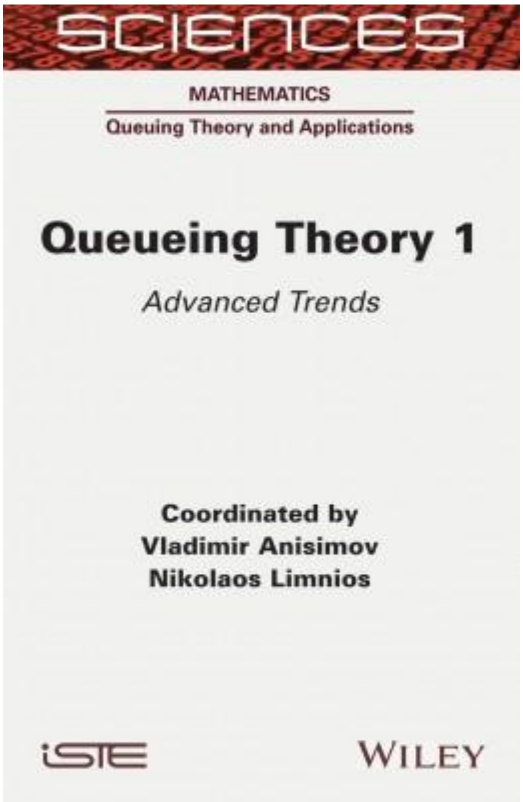
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9

Strong Stability of Queueing Systems and Networks: a Survey and Perspectives

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The analysis of the stability of queueing models aims at determining the conditions under which the mathematical model is a good representation of the real system despite approximation and estimation errors. A system is stable if small perturbations in its parameters generate at most a bounded deviation in its characteristics. The strong stability method has been used for the study of the sensitivity of diverse types of queues and queueing networks. In addition to the qualitative affirmation of stability, quantitative estimates of the perturbation error have been obtained in most cases. In this chapter, we review the application of the strong stability method to queues and queueing networks and provide directions for future research.

9.1. Introduction

Queueing models are useful for modeling and analysis of numerous systems such as communication systems, computer networks as well as production and manufacturing lines. The analysis of queueing models aims at evaluating a set of performance measures such as the utilization of resources, throughput and response time. However, real systems are usually very complicated and their representation by

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mathematical models is performed only through approximations. Hence, the real system is often replaced by another one, which is close to it in some sense but simpler in structure and/or components. This is necessary in order to obtain a model that is analytically tractable or can be solved by numerical methods. Moreover, the parameters of the model as well as the underlying probability distributions are estimated from empirical data by means of statistical methods. Usually, those approximations are performed with care in order to insure that the constructed model is sufficiently robust to resist the perturbations in its structure and parameters and that it remains a reliable representation of the real system. It is therefore very important to justify these approximations and estimate the resulting error.

In addition to the qualitative properties of the model, it is also important to have an estimation of the deviation in the characteristics (output) resulting from the perturbation of the parameters (input). Sensitivity analysis is a very important step in validating mathematical models.

Different mathematical methods have been elaborated in the study of the qualitative properties of stochastic systems, especially their stability. Here, we use the term "stability" to designate the ability of the system to resist perturbations (robustness, insensitivity). The first results were obtained by Rossberg (1965), Gnedenko (1970), Franken (1970) and Kennedy (1972). Kalashnikov and Tsitsiachvili (1972) proposed the method of test functions inspired by the classical Liapunov method initially applied to investigate the stability of differential equations (Kalashnikov and Tsitsiachvili 1972; Kalashnikov 1978). Stoyan (1977) investigated continuity properties of queueing models based on the weak convergence theory (see also Stoyan (1984)). Zolotariev (1975) and Rachev (1989) considered the stability problem as a continuity problem, which appears when applying some metric spaces in other spaces. Borovkov (1984) obtained theorems of ergodicity and stability with minimal conditions using renewal theory. Cao (1998) presented an approach based on Maclaurin series expansions of the stationary distribution of Markov chains to study the effect of parameter perturbation (see also Heidergott and Hordijk (2003)). Using operator-theoretic and probabilistic methods, Anisimov (1988) expressed the bounds for general Markov chains in terms of ergodicity coefficients of the iterated transition kernel, which are difficult to compute for infinite state spaces.

The strong stability method (Aïssani and Kartashov 1983a,b; Kartashov 1996) can be used to investigate the stability of a Harris recurrent Markov chain in general state space when the perturbation of its transition kernel is small with respect to a certain norm. A Markov chain is said to be strongly stable when small perturbations in the inputs (transition kernel) can lead to at most a bounded deviation of the outputs (stationary measure). Under this condition, approximations and parameter estimation errors result in a controlled deviation in the characteristics of the system in some sense. In addition to the qualitative affirmation of the stability (robustness) of the considered Markov chain, the strong stability method also allows the derivation of upper bounds on the deviation of the stationary characteristics resulting from the perturbation (approximation errors).

Queueing systems are among the first and most studied stochastic systems in the context of the strong stability theory. Many types of queues and queueing networks have been analyzed and their stability established. In most cases, quantitative estimates (perturbation bounds) are also obtained. This chapter focuses on the applicability of the strong stability method to queueing systems, reviews previous results and discusses future perspectives.

The remainder of this paper is organized as follows. In the first section, we introduce the notations and the basic definitions and theorems of the strong stability theory. Next, we review the application of the strong stability method to single queues, queueing networks as well as the use of non-parametric density estimation method in the study of those systems. The chapter will be finalized by a general conclusion with some future perspectives.

9.2. Preliminary and notations

Let $X = (X_t, t \geq 0)$, a homogeneous Markov chain with values in a measurable space $(\mathbb{E}, \mathcal{E})$ (where we assume that the σ-algebra $\mathcal{E}$ is countably generated), given by a regular transition kernel $P(x, A)$, $x \in \mathbb{E}$, $A \in \mathcal{E}$ and having a unique invariant measure π.

Denote by $m\mathcal{E}$ ($m\mathcal{E}^+$) the space of finite (nonnegative) measures on $\mathcal{E}$, $f\mathcal{E}$ ($f\mathcal{E}^+$) the space of bounded (nonnegative) measurable functions on $\mathbb{E}$.

Consider in the space $m\mathcal{E}$, the Banach space $\mathcal{M} = \{\mu \in m\mathcal{E} : \|\mu\| < \infty\}$ with norm $\|.\|$ compatible with the structural order in $m\mathcal{E}$, i.e.:

$$\|\mu_1\| \leq \|\mu_1 + \mu_2\| \text{ for } \mu_i \in \mathcal{M}^+, i = 1, 2.; \qquad [9.1]$$

$$\|\mu_1\| \leq \|\mu_1 - \mu_2\| \text{ for } \mu_i \in \mathcal{M}^+, i = 1, 2 \text{ and } \mu_1 \perp \mu_2; \qquad [9.2]$$

$$|\mu|(\mathbb{E}) \leq k\|\mu\| \text{ for } \mu \in \mathcal{M}; \qquad [9.3]$$

where $|\mu|$ is the variation of the measure μ, k is a finite positive constant and $\mathcal{M}^+ = \mathcal{M} \cap (m\mathcal{E}^+)$.

We introduce in $m\mathcal{E}$, the special family of norms:

$$\|\mu\|_v = \int_{\mathbb{E}} v(x)|\mu|(dx), \forall \mu \in m\mathcal{E}, \qquad [9.4]$$

where v is a measurable function bounded from below by a positive constant, (not necessary finite) on $\mathbb{E}$. Therefore, the induced norms on $f\mathcal{E}$ and $\mathcal{M}$ will have the following forms:

$$\|P\|_v = \sup\{\|\mu P\|_v, \|\mu\|_v \leq 1\} = \sup_{x\in\mathbb{E}} (v(x))^{-1} \int_{\mathbb{E}} |P(x,dy)|v(y), \qquad [9.5]$$

$$\|f\|_v = \sup\{|\mu f|, \|\mu\|_v \leq 1\} \quad = \sup_{x\in\mathbb{E}} (v(x))^{-1} |f(x)|. \qquad [9.6]$$

We associate to every transition kernel $P(x, A)$ in the space of bounded linear operators, the linear mappings $\mathcal{L}_P : m\mathcal{E} \to m\mathcal{E}$ and $\mathcal{L}_P^* : f\mathcal{E} \to f\mathcal{E}$, the values of which for $\mu \in m\mathcal{E}$ and $f \in f\mathcal{E}$ are, respectively:

$$\mu P(A) = \mathcal{L}_P(\mu)(A) = \int_{\mathbb{E}} \mu(dx)P(x,A), \ \ \forall A \in \mathcal{E},$$

$$Pf(x) = \mathcal{L}_P^*(f)(x) = \int_{\mathbb{E}} P(x,dy)f(y), \ \ \forall x \in \mathbb{E},$$

and with every function $f \in f\mathcal{E}$, we associate the linear functional $f : \mu \to \mu f$ such that:

$$\mu f = \int_{\mathbb{E}} \mu(dx)f(x).$$

For $\mu \in m\mathcal{E}$ and $f \in f\mathcal{E}$, $f \circ \mu$ is the transition kernel having the form:

$$f(x)\mu(A), x \in \mathbb{E}, A \in \mathcal{E},$$

where $\circ$ denotes the convolution between a measure and a function.

