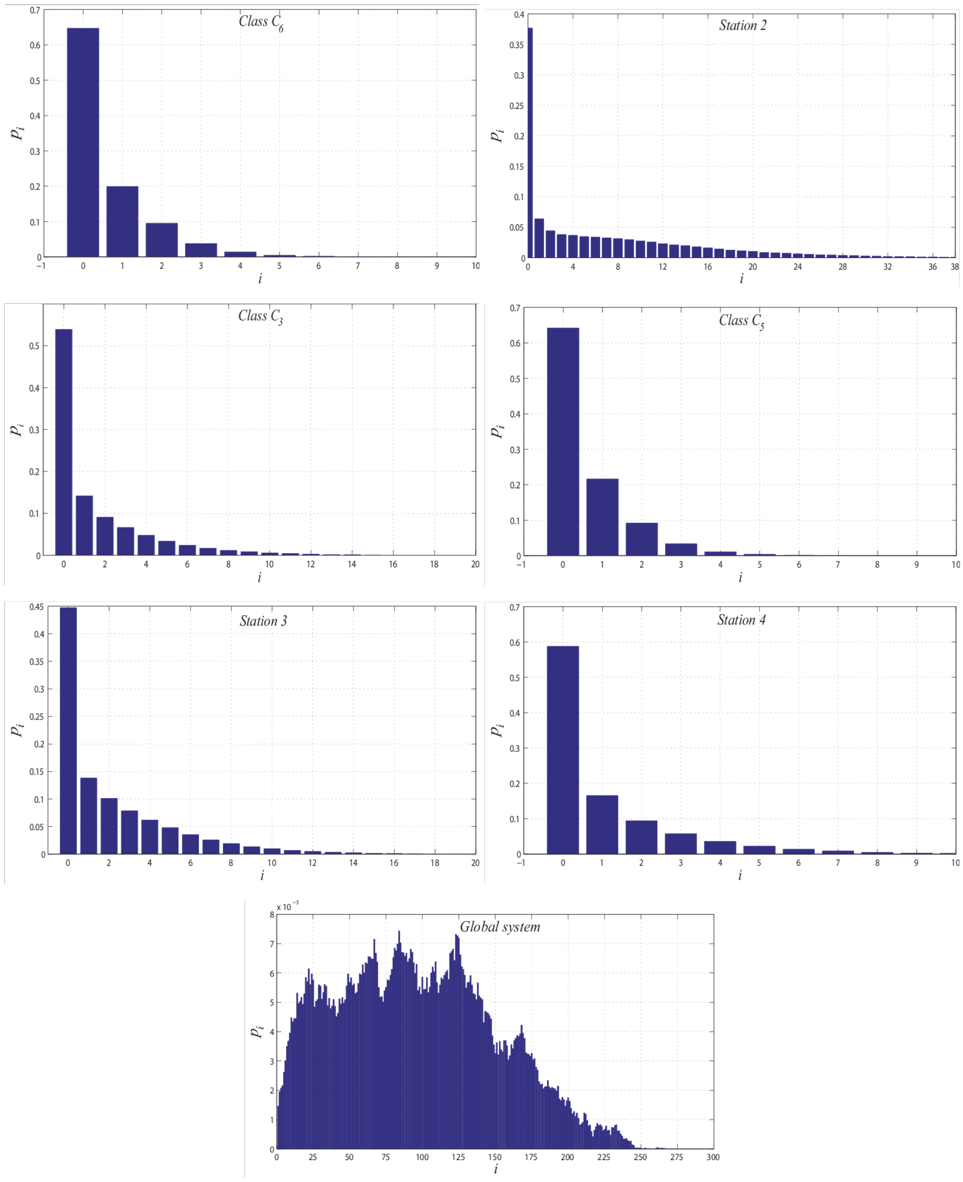


Figure 5 The state of the network when $\mu = [0.25, 0.1, 0.15, 0.1, 0.15, 0.2, 0.05]$ and $\alpha = 0.05$ (continued) (see online version for colours)



In conclusion, we present a summary table containing different situations of the network, based on its parameters. Table 8 permits us to constat that the state of the network is sensible to the variation of its parameters. Indeed, it moves from stability state (all conditions are fulfilled) to instability (if one of the conditions is not satisfied). For instance, when $(\mu_2, \mu_6) = (0.25, 0.05)$, the network is unstable because $(\rho_1 = 1.2 > 1)$.

