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Estimation of the strong stability in a $G/M/1$ queueing system

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Abstract. In this paper, we study the strong stability of the stationary distribution of the imbedded Markov chain in the $G/M/1$ queueing system, after perturbation of the service law (Aissani (1990), Kartashov (1981), Kartashov (1985)). We show that under some hypotheses, the characteristics of the $G/G/1$ queueing system can be approximated by the corresponding characteristics of the $G/M/1$ system.
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1 Strong stability in the $G/M/1$ queueing system

The applicability of the strong stability method for the $G/M/1$ system is not obvious (Aissani and Kartashov (1984)). In fact, the complexity of the $G/G/1$ system constrains us to work on more general and complex spaces that imposes on us the realization of some intermediate constructions which are of particular interest.

1.1 Preliminaries and notations

Consider a $G/G/1$ queueing system with a general service times distribution G and a general inter-arrival times probability distribution F. The following notations are used : θ_n (the arrival time of the n^{th} demand), ω_n (the departure time of the n^{th} customer), γ_n (the time interval from θ_n to the departure of the next customer) and $V_n = V(\theta_n - 0)$ (the number of customers found in the system immediately prior to θ_n).
Let be denote by, $\nu_{\theta_n} = \min\{m > 0, \omega_m \geq \theta_n\}$. Then, $\gamma_n = \omega_{\nu_{\theta_n}} - \theta_n$.
Let be consider the following sequence,

$$\begin{cases} T_0 = \omega_{\nu_{\theta_n}} - (\theta_n + \gamma_n) = 0 \\ and \\ T_k = T_{k-1} + \xi_{\nu_{\theta_n}+k} \ , \forall k > 0. \end{cases} \tag{1}$$

The sequence $\{T_k\}_{k\in\mathbb{N}}$ describes the departure process after θ_n.

Let's also consider a $G/M/1$ system with exponentially distributed service times with parameter μ and with the same distribution of the arrival flux than the $G/G/1$

one. We introduce the corresponding following notation : $\bar{\theta}_n$, $\bar{\omega}_n$, $\bar{\gamma}_n$ and $\bar{V}_n = \bar{V}(\bar{\theta}_n - 0)$ defined as above. We also define the process $\{\bar{T}_n\}_{n\in\mathbb{N}}$ as the sequence $\{T_n\}$.

In the sequel, when no domain of integration is indicated, an integral is extended over $\mathbb{R}^+$

1.2 Strong stability of the imbedded Markov chain in the $G/M/1$ queueing system

Lemma 11. *The sequence $X_n = (V_n, \gamma_n)$, forms a homogeneous Markov chain with state space $\mathbb{N}\times\mathbb{R}^+$ and transition operator $Q = \| Q_{ij} \|_{i,j\geq 0}$, defined by*

$$Q_{ij}(x,dy) = P(V_{n+1} = j, \gamma_{n+1} \in dy / V_n = i, \gamma_n = x)$$
$$= \begin{cases} q_{i-j}(x,dy) & \text{for } 1 \leq j \leq i,\ i \geq 1, \\ \sum_{k\geq i} q_k(x,dy) & \text{for } j = 0,\ i \geq 0, \\ p(x,dy) & \text{for } j = i+1,\ i \geq 0, \\ 0 & \text{for } j > i+1,\ i \geq 0. \end{cases}$$

where

$$\begin{cases} q_k(x,dy) = \int_x^\infty P(T_k \leq u - x < T_{k+1}, T_{k+1} - (u-x) \in dy) dF(u) \\ \text{and} \\ p(x,dy) = \int_0^x P(x - u \in dy) dF(u). \end{cases} \tag{2}$$

Lemma 12. *The sequence $\bar{X}_n = (\bar{V}_n, \bar{\gamma}_n)$, forms a homogeneous Markov chain with state space $\mathbb{N}\times\mathbb{R}^+$ and transition operator $\bar{Q} = \| \bar{Q}_{ij} \|_{i,j\geq 0}$, having a same form as Q (lemma 22) where,*

$$\bar{q}_k(x) = \int_x^\infty e^{-\mu(u-x)} \frac{[\mu(u-x)]^k}{k!} dF(u) \tag{3}$$

Remark 9. The assumption $\rho = \lambda/\mu < 1$ implies the existence of a stationary distribution $\bar{\pi}$ for the imbedded Markov chain $\bar{X}_n$ in the $G/M/1$ system. This distribution has the following form,

$$\bar{\pi}(\{k\}, A) = \bar{\pi}_k(A) = p_k E(A), \qquad \forall\ \{k\} \subset \mathbb{N} \text{ and } A \subset \mathbb{R}^+, \tag{4}$$

where $p_k = \lim_{n\to\infty} P(\bar{V}_n = k)$ is given by the following relation,

$$p_k = (1-\sigma)\sigma^k, \quad k = 0, 1, 2, \ldots \tag{5}$$

σ is the unique solution of the equation

$$\sigma = F^*(\mu - \mu\sigma) = \int_0^\infty e^{-(\mu-\mu\sigma)x} dF(x) \tag{6}$$