DEFINITION 9.1.– *The Markov chain X verifying $\|P\|_v < \infty$ is strongly $v-$stable, if every stochastic kernel Q in the neighborhood $\{Q : \|Q - P\|_v < \epsilon\}$ admits a unique stationary measure ν and:*

$$\|\nu - \pi\|_v \longrightarrow 0 \ \ \textit{when} \ \ \|Q - P\|_v \longrightarrow 0.$$

The following result (see Aïssani and Kartashov (1983a)) gives sufficient conditions for the strong $v-$stability of a Harris recurrent Markov chain.

THEOREM 9.1.– The Harris recurrent Markov chain X verifying $\|P\|_v < \infty$ is strongly v-stable, if the following conditions are satisfied:

1) $\exists \alpha \in \mathcal{M}^+, \exists h \in f\mathcal{E}^+$ such that : $\pi h > 0$, $\alpha \mathbb{I} = 1$, $\alpha h > 0$;

2) $T = P - h \circ \alpha$ is a non-negative kernel;

3) $\exists\, \rho < 1$ such that, $Tv(x) \ \leq \ \rho\, v(x), \forall x \in \mathbb{E}$;

where $\mathbb{I}$ is the function identically equal to 1.

One important feature of the strong stability method is the possibility of obtaining quantitative estimates. The following theorem (see Kartashov (1981, 1986c)) allows us to obtain an upper bound to the norm of the deviation of the stationary distribution of the strongly stable Markov chain X.

THEOREM 9.2.– Under the conditions of theorem 9.1 and for $\Delta = (Q - P)$ verifying the condition $\|\Delta\|_v < C^{-1}(1 - \rho)$, we have:

$$\|\nu - \pi\|_v \leq \|\Delta\|_v \|\pi\|_v C \left(1 - \rho - C\|\Delta\|_v\right)^{-1}, \qquad [9.7]$$

where

$$C = 1 + \|\mathbb{I}\|_v \|\pi\|_v \quad \text{and} \quad \|\pi\|_v \leq (\alpha v)(1 - \rho)^{-1}(\pi h).$$

Proofs of the above theorems, further conditions for the strong stability of homogeneous Markov chains and perturbation bounds as well as extensive results on the strong stability theory can be found in various studies (Aïssani 1990; Aïssani and Kartashov 1983a; Kartashov 1985, 1986a,b,c, 1996; Rabta and Aïssani 2008; Mouhoubi and Aïssani 2014; Rabta and Aïssani 2018).

Finally, we denote by $\mathbb{Z}$ the set of integer numbers and by $\mathbb{R}$ the set of real numbers. $\mathbb{A}^+$ denotes the non-negative part of the set $\mathbb{A}$ and $\mathbb{A}^* = \mathbb{A}/\{0\}$. Note that the notations introduced in each section are independent from those of the rest of the paper and might be redefined in another section.

9.3. Strong stability of queueing systems

Queueing models might have a large number of parameters that are subject to perturbations given that in practice those are unknown and must be approximated and/or fitted from empirical data. Obviously, the interarrival time and service time distributions are the first candidates. We distinguish the following kinds of perturbations.

– Parameter perturbations: only perturbations to individual parameters of the system are considered. For instance, in an $M/M/1$ queueing system, arrivals are Poisson distributed but the arrival rate has to be estimated. Considering the perturbation of the arrival rate while using a distribution of the same form (exponential) for the interarrival times, the perturbed system is also of type $M/M/1$.

– Distributional perturbations: here, the distribution of a constituent random variable is unknown and/or approximated by another distribution of a given (but different) form. In an $M/G/1$ queueing system, the general distribution of service times is unknown. Under certain conditions (distance), it might be replaced by an exponential distribution with the same mean. The resultant system is of type $M/M/1$.

– Non-parametric perturbations: we put in this category the applications of the strong stability method where the unknown distribution is fitted from data using non-parametric statistical methods such as kernel density estimation.

In all of the above kinds of perturbations, the same question is asked. The estimated parameters and/or distributions are imprecise, and hence, the mathematical representation (model) differs from the original system. We need to make sure that the estimation errors will not greatly impact the performance of our model and estimate the deviation in the performance measures between the original system and its mathematical representation, namely, the perturbation errors.

9.3.1. $M/M/1$ *queue*

The $M/M/1$ queueing system is the simplest queueing model. Customers arrive according to a Poisson process with rate λ and wait for service in front of a single server. We denote by E_λ the exponential distribution of the interarrival times. The queue capacity is infinite, the service durations follow an exponential distribution E_μ with rate μ and the service discipline is first-in-first-out (FIFO). The performance measures of this model are calculated in a closed form. In this section, we clarify the conditions under which the $M/M/1$ queueing model can be used as a good approximation in the case where the interarrival time distribution or the service distribution is different (but sufficiently close in some sense) from the exponential distribution.

9.3.1.1. *Perturbation of the interarrival time distribution*

Consider the queueing system $M/M/1$ $(FIFO, \infty)$ as described above. Let X_n be the number of customers in the queue just before the nth arrival. $X = \{X_n : n \geq 0\}$ is a homogeneous Markov chain with states in $\mathbb{Z}^+$ and transition matrix $P = (P_{ij})_{ij\geq 0}$ where

$$P_{ij} = \begin{cases} d_{i+1-j} = \frac{\lambda \mu^{i+1-j}}{(\lambda+\mu)^{i+2-j}} & \text{if } 1 \leq j \leq i+1, \\ 1 - \sum_{k=0}^{i} d_k = \left(\frac{\mu}{\mu+\lambda}\right)^i & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad [9.8]$$

Provided that the utilization $\frac{\lambda}{\mu} < 1$, the Markov chain X admits a unique stationary vector $\pi = (\pi_k)_{k \geq 0}$. Consider, on the other hand, a $GI/M/1(FIFO, \infty)$ queueing system where interarrival times are independent and identically distributed according to a general distribution H. Service times are distributed according to an exponential distribution E_μ. Let X_n^* be the number of customers in the queue just before the $n\text{th}$ arrival. It is very easy to show that $X^* = \{X_n^* : n \geq 0\}$ is a homogeneous Markov chain with states in $\mathbb{Z}^+$ and a transition matrix $P^* = (P_{ij}^*)_{i,j \geq 0}$ where

$$P_{ij}^* = \begin{cases} d_{i+1-j}^* = \int_0^\infty \frac{1}{(i+1-j)!} e^{-\mu t} (\mu t)^{i+1-j} \, dH(t) & \text{if } 1 \leq j \leq i+1, \\ 1 - \sum_{k=0}^{i} d_k^* & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \quad [9.9]$$

Again, a unique stationary vector $\pi^* = (\pi_k^*)_{k \geq 0}$ of the Markov chain X^* exists provided that the utilization is lower than 1. In order for the first system to be a good approximation of the latter, the interarrival time distributions of the two systems must be sufficiently close to each other. We measure the distance between the two distributions with the following metric:

$$w = w(H, E_\lambda) = \int_0^\infty |H - E_\lambda| \, (dt), \quad [9.10]$$

where $|a|$ is the variation of measure a.

THEOREM 9.3.– Suppose that the utilization of the $M/M/1$ queue $\frac{\lambda}{\mu} < 1$. Then, for each β such that $1 < \beta < \frac{\mu}{\lambda}$, the Markov chain X is stongly v-stable (with respect to the perturbation of the interarrival time distribution) with $v(k) = \beta^k$.

To prove the strong v-stability of the Markov chain X with respect to the function $v(k) = \beta^k$, $\beta > 1$, we check the conditions of theorem 9.1. Suppose that $\lambda/\mu < 1$ and let

$$h_i = \left(\frac{\mu}{\lambda + \mu}\right)^i \text{ for } i \geq 0 \quad \text{and} \quad \alpha_j = \begin{cases} 0 \text{ if } j \geq 1, \\ 1 \text{ if } j = 0. \end{cases}$$

The calculation is straightforward. In particular,

$$\rho = \frac{\beta\lambda}{\mu - \frac{\mu}{\beta} + \lambda} < 1 \text{ for } 1 < \beta < \mu/\lambda. \quad [9.11]$$

The following result provides an estimation of the distance (in the norm $\|.\|_v$) between the transition matrices of the two Markov chains. This could tell us how close the real system ($G/M/1$) is to its mathematical representation ($M/M/1$).

THEOREM 9.4.– For each β such that $1 < \beta < \mu/\lambda$, we have

$$\|P^* - P\|_v \leq (1+\beta)\,w,$$

where w is given by [9.10].