Table 8 The state of the network by varying μ_1 and μ_7

μ_1	μ_7	ρ_1	*Constatation*
0.25	0.05	1.2	Unstable
0.2	0.1	0.75	Stable
0.15	0.15	0.66	
0.1	0.2	0.75	

4.2.2 Variation of the service rates of the second station

Let us vary μ_2 and μ_6, fixed the arrival rate α, and the service rates of the first, the third and the fourth station. For $\alpha = 0.05$, $\mu_1 = 0.15$, $\mu_3 = 0.1$, $\mu_4 = 0.1$, $\mu_5 = 0.15$ and $\mu_7 = 0.15$. In the case of the two policies, the simulation results are summarised in Tables 9 to 11.

Table 9 Variation of the mean number of customers in the system in terms of μ_2 and μ_6

μ_2	0.3000	0.2500	0.2000	0.1500	0.1000
μ_6	0.0500	0.1000	0.1500	0.200	0.2500
α_e	0.1074	0.0899	0.0902	0.0899	0.0905
N_{11}	1.0587	1.2810	1.2275	1.2585	1.2831
N_{12}	0.7152	0.6030	0.6006	0.6014	0.6020
N_{21}	72.6008	1.6892	0.8582	0.9348	1.8614
N_{22}	638.6951	22.7448	5.3407	8.3180	155.6400
N_{31}	8.5491	4.5562	3.6572	3.4218	3.3426
N_{32}	9.5928	283.3280	296.0426	291.3640	157.2491
N_{41}	1.0470	1.1020	1.0601	1.0548	1.0312
N_{42}	1.0871	1.7127	1.7155	1.7641	1.7052
N_{51}	0.7706	0.8510	0.8231	0.8185	0.8140
N_{52}	0.7986	0.8510	1.1866	1.1928	1.1827
N_{61}	4.3566	1.0242	0.5903	0.4831	0.5680
N_{62}	4.6594	1.5667	0.8934	0.7788	0.9188
N_{71}	0.5167	0.6526	0.6472	0.6590	0.6838
N_{72}	0.6843	1.0459	1.0637	1.0878	1.1197
N_{81}	1.5754	1.9336	1.8747	1.9175	1.9670
N_{82}	1.3995	1.6489	1.6644	1.6891	1.7217
N_{91}	76.9573	2.7134	1.4485	1.4179	2.4294
N_{92}	643.3545	24.3115	6.2341	9.0968	156.5587
$N_{10,1}$	9.3197	5.4072	4.4803	4.2402	4.1566
$N_{10,2}$	10.3914	284.5099	297.2292	292.5568	158.4318
$N_{11,1}$	88.8994	11.1563	8.8637	8.6305	9.5841
$N_{11,2}$	656.2326	312.1830	306.8431	305.1068	318.4175

Table 10 Variation of the mean number of customers in the queues in terms of μ_2 and μ_6