F^* is the Laplace transform of the probability density function of the demands' inter-arrival times. We can show that, $0 < \sigma < 1$ (Kleinrock 1976).
Otherwise, note that,

$$\lim_{t\to\infty} P(X(t) = k) = \rho p_{k-1}, \quad \text{and} \quad \lim_{t\to\infty} P(X(t) = 0) = 1 - \rho \tag{7}$$

where, $X(t)$ represents the size of the $G/M/1$ system at time t and $k = 1, 2, \ldots$

The formulas (5) and (7) permit us to compute the stationary distribution of the queue length in a $G/M/1$ system. Unfortunately, for the $G/G/1$ system, these exact formulas are not known. So, if we suppose that the $G/G/1$ system is close to the $G/M/1$ system and if we show the strong stability in the $G/M/1$ queueing system (Aissani (1990), then we can use the formulas (5) and (7) to approximate the $G/G/1$ system characteristics with prior estimation of the corresponding approximation error.

Suppose that the service law of the $G/G/1$ system is close to the exponentially one with parameter μ. This proximity is characterized by the distance of variation,

$$W^* = W^*(G,E) = \int e^{\delta t}|G - E|(dt) \quad , \qquad \text{where} \qquad \delta > 0 \tag{8}$$

Let be consider the σ-Algebra $\mathcal{E}$, who represents the product $\mathcal{E}_1 \otimes \mathcal{E}_2$ ($\mathcal{E}_1$ is the σ-Algebra generated by the countable partition of $\mathbb{N}$ and $\mathcal{E}_2$ is the borelian σ-Algebra of $\mathbb{R}^+$).

We introduce in the space $m\mathcal{E}$ of finite measures on $\mathcal{E}$, the special family of norms $\| \cdot \|_v$,

$$\| \mu \|_v = \sum_{j\geq 0} \int v(j,y)|\mu_j|(dy) \tag{9}$$

where v is a measurable function on $\mathbb{N} \times \mathbb{R}^+$, bounded below away from zero (not necessary finite).

This norm induces a corresponding norm in the space $f\mathcal{E}$ of bounded measurable functions on $\mathbb{N} \times \mathbb{R}^+$, namely,

$$\| f \|_v = \sup_{k\geq 0} \sup_{x\geq 0} [v(k,x)]^{-1} |f(k,x)|, \ \forall f \in f\mathcal{E} \tag{10}$$

as well as a norm in the space of linear operators, namely,

$$\| P \|_v = \sup_{k\geq 0} \sup_{x\geq 0} [v(k,x)]^{-1} \sum_{j\geq 0} \int v(j,y)|P_{kj}(x,dy)| \tag{11}$$

We associate to each transition kernel P the linear mapping $P : f\mathcal{E} \to f\mathcal{E}$ acting on $f \in f\mathcal{E}$ as follows,

$$(Pf)(k,x) = \sum_{j\geq 0} \int P_{kj}(x,dy) f(j,y) \tag{12}$$

For $\mu \in m\mathcal{E}$ and $f \in f\mathcal{E}$ the symbol μf denotes the integral

$$\mu f = \sum_{j\geq 0} \int \mu_j(dx) f(j,x) \tag{13}$$

and $f \circ \mu$ denotes the transition kernel having the form

$$(f \circ \mu)_{ij}(x,A) = f(i,x)\mu_j(A) \tag{14}$$

We apply the theorem 2 (Kartashov 1981) to the imbedded Markov chain $\bar{X}_n$ (defined in lemma 12).
Consider the test function,

$$\begin{aligned} v : \mathbb{N} \times \mathbb{R}^+ &\to \quad \mathbb{R}^+ \\ (k, x) \ &\mapsto v(k, x) = \beta^k e^{\delta x} \end{aligned} \tag{15}$$

where, $1 < \beta < 1/\sigma$ and $0 < \delta = \mu - \frac{\mu}{\beta} < \mu$, ($\sigma$ is given by relation (6)).

Let α be a measure defined as follows, for $\{j\} \times dy \in \mathcal{E}$, we have,

$$\alpha(\{j\} \times dy) = \alpha_j(dy) = \begin{cases} E(dy) & \text{for } j = 0, \\ 0 & \text{for } j \neq 0. \end{cases} \tag{16}$$

And the measurable function

$$\begin{aligned} h : \mathbb{N} \times \mathbb{R}^+ &\to \quad \mathbb{R} \\ (i, x) \ &\mapsto h_i(x) = h(i, x) = \textstyle\sum_{k \geq i} \bar{q}_k(x) \end{aligned} \tag{17}$$

where $\bar{q}_k(x)$ is defined by the relation (3).