Now, we are ready to estimate the deviation of stationary distribution that results from the considered perturbation of the interarrival time distribution. The following theorem gives an upper bound to the difference between the stationary distributions π and π^* with respect to the norm $\|.\|_v$.

THEOREM 9.5.– Under the conditions of theorem 9.3 and for each distribution H satisfying

$$w < \frac{(1-\rho)(\mu-\lambda\beta)}{(1+\beta)(2\mu-\lambda(1+\beta))},$$

we have

$$\|\pi^* - \pi\|_v \leq \frac{(1+\beta)(2\mu-\lambda(1+\beta))(\mu-\lambda)\,w}{\frac{(\beta-1)(\mu-\lambda\beta)^3}{(\beta-1)\mu+\lambda\beta} - (2\mu-\lambda(1+\beta))(1+\beta)(\mu-\lambda\beta)\,w},$$

where ρ is given by [9.11].

The proof of this result is based on theorem 9.2. For detailed calculations, see Bouallouche and Aïssani (2006a).

9.3.1.2. *Perturbation of the service time distribution*

Consider again the queueing system of type $M/M/1\ (FIFO,\infty)$ described above. This time, we consider the random variable X_n representing the number of customers in the queue just after the $n\mathrm{th}$ departure. $X = \{X_n : n \geq 1\}$ is a homogeneous Markov chain with states in $\mathbb{Z}^+$ and transition matrix $P = (P_{ij})_{ij\geq 0}$ where

$$P_{ij} = \begin{cases} f_j & \text{if } i = 0 \\ f_{j-i+1} & \text{if } 1 \leq i \leq j+1 \\ 0 & \text{otherwise,} \end{cases} \qquad [9.12]$$

where

$$f_k = \frac{\mu\lambda^k}{(\lambda+\mu)^{k+1}}.$$

Under the usual utilization condition $\frac{\lambda}{\mu} < 1$, X is irreducible and aperiodic. Therefore, it admits a unique stationary vector π.

Consider the $M/G/1$ $(FIFO, \infty)$ queueing system obtained by replacing the service time distribution in the previous model by a general distribution F with the same mean. Let X_n^* be the number of customers in the queue in this new model just after the $n\mathrm{th}$ departure. $X^* = \{X_n^* : \ n \geq 1\}$ is a homogeneous Markov chain with states in $\mathbb{Z}^+$ and transition matrix $P^* = (P_{ij}^*)_{ij\geq 0}$ where

$$P_{ij}^* = \begin{cases} f_j^* & \text{if } i = 0, \\ f_{j-i+1}^* & \text{if } 1 \leq i \leq j+1, \\ 0 & \text{otherwise.} \end{cases} \qquad [9.13]$$

with

$$f_k^* = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dF(t) .$$

Suppose that the distribution of service durations in the $M/G/1$ system is close to the exponential service time distribution of the $M/M/1$ system. We measure the distance between the two distributions by

$$w = w(F, E_\mu) = \int_0^\infty |F - E_\mu| (dt) . \qquad [9.14]$$

First, we prove the following result.

THEOREM 9.6.– Suppose that the utilization of the $M/M/1$ system $\lambda/\mu < 1$. Then, for every β such that $1 < \beta < \mu/\lambda$, the Markov chain X is strongly v-stable for a test function $v(k) = \beta^k$ with respect to the perturbation of the service time distribution.

To prove this result, we choose

$$h(i) = \delta_{i0} = \begin{cases} 1 \text{ if } i = 0, \\ 0 \text{ if } i > 0, \end{cases}$$

and

$$\alpha_j = f_j .$$

Then, we check the conditions of theorem 9.1. In particular, it emerges that if $\beta < \mu/\lambda$, then

$$\rho = \frac{\mu}{\beta\,(\lambda + \mu - \beta\lambda)} < 1. \qquad [9.15]$$

This intermediary result is necessary to prove the next theorems.

LEMMA 9.1.– Under the conditions

$$\int_0^\infty t^2\,|F - E_\mu|\,(dt) < +\infty.$$

and

$$\int_0^\infty t\,|F - E_\mu|\,(dt) < w/\lambda,$$

where w is given by [9.14], there exists $\beta > 1$ such that

$$\int_0^\infty e^{\lambda(\beta-1)t}\,|F - E_\mu|\,(dt) < \beta w.$$

The distance between the transition matrices of the two systems is estimated by the following result.

THEOREM 9.7.– Under the conditions of lemma 9.1, we have

$$\|P - P^*\|_v \leq \beta_o w,$$

with $v(k) = \beta^k$ and

$$\beta_o = \max\left(\beta \;:\; 1 < \beta < \frac{\mu}{\lambda} \text{ and } \int_0^\infty e^{\lambda(\beta-1)t}\,|F - E_\mu|\,(dt) < \beta w\right). \qquad [9.16]$$

Now, we apply theorem 9.2 to estimate the deviation of the stationary vector with respect to the perturbation of the service time distribution.

THEOREM 9.8.– Under the conditions of theorem 9.6 and lemma 9.1, for every β such that $1 < \beta < \mu/\lambda$ and if

$$w \leq \frac{(1-\rho)}{C\beta_o},$$

we have

$$\|\pi - \pi^*\|_v \leq \beta_o w C C^{'} \left(1 - \rho - \beta_o w C\right)^{-1} = e(\beta).$$

with $v(k) = \beta^k$, β_o is given by [9.16] and ρ is given by [9.15],

$$C^{'} = \frac{\mu - \lambda}{\mu - \lambda\beta} \text{ and } C = \frac{2\mu - \lambda(1+\beta)}{\mu - \lambda\beta}.$$

Additionally to the stationary probabilities, the deviation of the other performance measures can be obtained. For example, let N_s (respectively, N_s^*) be the average number of customers in the system, N_q (respectively, N_q^*) be the average number of customers in the queue and T_s (respectively, T_s^*) be the average response time while T_q (respectively, T_q^*) is the average waiting time in the $M/M/1$ (respectively, $M/G/1$) system.

THEOREM 9.9.– Under the same conditions as theorem 9.8, we have the inequalities:

$$|N_s - N_s^*| = \left|N_q - N_q^*\right| < e(\beta)/\ln(\beta), \qquad [9.17]$$

$$|T_s - T_s^*| = \left|T_q - T_q^*\right| < e(\beta)/(\lambda \ln(\beta)), \qquad [9.18]$$

with $e(\beta) = \beta_o w C C^{'} \left(1 - \rho - \beta_o w C\right)^{-1}$.

The detailed proofs of the results of this section can be found in Bouallouche and Aïssani (2006b).

9.3.2. $PH/M/1$ and $M/PH/1$ queues

In the previous section, the interarrival time distribution (respectively, the service time distribution) is approximated by an exponential one having the same mean. This is a rather strict condition because in practice not many distributions can be approximated by the exponential distribution at an acceptable precision level. It would be very helpful if we could find a family of distributions that can approximate a large set of general distributions with a good level of precision and still have properties to allow analytical solutions of the underlying queueing model. This is precisely what the family of phase type (PH) distributions can provide us. Many PH queueing problems can be analytically solved using matrix–geometric techniques (Latouche and Ramaswami 1999). Additionally, the set of PH distributions is dense in the set of positive distributions allowing one (in theory) to approximate any positive distribution by a PH distribution at any desired precision level (Asmussen 2003).

In Djabali *et al.* (2018), the $M/PH/1$ queueing model is used to represent an $M/G/1$ system by approximating the general service time distribution of the latter by a PH distribution. This is achieved by matching the first two moments of both distributions. The matched PH distribution is chosen among the family of hyperexponential or hypoexponential distributions depending on the value of variability coefficient of the original distribution. The confirmation of the strong stability of the underlying Markov chain as well as quantitative estimates of the perturbation error are obtained in each case. Similar results (Djabali *et al.* 2015) exist for the $PH/M/1$ queue when perturbing the interarrival time distribution.

9.3.3. $G/M/1$ and $M/G/1$ queues

This section is concerned with the perturbation of the service time (respectively, interarrival time) distribution in a $G/M/1$ (respectively, $M/G/1$) queueing system. The exponential distribution is replaced by a general distribution with the same mean and the result of this perturbation is a $G/G/1$ queueing model. The conditions of the strong stability of the underlying Markov chain are clarified in each case and upper bounds to the deviation of the stationary vectors are obtained.

9.3.3.1. *Strong stability in the $G/M/1$ queueing system*

Consider a $G/G/1$ queueing system with a general service times distribution G and a general interarrival times probability distribution F. The following notations are used: θ_n (the arrival time of the $n\text{th}$ demand), ω_n (the departure time of the $n\text{th}$ customer), γ_n (the time interval from θ_n to the departure of the next customer) and $V_n = V(\theta_n - 0)$ (the number of customers found in the system immediately prior to θ_n).

Let us denote by $\nu_{\theta_n} = \min\{m > 0, \omega_m \geq \theta_n\}$. Then, $\gamma_n = \omega_{\nu_{\theta_n}} - \theta_n$.