μ_2	0.3000	0.2500	0.2000	0.1500	0.1000
μ_6	0.0500	0.1000	0.1500	0.2000	0.2500
α_e	0.1074	0.0899	0.0902	0.0899	0.0905
Q_{11}	0.5670	0.7498	0.7021	0.7300	0.7525
Q_{12}	0.0000	0.0000	0.0000	0.0000	0.0000
Q_{21}	71.6255	1.1508	0.4358	0.4774	1.2416
Q_{22}	637.6981	21.8258	4.5527	7.4597	154.6435
Q_{31}	7.8819	3.7893	2.8961	2.6644	2.5850
Q_{32}	8.9089	282.3325	295.0450	290.3666	156.2545
Q_{41}	0.6264	0.6053	0.5629	0.5570	0.5346
Q_{42}	0.6553	1.1167	1.1154	1.1616	1.1086
Q_{51}	0.3185	0.3284	0.3042	0.3016	0.2967
Q_{52}	0.3355	0.5323	0.5362	0.5417	0.5336
Q_{61}	3.4967	0.4937	0.2090	0.1520	0.2042
Q_{62}	3.7881	0.9075	0.3918	0.3159	0.4064
Q_{71}	0.1635	0.2371	0.2359	0.2498	0.2791
Q_{72}	0.2391	0.4778	0.4980	0.5275	0.5728
Q_{81}	0.9588	1.2647	1.2111	1.2527	1.3029
Q_{82}	0.3995	0.6489	0.6644	0.6891	0.7217
Q_{91}	75.9715	2.0124	0.8648	0.8328	1.7306
Q_{92}	642.3568	23.3649	5.3834	8.2021	155.5616
$Q_{10,1}$	8.6145	4.5764	3.6480	3.4098	3.3252
$Q_{10,2}$	9.6727	283.5134	296.2311	291.5587	157.4356
$Q_{11,1}$	87.8995	10.1607	7.8726	7.6400	8.5913
$Q_{11,2}$	655.2326	311.1830	305.8431	304.1068	317.4175

Table 11 Variation of the load of the system in terms of μ_2 and μ_6

μ_2	0.3000	0.2500	0.2000	0.1500	0.1000
μ_6	0.0500	0.1000	0.1500	0.2000	0.2500
α_e	0.1074	0.0899	0.0902	0.0899	0.0905
C_{11}	49.1684	53.1218	52.5445	52.8489	53.0629
C_{12}	71.5172	60.2999	60.0637	60.1370	60.1982
C_{21}	97.5310	53.8347	42.2434	45.7444	61.9838
C_{22}	99.6968	91.8981	78.8088	85.8292	99.6431
C_{31}	66.7137	76.6885	76.1054	75.7430	75.7628
C_{32}	68.3900	99.5501	99.7589	99.7421	99.4593
C_{41}	42.0638	49.6731	49.7235	49.7820	49.6546
C_{42}	43.1798	59.5967	60.0038	60.2487	59.6664
C_{51}	45.2146	52.2598	51.8902	51.6811	51.7297

Table 11 Variation of the load of the system in terms of μ_2 and μ_6 (continued)

C_{52}	46.3121	64.9582	65.0430	65.1050	64.9109
C_{61}	85.9871	53.0533	38.1318	33.1047	36.3828
C_{62}	87.1341	65.9186	50.1596	46.2909	51.2326
C_{71}	35.3228	41.5530	41.1259	40.9190	40.4722
C_{72}	44.5253	56.8093	56.5768	56.0267	54.6899
C_{81}	61.6549	66.8929	66.3618	66.4738	66.4124
C_{82}	100.0000	100.0000	100.0000	100.0000	100.0000
C_{91}	98.5839	70.0996	58.3665	58.5150	69.8806
C_{92}	99.7674	94.6594	85.0746	89.4648	99.7095
$C_{10,1}$	70.5120	83.0829	83.2273	83.0392	83.1352
$C_{10,2}$	71.8760	99.6542	99.8107	99.8047	99.6139
$C_{11,1}$	99.9908	99.5598	99.1070	99.0451	99.2803
$C_{11,2}$	100.0000	100.0000	100.0000	100.0000	100.0000

Following the numerical results given in Tables 9 to 11, we constat that:

- In the case of the second policy:
  1. For $(\mu_2, \mu_6) = (0.3, 0.05)$, the mean number of customers as well as the load in the second class and in the second station in the case of the first policy is considerable and it is very high in the case of the second policy which causes a bottleneck in the overall system; equation (2) is not verified ($m_1 + m_7 < m_2 + m_6$).
  2. For (μ_2, μ_6) varying from (0.25, 0.1) to (0.2, 0.15), the load is very high in the third class and the mean number of customers in the third station increases.
  3. For (μ_2, μ_6) varying from (0.15, 0.2) to (0.1, 0.25), the mean number of customers deceases.
  4. Whatever the variation of (μ_2, μ_6), the first station 1 and the global system are clogged which will also cause a congestion in the station 2.
- In the case of the first policy: For (μ_2, μ_4) varying from (0.25, 0.1) to (0.1, 0.25), the mean number of customers is almost equally distributed.

For some results of Tables 9 to 11, the graphs concerning the first policy are illustrated in Figures 6 and 7.

Figure 6 The state of the network when $\mu = [0.15, 0.25, 0.1, 0.1, 0.15, 0.1, 0.15]$ and $\alpha = 0.05$ (see online version for colours)

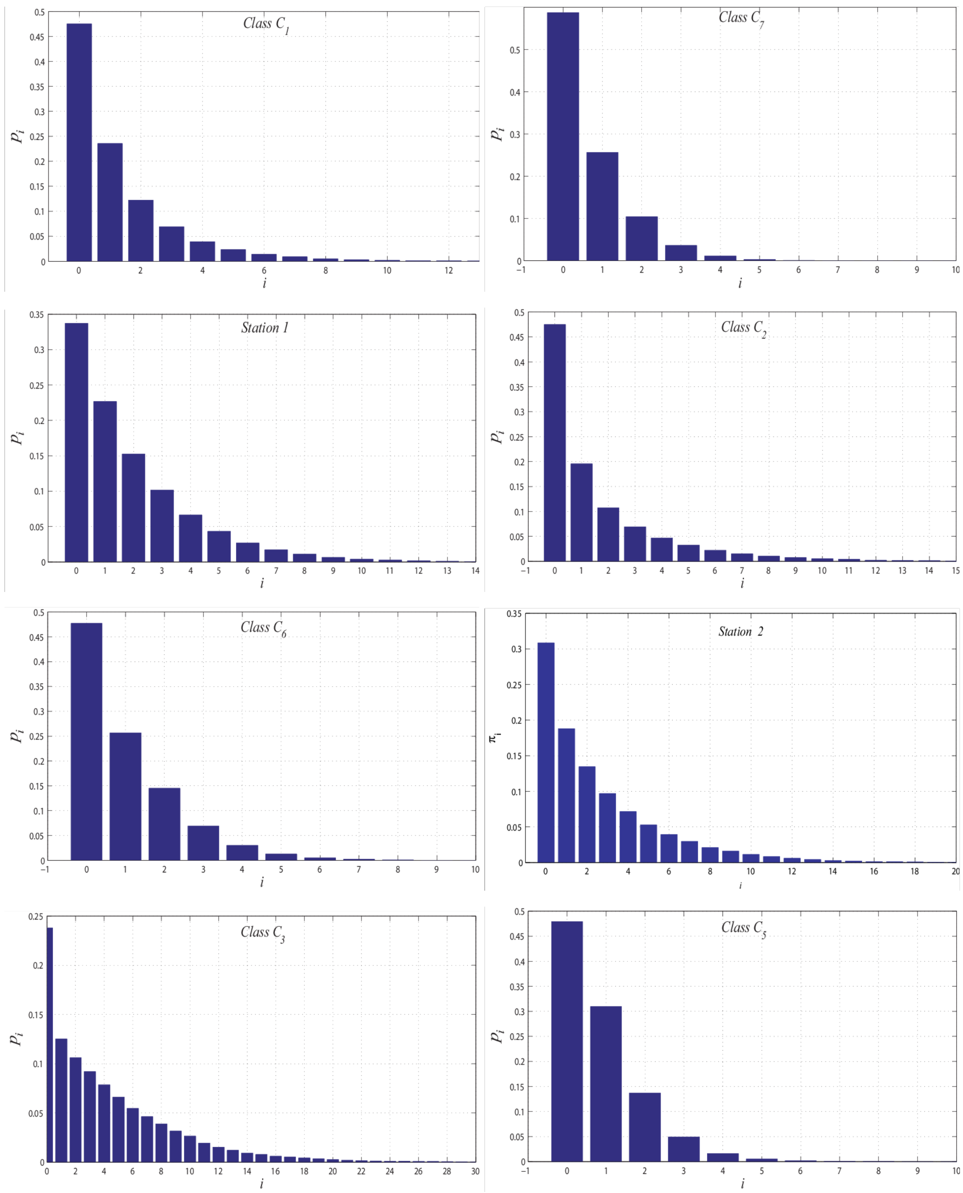


Figure 6 The state of the network when μ = [0.15, 0.25, 0.1, 0.1, 0.15, 0.1, 0.15] and α = 0.05 (continued) (see online version for colours)

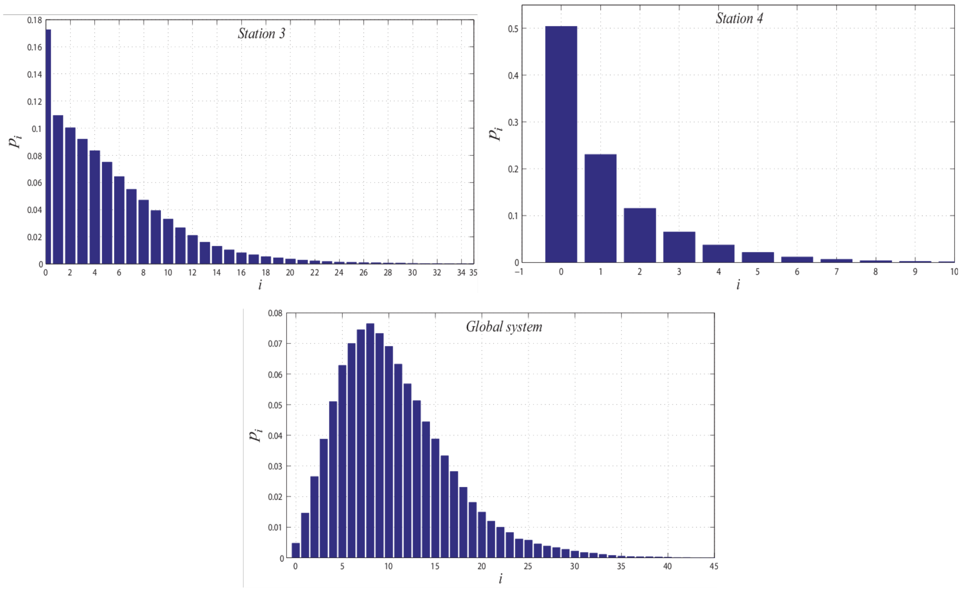


Figure 7 The state of the network when μ = [0.15, 0.3, 0.1, 0.1, 0.15, 0.05, 0.15] and α = 0.05 (see online version for colours)