Lemma 13. *Let $\bar{X}_n$ be the Markov chain defined in lemma 12. Then, the operator $T = \bar{Q} - h \circ \alpha = \| T_{ij}(x, dy) \|_{i,j \geq 0}$ is non-negative.*

Proof. In fact, it is easily seen that,

$$T_{ij}(x, dy) = \begin{cases} 0 & \text{for } j = 0, \\ \bar{Q}_{ij}(x, dy) & \text{for } j > 0. \end{cases}$$

hence the result.

Lemma 14. *$\forall x > 0$, $\forall \beta > 1$ and for $\delta = \mu - \frac{\mu}{\beta} > 0$, the following inequality holds.*

$$\int_0^x dF(u) \int_0^x e^{\delta y} P(x - u \in dy) \leq \int_0^x e^{\delta(u-x)} dF(u). \tag{18}$$

Proof. It's sufficient to notice that,

$$\begin{aligned} \int_0^x dF(u) \int_0^x e^{\delta y} P(x - u \in dy) &= \int_0^x dF(u) \int_0^x e^{\delta y} P(y < x - u \leq y + dy) \\ &\leq \int_0^x e^{\delta(x-u)} dF(u) \end{aligned}$$

Lemma 15. *Suppose that in the $G/M/1$ system, the following geometric ergodicity condition holds,*

$$\mu \bar{\tau} > 1 \tag{19}$$

Then, $\forall \beta \in \mathbb{R}$ such that, $1 < \beta < 1/\sigma$, the following inequality is true,

$$\beta F^*(\mu - \frac{\mu}{\beta}) < 1 \tag{20}$$

σ and F^* have been defined in the relation (6) and $\bar{\tau}$ is the mean inter-arrival time in the $G/M/1$ system.

Proof. Let's consider the function

$$\begin{array}{rcl}\psi : [1, 1/\sigma] & \rightarrow & \mathbb{R}^{+} \\ \beta & \mapsto & \beta F^{*}(\mu - \frac{\mu}{\beta})\end{array} \quad (21)$$

From the convexity of ψ and from the relation (6), we have the result.

Lemma 16. *For a function v such that, $v(k, x) = \beta^{k} e^{\delta x}$, with $1 < \beta < 1/\sigma$ and $\delta = \mu(1 - 1/\beta) > 0$, The following inequality holds,*

$$(Tv)(k, x) \leq \rho v(k, x) \quad (22)$$

where $\rho = \beta F^{*}(\mu - \frac{\mu}{\beta}) < 1$

Proof. From the relation (12) and from the lemmas 22, 12 and 13, we have,

$$\begin{aligned}(Tv)(k, x) &= \sum_{j \geq 0} \int T_{kj}(x, dy) v(j, y) = \sum_{j > 0} \int \bar{Q}_{kj}(x, dy) v(j, y) \\ &\leq \beta^{k+1} [\int_{x}^{\infty} e^{-\mu(u-x)} e^{\frac{\mu}{\beta}(u-x)} dF(u) + \int_{0}^{x} e^{\delta(x-u)} dF(u)] \\ &\leq \beta^{k} e^{\delta x} \beta \int_{0}^{\infty} e^{-\delta u} dF(u)\end{aligned}$$

from which we obtain the result.

Lemma 17. *Let $\bar{Q}$ be the transition kernel of the imbedded Markov chain $\bar{X}_n$ in the $G/M/1$ system. Then, $\| \bar{Q} \|_{v} < \infty$*

Proof. From the relations (12) and (11), the proof can be easily established by the lemmas 13 and 16.

All the conditions of the theorem 2 (Kartashov 1981) are satisfied, then we can state the following result.

Theorem 1. *Suppose that in the $G/M/1$ system, the geometric ergodicity condition (19) holds.*
Then, $\forall \beta \in \mathbb{R}^{+}$ such that, $1 < \beta < \frac{1}{\sigma}$, the Markov chain $\bar{X}_n$ is strongly v-stable for a function $v(k, x) = \beta^{k} e^{\delta x}$.
where, $0 < \delta = \mu - \frac{\mu}{\beta} < \mu$ and $\rho = \beta F^{*}(\mu - \frac{\mu}{\beta}) < 1$.

Proof. The proof of this theorem is completely established from the previous lemmas.

Remark 10. To "measure" the performances of the strong stability method in a $G/M/1$ queueing system, after disturbing the service duration, we can use a general approach based on discret-event simulation (Banks 1996). We choose, for example, the Weibull probability distribution for modeling the demands' inter-arrival duration in both systems ($G/G/1$ *and* $G/M/1$) and the Cox probability distribution for the service duration in $G/G/1$ queueing system.

From the results obtained by repetition of these simulations, we estimate the margin between the corresponding characteristics of the simulated queueing systems. finally, we use the student's test to construct the confidence intervals for these margins.

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