Define recursively the following sequence,

$$\begin{cases} T_0 = \omega_{\nu_{\theta_n}} - (\theta_n + \gamma_n) = 0, \\ T_k = T_{k-1} + \omega_{\nu_{\theta_n}+k} \ , \forall k > 0. \end{cases} \quad [9.19]$$

The sequence $\{T_k\}_{k\in\mathbb{Z}^+}$ describes the departure process after θ_n.

Let us also consider a $G/M/1$ system with exponentially distributed service times with parameter μ and with the same distribution of the interarrival times as the $G/G/1$ one. We introduce the corresponding following notation: $\bar{\theta}_n, \bar{\omega}_n, \bar{\gamma}_n$ and $\bar{V}_n = \bar{V}(\bar{\theta}_n - 0)$ defined as above. We also define the process $\{\bar{T}_n\}_{n\in\mathbb{N}}$ as the sequence $\{T_n\}$.

In what follows, when no domain of integration is indicated, an integral is extended over $\mathbb{R}^+$.

LEMMA 9.2.– The sequence $X_n = (V_n, \gamma_n)$ forms a homogeneous Markov chain with state space $\mathbb{Z}^+ \times \mathbb{R}^+$ and transition operator $Q = (Q_{ij})_{i,j\geq 0}$ defined by

$$Q_{ij}(x, dy) = P(V_{n+1} = j, \gamma_{n+1} \in dy / V_n = i, \gamma_n = x),$$
$$= \begin{cases} q_{i-j}(x, dy) & \text{for } 1 \leq j \leq i,\ i \geq 1, \\ \sum_{k\geq i} q_k(x, dy) & \text{for } j = 0,\ i \geq 0, \\ p(x, dy) & \text{for } j = i+1,\ i \geq 0, \\ 0 & \text{for } j > i+1,\ i \geq 0. \end{cases}$$

where

$$\begin{cases} q_k(x, dy) = \int_x^\infty P(T_k \leq u - x < T_{k+1}, T_{k+1} - (u-x) \in dy) dF(u), \\ p(x, dy) = \int_0^x P(x - u \in dy) dF(u). \end{cases} \qquad [9.20]$$

LEMMA 9.3.– The sequence $\bar{X}_n = (\bar{V}_n, \bar{\gamma}_n)$ forms a homogeneous Markov chain with state space $\mathbb{Z}^+ \times \mathbb{R}^+$ and transition operator $\bar{Q} = (\bar{Q}_{ij})_{i,j\geq 0}$ having the same form as Q (Lemma 9.2), where

$$\bar{q}_k(x) = \int_x^\infty e^{-\mu(u-x)} \frac{[\mu(u-x)]^k}{k!} dF(u). \qquad [9.21]$$

REMARK 9.1.– The assumption $\bar{\tau}\mu > 1$, where $\bar{\tau}$ is a mean time between arrivals in the $G/M/1$ queueing system, implies the existence of a stationary distribution $\bar{\pi}$ for the embedded Markov chain $\bar{X}$. This distribution has the following form:

$$\bar{\pi}(\{k\}, A) = \bar{\pi}_k(A) = p_k E_\mu(A), \qquad \forall \{k\} \subset \mathbb{Z}^+ \textit{ and } A \subset \mathbb{R}^+, \qquad [9.22]$$

where $p_k = \lim_{n\to\infty} P(\bar{V}_n = k)$ is given by the following relation:

$$p_k = (1-\sigma)\sigma^k, \quad k = 0, 1, 2, \ldots \qquad [9.23]$$

σ is the unique solution of the equation

$$\sigma = F^*(\mu - \mu\sigma) = \int_0^\infty e^{-(\mu-\mu\sigma)x} dF(x), \qquad [9.24]$$

F^* is the Laplace transform of the probability density function of the demands' inter-arrival times. We can show that $0 < \sigma < 1$ (Kleinrock 1975).

Otherwise, note that

$$\lim_{t\to\infty} P(X(t)=k) = \frac{1}{\bar{\tau}\mu} p_{k-1}, \;\; k=1,2,\dots \quad and \quad \lim_{t\to\infty} P(X(t)=0) = 1-\frac{1}{\bar{\tau}\mu}, \qquad [9.25]$$

where $X(t)$ represents the size of the $G/M/1$ system at time t.

The formulas [9.23] and [9.25] permit us to compute the stationary distribution of the queue length in a $G/M/1$ system. Unfortunately, for the $G/G/1$ system, these exact formulas are not known. So, if we suppose that the $G/G/1$ system is close to the $G/M/1$ system, then we can use the formulas [9.23] and [9.25] to approximate the $G/G/1$ system characteristics with prior estimation of the corresponding approximation error.

Suppose that the service time distribution of the $G/G/1$ system is close to the exponential one with parameter μ. This proximity is characterized by the distance of variation,

$$W^* = W^*(G, E_\mu) = \int e^{\delta t} |G - E_\mu|(dt) \;, \qquad \text{where} \quad \delta > 0. \qquad [9.26]$$

Let also consider the following deviation:

$$W_0 = W_0(G, E_\mu) = \int |G - E_\mu|(dt). \qquad [9.27]$$

We apply theorem 9.1 to the imbedded Markov chain $\bar{X}$. Consider the test function

$$\begin{array}{rl} v : \mathbb{Z}^+ \times \mathbb{R}^+ \to & \mathbb{R}^+ \\ (k, x) & \mapsto v(k,x) = \beta^k e^{\delta x} \end{array} \qquad [9.28]$$

where $1 < \beta < 1/\sigma$ and $0 < \delta = \mu - \frac{\mu}{\beta} < \mu$ and σ is given by relation [9.24].

Let α be a measure defined as follows, for $\{j\} \times dy \in \mathcal{E}_\mu$, we have,

$$\alpha(\{j\}, dy) = \alpha_j(dy) = \begin{cases} E_\mu(dy) & \text{for } j = 0, \\ 0 & \text{for } j \neq 0. \end{cases} \qquad [9.29]$$

And the measurable function

$$h : \mathbb{Z}^+ \times \mathbb{R}^+ \to \mathbb{R} \qquad (i,x) \mapsto h_i(x) = h(i,x) = \sum_{k \geq i} \bar{q}_k(x) \qquad [9.30]$$

where $\bar{q}_k(x)$ is defined by the relation [9.21].

Now, we can state the following result.

THEOREM 9.10.– Suppose that the geometric ergodicity condition $\mu\bar{\tau} > 1$ holds. Then, $\forall \beta \in \mathbb{R}^+$ such that, $1 < \beta < \frac{1}{\sigma}$, the Markov chain $\bar{X}$ is strongly v-stable for a function $v(k,x) = \beta^k e^{\delta x}$ where, $0 < \delta = \mu - \frac{\mu}{\beta} < \mu$.

To prove this result, we check that all the conditions of theorem 9.1 are satisfied (Benaouicha and Aïssani 2005). In particular,

$$\rho = \beta F^*(\mu - \frac{\mu}{\beta}) < 1.$$

The following result gives us the quantitative estimation of the deviation of the norm of transition operator in the $G/M/1$ system, after perturbation of the service time distribution. The proof is based on a series of lemmas demonstrated in Benaouicha and Aïssani (2005).

THEOREM 9.11.– Let Q and $\bar{Q}$ be the transition kernels of the Markov chains X and $\bar{X}$, respectively. Suppose that, for each β such that $1 < \beta < 1/\sigma$, the following conditions hold:

1) $G^* = \int e^{\delta t} G(dt) < +\infty$;

2) $\exists a > 0$ such that $\int e^{au} dF(u) = N < +\infty$;

3) $W_0 = \int |G - E_\mu|(dt) < \frac{a}{a+\mu}$;

4) the geometric ergodicity condition $\mu\bar{\tau} > 1$.

Then, the following inequality holds:

$$\|Q - \bar{Q}\|_v \leq W^*(1 + \mu\bar{\tau}) + W_0 G^* \frac{N + \mu M}{1 - C_0},$$

where $C_0 = W_0 + \frac{\mu}{a+\mu} < 1$ and $M = \int u e^{au} dF(u) < +\infty$.

The upper bound on the norm of the deviation of the stationary measure of the Markov chain $\bar{X}$ is given by the following theorem.

THEOREM 9.12.– Let π *and* $\bar{\pi}$ be the stationary measures of X and $\bar{X}$, respectively. If

$$W^* = W^*(G, E_\mu) < \frac{1-\rho}{2C(1+\mu\bar{\tau}+C_1)},$$

and if

$$W_0 < \frac{a}{a+\mu},$$

then the following inequality holds:

$$\|\pi - \bar{\pi}\|_v \leq 2[(1+\mu\bar{\tau})W^* + C_1 W_0]\frac{C(C-1)}{1-\rho},$$

where

$$C = 1 + \|\bar{\pi}\|_v = \frac{1+\beta(1-2\sigma)}{1-\beta\sigma}, \qquad C_1 = \frac{N+\mu M}{1-C_0}G^*. \qquad [9.31]$$

The proof of this theorem is based on theorem 9.2 and uses a series of intermediary results. Further details can be found in previous studies (Aïssani 1982, 1987a,b; Benaouicha and Aïssani 2005).