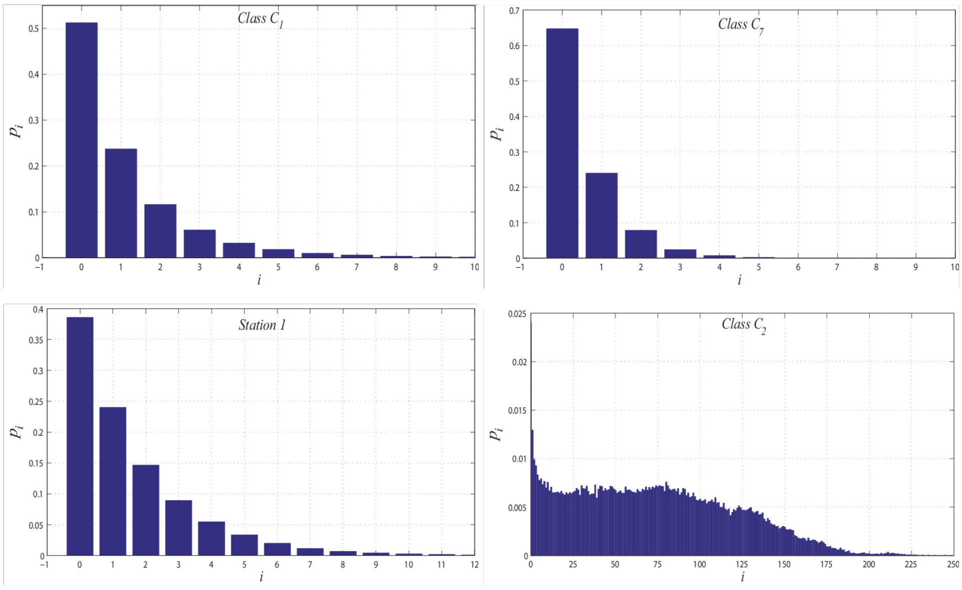
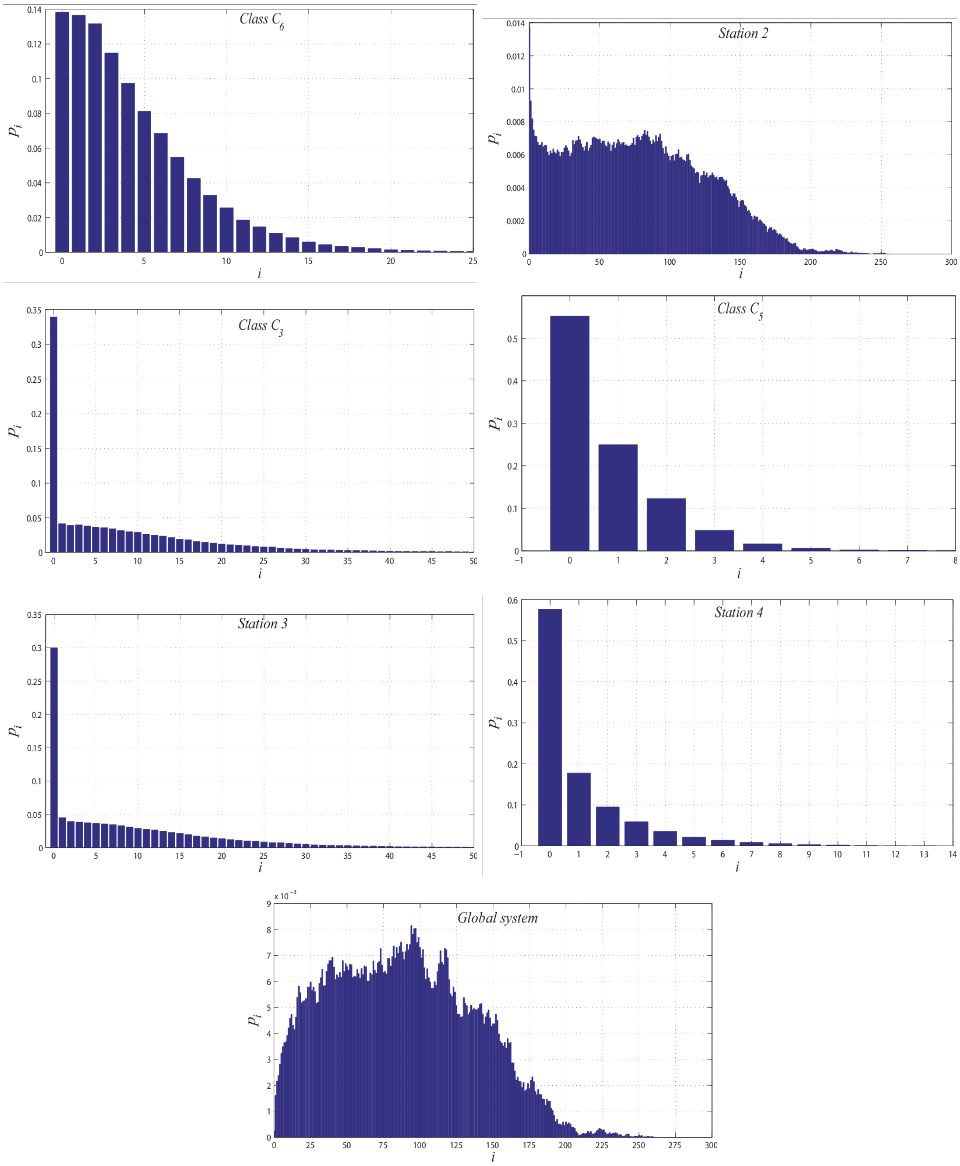