9.3.3.2. *Strong stability of the $M/G/1$ queue*

Consider a $G/G/1$ (FIFO,∞) queueing model. We denote by τ'_n the interarrival time between the arrival dates of the $(n-1)\mathrm{th}$ and the $n\mathrm{th}$ customers. The sequence of independent and identically distributed service times is denoted by $\{\xi_n\}$. Let $\theta'_n = \tau'_1 + \cdots + \tau'_n$ be the arrival moment of the $(n+1)\mathrm{th}$ customer with $\theta'_0 = 0$, and consider the distributions $G(t) = P(\tau'_n < t)$ and $F(t) = P(\xi_n < t)$ with $m = \mathrm{E}\xi_n$.

Let q_n be the number of customers in the system at the end of service of the $n\mathrm{th}$ customer, γ_n the (residual) time until the next arrival. The sequence $X_n = (q_n, \gamma_n)$ is therefore a homogeneous Markov chain with states in $\mathbb{Z}^+ \times \mathbb{R}^+$ and a transition kernel

$$Qf(n,x) = \mathrm{E}(f(n-1,\ x-\xi_1),\ \xi_1 < x) +$$
$$+\sum_{k=1}^{\infty} \mathrm{E}(f(n-1+k,\ \theta'_k + x - \xi_1),\ \ \theta'_{k-1} \leq \xi_1 - x < \theta'_k), \qquad [9.32]$$

where $n > 0$ and $Qf(n,x) = \mathrm{E}Qf(1,\tau'_1)$. Q can be obtained from the distributions G and F.

On the other hand, consider a $M/G/1$ (FIFO,∞) queueing model with (i.i.d) exponential interarrival times $\{\theta_n\}_{n\geq 1}$, $\theta_0 = 0$ of mean $1/\lambda$. The service time distribution is F. Let P be the transition kernel of the Markov chain $\{X_n\}$ in the $M/G/1$ system.

In what follows, we assume that the following conditions are satisfied:

$$\lambda \, \mathrm{E} \, \xi_1 \; < 1 \, , \; \mathrm{E} \, exp \, (a\xi_1) < +\infty, \qquad [9.33]$$

for a certain $a > 0$.

Suppose that the interarrival time distribution G in the $G/G/1$ system is close to the exponential interarrival time distribution E_λ of the $M/G/1$ model. The distance between the two distributions is measured by

$$w(G, E_\lambda) = \int \exp(ct)|G - E_\lambda|(dt), \qquad [9.34]$$

where $0 < c < \lambda$ is a fixed parameter.

Consider the measurable test function $v(k, x)$ on $\mathbb{Z}^+ \times \mathbb{R}^+$ defined by

$$v(k, x) = \beta^k \exp{(\varepsilon x)},$$

and the corresponding $v-$norm $\|.\|_{\beta\varepsilon}$.

It is easy to prove that $\|P\|_{\beta\varepsilon} < \infty$ under the condition $\lambda/\mu < 1$ and for every $0 < \varepsilon < \lambda$ and $1 < \beta < 1 + a\lambda^{-1}$. Additionally, we have the following result:

LEMMA 9.4.– Suppose that the condition $\lambda/\mu < 1$ is satisfied and let $0 < c < \lambda$ be a fixed constant. Then, there exist $\beta_1 = \beta_1(\lambda, c, F) > 1$ and $L_1(\beta) < \infty$ such that $\|Q - P\|_{\beta c} \leq L_1(\beta)w(G, E_\lambda)$ for every $1 < \beta < \beta_1$ and every distribution G such that $w(G, E_\lambda)$ is sufficiently small.

Let $v(n, x) = \beta^n(exp\,(cx) + bexp(-\delta x))$. We show that the Markov chain X is strongly stable with respect to the norm $\|.\|_v$. By the fact that $b^{-1}\|T\|_{\beta c} \leq \|T\|_v \leq (1 + b)\|T\|_{\beta c}$, the chain X_n is strongly stable with respect to the norm $\|.\|_{\beta c}$.

Let us check the conditions of the strong stability of the Markov chain X.

– it is easy to check that $\|P\|_v < \infty$;

– define the measure α and the measurable function h on $I \times \mathbb{R}_+$ as follows:

$h(n, x) = 0$ for $n > 0, h(0, x) = 1,$

$\alpha(\{n\} \times A) = P(\theta_n \leq \xi_1 < \theta_{n+1},\ \theta_{n+1} - \xi_1 \in A).$

We may easily show that for the operator $T = P - ho\alpha$, $Tf(n,x) = Pf(n,x)$ for $n > 0$ and $Tf(0,x) = 0$. Clearly, T is non-negative.

Observe that for $n > 0$ and $v(k,x) = \beta^k \, exp\,(-\delta x)$,

$$Tv(n,x) = \beta^{n-1}\mathrm{E}(exp\,(\delta\xi_1 - \delta x),\ \xi_1 < x) + \sum_{k\geq 1}\beta^{n-1+k}\times$$
$$\times\mathrm{E}(exp(-\delta\theta_k - \delta x + \delta\xi_1),\ \theta_{k-1} \leq \xi_1 - x < \theta_k) = \beta^{n-1}exp(-\delta x)\times$$
$$\times\mathrm{E}(exp(\delta\xi_1 - \delta x),\ \xi_1 < x) + \lambda(\lambda+\delta)^{-1}\beta^n\mathrm{E}(exp(\lambda(\beta-1)(\xi_1 - x), (\xi_1 \geq x). \quad [9.35]$$

For $\delta = \lambda(\beta-1)$, we obtain $Tv(n,x) = \rho_0 v(n,x)$, where $\rho_0 = \beta^{-1}\mathrm{E}\,exp(\lambda(\beta-1)\xi_1)$. Given that $\lambda/\mu < 1$, it is easy to establish that $\rho_0 < 1$ for $\beta < 1$ if sufficiently small.

By choosing β and setting $\delta = \lambda(\beta-1)$ in the expression of $v(n,x)$, we obtain for $n > 0$

$$Tv(n,x) \leq (b\rho_0 + \lambda(\lambda - c)^{-1}\beta)\,\beta^n\,exp\,(-\delta x) + \beta^{n-1}exp(cx) \leq \rho V(n,x),$$

where $\rho = \rho_0 + b^{-1}\lambda(\lambda - C)^{-1}\beta$. In addition, $Tv(0,x) = 0 < \rho v(0,x)$.

It is sufficient to choose $\rho < 1$. This is possible because $\rho_0 < 1$ and the constant b can be taken sufficiently large.

9.3.4. *Other queues*

In the same way, a multitude of queueing systems have been studied. Hence, strong stability results and perturbation bounds were also obtained for the following queueing systems:

– group arrival queues: perturbation of the distribution of the size of the group (Boukir *et al.* 2009);

– retrial queues: perturbation of the retrial rate (Berdjoudj and Aïssani 2003);

– multiple server $M/M/m$ queue: perturbation of the interarrival time distribution (Issaadi *et al.* 2016);

– queues with unreliable server: perturbation of failure rate (Abbas and Aïssani 2010a,c);

– queues with server vacation: perturbation of vacation rate (Rahmoune and Aïssani 2008, 2014);

– $GI/M/\infty$ queue: perturbation of the size of the system (Aïssani 1992a,b; Bareche *et al.* 2016);

– $M_2/G_2/1$ queue with priority: perturbation of the rate of priority arrivals (Aïssani 1991; Bouallouche and Aïssani 2008; Hamadouche and Aïssani 2011);

– $GI/M/1$ queue with negative customers: perturbation of the negative arrival rate (Abbas and Aïssani 2010b).

9.3.5. *Queueing networks*

The application of the strong stability method to queueing networks poses numerous challenges. Except from Jackson networks, the performance measures cannot be obtained in closed form. The approximation of general queueing networks by Jackson networks also poses problems. First, the dimension of the underlying Markov chain might be high. Additionally, the strong stability method supposes that the perturbed process is also Markovian. The interconnection of the network nodes and the complex dynamics of such systems make it very difficult to maintain the Markov property or to define an embedded Markov chain in the perturbed system.

9.3.5.1. *Jackson networks with two tandem stations*

Consider the following Jackson network with two tandem queues $[M/M/1 \rightarrow M/M/1]$. Customers arrive at the first station according to a Poisson process with rate λ. Denote by E_λ the exponential distribution of the interarrival times. Service durations follow an exponential distribution with rate μ in the first station and an exponential distribution with rate μ_1 in the second one. The performance measures of this model can be calculated in the closed form by exploiting the product form property (Jackson 1957). In this section, we clarify the conditions under which this type of queueing network can be used as a good approximation in the case where the service distribution in the first station is different (but sufficiently close in some sense) from the exponential distribution.

The state of the tandem network above is completely described by the Markovian bidimensional process $\overline{V}(t) = (\overline{X}(t), \overline{Y}(t))$, where $\overline{X}(t)$ is the number of customers in the first station at time t, and $\overline{Y}(t)$ is the number of customers in the second station at the same time t.

Denote by $\overline{d}_n^+$ the moment just after the departure of the nth customer from the first station. Then, the embedded sequence $(\overline{V}_n)_{n\geq 0}$ of random variables where $\overline{V}_n =$

$\overline{V}(\overline{d}_n^+), \overline{V}_0 = 0$ is a Markov chain. Under the condition $\lambda \leq \min(\mu_1, \mu)$, the transition probabilities of the Markov chain $\overline{V}_n$ are given by:

$$\overline{Q}_{ij}(k,l) = \begin{cases} \overline{P}_j \overline{q}_{kl}, & \text{if } i = 0, j \geq 0, 1 \leq l \leq k+1, k \geq 0, \\ \overline{P}_j \overline{q}_{k0}, & \text{if } i = 0, j \geq 0, l = 0, k \geq 0, \\ \overline{P}_{j-i+1} \overline{q}_{kl}, & \text{if } 1 \leq i \leq j+1, j \geq 0, 1 \leq l \leq k+1, k \geq 0, \\ \overline{P}_{j-i+1} \overline{q}_{k0}, & \text{if } 1 \leq i \leq j+1, j \geq 0, l = 0, k \geq 0, \\ 0, & \text{otherwise} \end{cases} \quad [9.36]$$

where:

$$\overline{P}_r = \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^r}{r!} dE_{\mu_1}(x), \forall r \in \mathbb{N},$$

$$\overline{q}_{kl} = \int_0^\infty \exp(-\mu x) \frac{(\mu x)^{k+1-l}}{(k+1-l)!} dE_\lambda(x),$$

$$\overline{q}_{ko} = 1 - \sum_{l=1}^{k+1} \overline{q}_{kl}.$$

On the other hand, consider the two stations tandem network $[M/G/1 \to \cdot/M/1]$. Arrivals to the first station are Poisson distributed with the same rate λ as before while the service times distribution H is general. Consequently, the arrival process to the second station is not Poissonian anymore. Denote by F the distribution of interarrival times to this station. Service times at the second station are exponentially distributed having common distribution E_μ with rate μ. Let $V(t) = (X(t), Y(t))$ be the two-dimensional process, where $X(t)$ is the number of customers in the first station at time t, $Y(t)$ is the number of customers in the second station at time t. $V(t)$ completely describes the state of the considered network $[M/G/1 \to ./M/1]$. However, $V(t) = (X(t), Y(t))$ is not a Markovian process. Denote by d_n^+ the moment just after the departure of the nth customer from the first station. Then, the sequence of random variables $V_n = (\overline{X}_n, \overline{Y}_n)$ at these times is an embedded Markov chain that describes the state of the network $[M/G/1 \to \cdot/M/1]$ at those specific dates. Under the condition $\lambda < \min(\mu_1, \mu_2)$ the transition probabilities of the chain V_n write:

$$Q_{ij}(k,l) = \begin{cases} P_j q_{kl}, & \text{if } i = 0, j \geq 0, 1 \leq l \leq k+1, k \geq 0, \\ P_j q_{k0}, & \text{if } i = 0, j \geq 0, l = 0, k \geq 0, \\ P_{j-i+1} q_{kl}, & \text{if } 1 \leq i \leq j+1, j \geq 0, 1 \leq l \leq k+1, k \geq 0, \\ P_{j-i+1} q_{k0}, & \text{if } 1 \leq i \leq j+1, j \geq 0, l = 0, k \geq 0, \\ 0, & \text{otherwise} \end{cases} \quad [9.37]$$

where

$$P_r = \int_0^{\infty} \exp(-\lambda x) \frac{(\lambda x)^r}{r!} dH(x), \forall r \in \mathbb{N},$$

$$q_{kl} = \int_0^{\infty} \exp(-\mu x) \frac{(\mu x)^{k+1-l}}{(k+1-l)!} dE_\lambda(x),$$

$$q_{ko} = 1 - \sum_{l=1}^{k+1} q_{kl}.$$

Let α be the measure defined as follows:

$$\alpha(\{i\},\{j\}) = \begin{cases} P_i q_{kj} & \text{if } \ 0 \le l \le k+1, \\ 0 & \text{otherwise.} \end{cases} \qquad [9.38]$$

Consider as well the measurable function h defined by:

$$h(i,k) = h_i(k) = \mathbf{1}_{\{i=0\}}. \qquad [9.39]$$

Using them in theorem 9.1, the following result can be proved.

THEOREM 9.13.– Under the condition $\lambda < \min(\mu_1, \mu)$, the Markov chain $\{V_n\}_{n\ge 0} = \{X_n, Y_n\}_{n\ge 0}$ is strongly v-stable with respect to the test function $v(i,j) = \gamma^i \beta^j$, for every γ, β such that

$$1 < \gamma < \frac{\mu_1}{\lambda}, \text{and} 1 < \beta < \gamma\Big(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\Big).$$

9.3.5.2. *Tandem queues with constant retrials*

Consider the tandem queueing network $[M/G/1 \to ./G/1/1]$ with blocking after service, consisting of a sequence of two service stations without intermediate queue. Customers arrive to the first station according to a Poisson process with intensity λ. Each customer receives service at station 1 and then proceeds to station 2 for an additional service. Since there is no intermediate waiting room, a customer whose service in the station 1 is completed cannot proceed to the second station if the latter is busy. Instead, the customer remains at station 1 that is blocked until station 2 becomes empty. The arriving customer who finds station 1 busy or blocked behaves like a retrial customer, i.e., he does not join a queue but instead he is placed in a hypothetical retrial queue (orbit) of infinite capacity and retries for service under the constant retrial policy. According to this policy, the parameter of the exponential time of each customer in the retrial group is $\frac{\mu}{n}$, where n is the size of the retrial group. Thus, the total intensity is μ. If the server of station 1 is free at the time of an attempt, the customer at the head of the retrial group receives service immediately. Otherwise, they repeat their demand later.

Service times at stations 1 and 2 are independent and arbitrarily distributed random variables with probability density functions $b_i(x)$, distribution functions $B_i(x)$ and finite mean values $1/\mu_i$, for $i = 1, 2$, respectively.

Let $X(t)$ represent the number of customers in the orbit at time t, and for $l = 1, 2$:

$$\xi^l(t) = \begin{cases} 0 & \text{if the } l^{\text{th}} \text{ server is idle at time } t, \\ 1 & \text{if the } l^{\text{th}} \text{ server is working at time } t, \\ 2 & \text{if the } l^{\text{th}} \text{ server is blocked at time } t. \end{cases}$$

The considered model is completely described by the regenerative process:

$$V(t) = \left(X(t), \xi^1(t), \xi^2(t)\right).$$

However, this process is not Markovian. We denote by $d_n, n \in \mathrm{N}$, the instant of the $n\mathrm{th}$ departure from station 1. We assume, without loss of generality, that $d_0 = 0$, and we note that:

$$V_n = V(d_n + 0) = \left(X(d_n + 0), \xi^1(d_n + 0), \xi^2(d_n + 0)\right) = (X_n, 0, 0).$$

Thus, V_n is a semiregenerative process with embedded Markov renewal process $(X, D) = \{X_n, d_n : n \in \mathrm{N}\}$. The process $\{X_n\}$ is an homogeneous, irreductible and aperiodic Markov chain with the transition matrix $\mathbf{P} = \{p_{ij}\}_{i,j \geq 0}$, where:

$$p_{ij} = \begin{cases} \int_0^{+\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} f_0(t) dt, & \text{for } i = 0, \\ \int_0^{+\infty} \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t} f_1(t) dt \\ \quad + \int_0^{+\infty} \frac{(\lambda t)^{j-i+1}}{(j-i+1)!} e^{-\lambda t} f_2(t) dt, & \text{for } 1 \leq i < j+1, \\ \int_0^{+\infty} e^{-\lambda t} f_2(t) dt & \text{for } i = j+1, \\ 0, & \text{otherwise}, \end{cases} \qquad [9.40]$$

$$\text{with: } \begin{cases} f_0(t) = \int_0^{+\infty} \lambda e^{-\lambda w} \frac{d}{dt} \left(B_1(t) B_2(t+w)\right) dw; \\ f_1(t) = \int_0^{+\infty} \lambda e^{-(\lambda+\mu) w} \frac{d}{dt} \left(B_1(t) B_2(t+w)\right) dw; \\ f_2(t) = \int_0^{+\infty} \mu e^{-(\lambda+\mu) w} \frac{d}{dt} \left(B_1(t) B_2(t+w)\right) dw. \end{cases}$$

Let $\psi_u(s)$ be the function defined as:

$$\psi_u(s) = \int_0^{+\infty} ue^{-uw}dw \int_0^{+\infty} e^{-sx}d_x\left(B_1(x)B_2(x+w)\right). \qquad [9.41]$$

Assume that the mean retrial rate in the above tandem queues tends to infinity, i.e. the customers in the retrial group try continuously to find a position for service and they become ordinary customers. It means that if $\mu \to +\infty$, the tandem network with constant retrials becomes similar to the classical model of two queues in tandem without intermediate room.