Figure 7 The state of the network when μ = [0.15, 0.3, 0.1, 0.1, 0.15, 0.05, 0.15] and α = 0.05 (continued) (see online version for colours)



As conclusion, we present a summary table (see Table 12) containing different network situations, depending on its parameters. From Table 12, we constat that the state of the system is sensible to the variation of its parameters. In fact, we have a stable state when all the conditions are satisfied, and unstable state when one of the conditions is not satisfied. For example, when $(\mu_2, \mu_6) = (0.3, 0.05)$, the system is unstable because $(\rho_2 = 1.16 > 1)$.

Table 12 The state of the system by varying μ_2 and μ_6 in the case of the two policies

μ_2	μ_6	ρ_2	*Constatation*
0.30	0.05	1.60	Unstable
0.25	0.10	0.70	Stable
0.20	0.15	0.58	
0.15	0.20	0.58	
0.10	0.25	0.70	

4.3 Concluding remarks for this section

- A detailed simulation of a type of multi-station manufacturing systems (four stations and seven classes) is presented, considering two disciplines.
- Using Monte Carlo techniques, the properties for system trajectories are well-established. The effectiveness of the proposed Monte Carlo algorithm is examined by a series of simulation experiments and is found to be gratifying.
- It is shown how perfect samples can be used to analyse and solve complex multi-server queue models, as well as to highlight selected properties and characteristics of them. The theoretical result given in Section 3 is illustrated and the effect of various parameters on the performance of the system have been examined.
- Simulation analysis from the stationary distributions should output information about the facial characteristics of any system, examples are information about bottlenecks.

5 Conclusions and final remarks

In the current paper, we present an analysis of a priority multi-station manufacturing system modelled by re-entrant controlled queueing system, these sort of systems are fundamental of study in operations research and applied probability as they provide sensible models for a variety of engineering, communications, telecommunication, and service situations. We establish the stability condition (condition 2) of Theorem 1 for our model by using Foster criterion and fluid approach. This is a generalisation of the result given by Weiss (2004).

In this present work, it is also demonstrated that a Monte Carlo simulation can be used to deal with this type of systems. From the preliminary analysis, it was shown that the simulation model is capable of analysing a complex multi-station manufacturing system. The model developed in this study was aimed to understand and improve the performances of the priority production system. From the results, it can be concluded that the performances obtained for a given system from our technique and theoretical results are very close to each other. Still the coherence in results obtained is very significant, hence it can be said that these techniques are applicable to any manufacturing system to evaluate and confirms its performances.

For further work, it is interesting to study our system with general processing times and investigate their potential in terms of performance. For a general re-entrant line with infinite supply of work, a general approach may be needed for hunting stable conditions.

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