Now, let $\overline{X}(t)$ denote the number of customers in the first queue of the classical tandem network at time t and for $l = 1, 2$, we consider:

$$\overline{\xi}^l(t) = \begin{cases} 0 & \text{if the } l^{\text{th}} \text{ server is idle at time } t, \\ 1 & \text{if the } l^{\text{th}} \text{ server is working at time } t, \\ 2 & \text{if the } l^{\text{th}} \text{ server is blocked at time } t. \end{cases}$$

The state of the ordinary tandem network $[M/G/1 \to ./G/1]$ is completely described by the process $\overline{V}(t) = \left(\overline{X}(t), \overline{\xi}^1(t), \overline{\xi}^2(t)\right)$. It is clear that $\overline{X}_n = (\overline{X}, D) = \{\overline{X}_n, \overline{d}_n, n \geq 0\}$ is the embedded Markov renewal process of the semiregenerative process $\left(\overline{X}(t), \overline{\xi}^1(t), \overline{\xi}^2(t)\right)$ at the instant $\overline{d}_n$ of the nth departure from station 1.

Now, suppose that $\rho = \lim_{\mu \to +\infty} \rho^* < 1$. Then $\overline{X}_n$ is an irreductible and homogeneous, aperiodic, positive recurrent Markov chain with transition matrix $\overline{\mathbf{P}} = \{\overline{p}_{ij}\}_{i,j \geq 0}$:

$$\overline{p}_{ij} = \begin{cases} \int_0^{+\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} f_0(t)dt, & i = 0, \\ \\ \int_0^{+\infty} \frac{(\lambda t)^{j-k+1}}{(j-k+1)!} e^{-\lambda t} d_t\left(B_1(t)B_2(t)\right), & i \in [i, j+1], \\ \\ 0, & \text{otherwise}. \end{cases} \qquad [9.42]$$

It is easy to show that:

$$\lim_{\mu \to +\infty} \psi_{\lambda+\mu}(s) = \psi(s) = \int_0^{+\infty} e^{-st} d_t\left(B_1(t)B_2(t)\right); \quad \lim_{\mu \to +\infty} v_{\lambda+\mu} = \frac{\rho}{\lambda};$$

$$\lim_{\mu \to +\infty} \rho^* = \rho = -\lambda \frac{d\psi(s)}{ds} \mid_{s=0} = \lambda \int_0^{+\infty} t d_t\left(B_1(t)B_2(t)\right).$$

Suppose that the mean retrial rate in the tandem network with constant retrials tends to infinity. The distance between the two distribution's interarrivals is measured by: $W = \int_0^{+\infty} |f_2(t) - \frac{d}{dt}(B_1(t)B_2(t))|dt$.

Consider the test function $v(j) = \beta^j, \beta > 1$, the measure α and the measurable function h defined by:

$$\alpha(\{j\}) = \alpha_j = \overline{p}_{0j}, \qquad h(i) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By using them in theorem 9.1, and based on a series of lemmas demonstrated in a previous study (Lekadir and Aïssani 2008a), the following theorems can be proved.

THEOREM 9.14.– In the two stations' tandem network with blocking $[M/G/1 \to ./G/1/1]$, the Markov chain $\overline{X}_n$ representing the number of customers in the first station at the instant of the $n\text{th}$ departure from the first station is strongly v-stable with respect to the function $v(k) = \beta^k$ for all β such that $1 < \beta \leq \beta_0$, where β_0 is given by: $\beta_0 = \sup\left\{\beta : \gamma(\beta) = \frac{\psi(\lambda - \lambda\beta)}{\beta} < 1\right\}$.

THEOREM 9.15.– Let π (respectively, $\overline{\pi}$) be the stationary distribution of the Markov chain X_n (resp. $\overline{X}_n$). For β such that $1 < \beta < \beta_0$, we have:

$$\|\pi - \overline{\pi}\|_v \leq c_0(1 + c_0)\|\Delta\|_v\big(1 - \gamma - (1 + c_0)\|\Delta\|_v\big)^{-1},$$

where: $c_0 = \frac{\psi_\lambda(\lambda\beta - \lambda) - \gamma}{1 - \gamma}$.

9.3.5.3. *Tandem queues with non-preemptive priority*

Consider the tandem queueing network $[M_2/G_2/1 \to ./G/1/1]$ with non-preemptive priority. Customers arrive to the first station according to a Poisson process. Denote by $(\theta\lambda)$ (respectively, (λ)) the arrival rate of the priority (respectively, the non-priority) customers. The distribution function of the service times of the priority (respectively, non-priority) customers in the first station is $C_1(x)$ (respectively, $C_2(x)$)) and the corresponding density function is $c_1(x)$ (respectively, $c_2(x)$). Moreover, service times in the second station are independent having distribution function $D(x)$ and density function $d(x)$ common for both priority classes. Now, let $\{(X^i(t)\}_{i=1,2}$ represent the number of customers in station i at the departure time of the $n\text{th}$ customer. The state of the considered model is completely described by the bidimentional process $V(t) = \big(X^1(t), X^2(t)\big)$. Suppose that θ tends to zero, the part of the tandem queue related to priority customers can be interpreted as a perturbation of the network $[M/G/1 \to ./G/1/1]$ of two stations in tandem with only non-priority customers. The Markov chain $\left(X_n^{(1)}, X_n^{(2)}\right)_{n\geq 0}$ has transition probabilities $P_{k,l}(i,j,\theta)$ defined by:

$$\begin{cases} \int_0^\infty \left(\frac{(\lambda\theta x)^{i-k+1}}{(i-k+1)!}\right)\left(\frac{(\lambda x)^{j-l}}{(j-l)!}\right) e^{-\lambda(\theta+1)x} d\left[C_1(x)D(x)\right]; \text{if } k\neq 0, l\geq 0; i\geq k-1, j\geq l, \\ \int_0^\infty \left(\frac{(\lambda\theta x)^{i}}{i!}\right)\left(\frac{(\lambda x)^{j-l+1}}{(j-l+1)!}\right) e^{-\lambda(1+\theta)x} \left[\int_0^\infty \lambda e^{-\lambda t} d\left(C_2(x)D(x+t)\right)dt\right]; \\ \qquad \text{if } k=0, l\neq 0; i\geq 0, j\geq l-1, \\ \frac{\theta}{1+\theta}\int_0^\infty \left(\frac{(\lambda\theta x)^{i}}{i!}\right)\left(\frac{(\lambda x)^{j}}{j!}\right) e^{-\lambda(1+\theta)x} \left[\int_0^\infty \lambda e^{-\lambda t} d\left(C_1(x)D(x+t)\right)dt\right] + \\ \quad + \frac{1}{1+\theta}\int_0^\infty \left(\frac{(\lambda\theta x)^{i}}{i!}\right)\left(\frac{(\lambda x)^{j}}{j!}\right) e^{-\lambda(1+\theta)x} \left[\int_0^\infty \lambda e^{-\lambda t} d\left(C_2(x)D(x+t)\right)dt\right]; \\ \qquad \text{if } k=0, l=0; i\geq 0, j\geq 0, \\ 0; \qquad \text{otherwise.} \end{cases}$$

The transition probabilities $P_{k,l}(i,j,0)$ of the Markov chain $\left(\overline{X}_n^{(1)}, \overline{X}_n^{(2)}\right)_{n\geq 0}$ describing the state of the tandem network $[M/G/1 \to ./G/1/1]$ are given by:

$$\begin{cases} \int_0^\infty \left(\frac{(\lambda x)^{j-l}}{(j-l)!}\right) e^{-\lambda x} d\left[C_1(x)D(x)\right]; & \text{if } k\neq 0, l\geq 0; i=k-1, j\geq l, \\ \int_0^\infty \left(\frac{(\lambda x)^{j-l+1}}{(j-l+1)!}\right) e^{-\lambda x} \left[\int_0^\infty \lambda e^{-\lambda t} d\left(C_2(x)D(x+t)\right)dt\right]; & \text{if } k=0, l\neq 0; i=0, j\geq l-1, \\ \int_0^\infty \left(\frac{(\lambda x)^{j}}{j!}\right) e^{-\lambda x} \left[\int_0^\infty \lambda e^{-\lambda t} d\left(C_2(x)D(x+t)\right)dt\right]; & \text{if } k=0, l=0; i=0, j\geq 0, \\ 0; & \text{otherwise.} \end{cases}$$

The conditions of the v-stability of the network $[M/G/1 \to ./G/1/1]$ are clarified by the following theorem:

THEOREM 9.16.– Suppose that in the tandem network $[M/G/1 \to ./G/1/1]$, the following the assumptions holds:

– $\lambda E(\xi_2) < 1$ (geometric ergodicity condition), ξ_i is the random variable representing the service times of customers with priority i;

– $\exists a > 0/E(e^{a\xi_2}) < \infty$ (Cramer condition).

Then, for all β such that $1 < \beta < \beta_0$ and $\delta > 1$, the Markov chain $\left(\overline{X}_n^{(1)}, \overline{X}_n^{(2)}\right)_{n\geq 0}$ is v-strongly stable with respect to the function $v(i,j) = \delta^i \beta^j$ with , $\delta = \frac{E(e^{x\xi_1})}{\gamma_2}$, where $\gamma_2 = \frac{E(e^{x\xi_2})}{\beta}$ and $\beta_0 = \sup\{\beta / E(e^{x\xi_2}) < \beta\}$.

To prove theorem 9.16, it is sufficient to check the conditions of theorem 9.1 using the test function:

$$
\begin{aligned}
& v : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}_+^* \\
& \qquad (i,\ j) \longrightarrow V(i,\ j) = \delta^i \times \beta^j;\ \text{with}\ \ \delta > 1,\ \beta > 1.
\end{aligned}
$$

The measure α and the measurable function h are defined by:

$$
\begin{aligned}
& \alpha : \sigma(N) \times \sigma(N) \longrightarrow \mathbb{R}^+ && h : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R} \\
& \qquad (\{i\}, \{j\}) \longrightarrow P_{0,0}(i,j,0) && \qquad (k,\ l) \longrightarrow 1_{\{k=0,\ l=0\}}.
\end{aligned}
$$

Note $f_i(x) = \displaystyle\int_0^{+\infty} \lambda e^{-\lambda t} \frac{d}{dx}\Big(C_i(x) D(x+t)\Big) dt,\ i = \overline{1,2}$, then:

$$
\alpha\big(\{i\}, \{j\}\big) = \begin{cases} P_{0,0}(0,j,0) = \displaystyle\int_0^{+\infty} \frac{(\lambda x)^j}{j!} e^{-\lambda x} f_2(x) dx,\ \text{if}\ i = 0; \\ P_{0,0}(i,j,0) = 0,\ \text{if}\ i \neq 0. \end{cases}
$$

From the above theorem, the tandem network $[M/G/1 \to ./G/1/1]$ is strongly v-stable. It means that its characteristics can approximate the tandem network $[M_2/G_2/1 \to ./G/1/1]$ under the condition that the arrival rate of priority customers θ tends to zero. To characterize this proximity, it is essential to estimate the deviation between the stationary distributions of the chains X_n and $\overline{X}_n$. To do so, we first estimate the deviation $\|\Delta\|_v$ of the transition operator $\mathbf{P}$.

LEMMA 9.5.–

$$
\begin{aligned}
& \|\Delta\|_v = \|\mathbf{P} - \overline{\mathbf{P}}\|_v = \sup_{k\geq 0} \sup_{l\geq 0} \frac{1}{\delta^k \beta^l} \sum_{i\geq 0} \sum_{j\geq 0} \delta^i \beta^j \left|\Delta_{kl}(i,j,\theta)\right|, \\
& \quad \text{with} \quad : \left|\Delta_{kl}(i,j,\theta)\right| = \left|P_{kl}(i,j,\theta) - P_{kl}(i,j,0)\right|.
\end{aligned}
$$

Denote by $\Theta = \sup\{E_{00}, E_{0l}, E_{kl}\}$. Then:

$$
\|\Delta\|_v \leq \Theta; \tag{9.43}
$$

where:

$$
\begin{aligned}
E_{00} &\leq \phi_2(\lambda - \lambda\beta) + \frac{1}{1+\theta}\phi_2(\lambda\theta + \lambda - \lambda\beta - \lambda\delta\theta) \\
&\quad - \left(\frac{1}{1+\theta} + 1\right)\phi_2(\lambda\theta + \lambda - \lambda\beta) + \frac{\theta}{1+\theta}\phi_1(\lambda\theta + \lambda - \lambda\beta - \lambda\theta\delta); \\
E_{0l} &= \frac{1}{\beta}\left[\psi_2(\lambda - \lambda\beta) + \psi_2(\lambda + \lambda\theta - \lambda\beta - \lambda\delta\theta) - 2\psi_2(\lambda + \lambda\theta - \lambda\beta)\right]; \\
E_{kl} &= \frac{1}{\delta}\left[\psi_1(\lambda + \lambda\theta - \lambda\theta\delta - \lambda\beta) + \psi_1(\lambda - \lambda\beta) - 2\psi_1(\lambda + \lambda\theta - \lambda\beta)\right]; \\
\phi_i(s) &= \int_0^\infty \lambda e^{-\lambda t} dt \int_0^\infty e^{-sx} d[C_i(x)D(x+t)], \\
\psi_i(s) &= \int_0^\infty e^{-sx} d[C_i(x)D(x)].
\end{aligned}
$$

The following lemma is also needed to prove the next theorem.

LEMMA 9.6.– Let us consider the constant ϖ defined by $\varpi = \beta P_0 \left(\frac{\gamma}{1-\gamma}\right)$, where: $P_0 = \frac{1+\lambda\psi_1'(0)+\theta\lambda\phi_1'(0)}{1+\lambda\psi_1'(0)+\theta\lambda\psi_2'(0)-\lambda\phi_1'(0)-\theta\lambda\phi_1'(0)}$, then $\|\overline{\pi}\|_v \leq \varpi$.

The upper bound on the norm of the deviation of the stationary measure of the Markov chains of the tandem queues $[M/G/1 \rightarrow ./M/1/1]$ and $[M_2/G_2/1 \rightarrow ./M/1/1]$ is given by the following theorem.

THEOREM 9.17.– Let $\overline{\pi}$ (respectively, π) be the stationary distribution of $[M/G/1 \rightarrow ./M/1/1]$ (respectively, $[M_2/G_2/1 \rightarrow ./M/1/1]$). Under the condition $\Theta < \frac{1-\gamma}{C}$, we have:

$$
\|\pi - \overline{\pi}\|_v \leq \frac{\Theta\varpi\,(1+\varpi)}{1-\gamma-(1+\varpi)\Theta} \qquad [9.44]
$$

where γ, Θ, ϖ and P_0 are defined previously. $1 < \beta < \beta_0$ where β_0 is defined by: $\beta_0 = \sup\left\{\beta : \frac{\psi_2(\lambda\beta-\lambda)}{\beta} < 1\right\}, 1 < \beta_0 < +\infty$.

The proof of this theorem is based on theorem 9.2 and uses a series of intermediary results (for more details, see Lekadir and Aïssani (2011)).

9.3.5.4. *Tandem queues with blocking*

A two-station tandem network of type $[M/M/1/\infty \leftrightarrow M/M/1/N]$ with blocking was considered in a previous study (Adel-Aissanou *et al.* 2012). The infinite buffer of the first station is truncated by rejecting arrivals when the queue length reaches a given level Q. It is expected that such a truncation well approximates the original model as the truncation level (or size) becomes large. The conditions that guarantee that the steady-state joint queue length distribution of the original tandem queueing system is well approximated by the finite buffer truncation are clarified and error bounds on the stationary queue length distributions are obtained.

9.3.6. *Non-parametric perturbation*

The above results are concerned with the use of parametric distributions with known or unknown form. In other works, the authors explore the use of non-parametric distributions such as kernel density functions fitted from empirical data. Combining statistics theory results with the strong stability method, one is able to estimate the impact of the use of such technique.

Note that, in practice, all model parameters are imprecisely known because they are obtained by means of statistical methods. That is why the strong stability inequalities will allow us to numerically estimate the uncertainty shown during this analysis.

In this sense, one aspect which is of interest is when a distribution governing a queueing system is unknown and we resort to non-parametric methods to estimate its density function. For instance, if one had real data, then one could apply the kernel density method to estimate the related density function.

Moreover, it is very often the case that the natural domain of definition of a density to be estimated is not the whole real line but an interval bounded on one or both sides. For example, when the data are measurements of positive quantities, it will be preferable to obtain density estimates that take the value zero for all negative numbers.

Indeed, the strong stability method states that the perturbation must be small in the sense that the general law G of arrivals (respectively, service times) must be close but not equal to the Poisson (respectively, exponential) law. Consequently, the density function of the law G must be close to the exponential density, which is defined on a bounded support. Thus, the boundary effects must be taken into consideration when using the kernel density method.

By combining the techniques of correction of boundary effects with the calculation of the variation distance characterizing the proximity of the quoted systems, one will be able to check whether this density is sufficiently close to that of the Poisson law (or that of the exponential law), and then apply the strong stability method to approximate the characteristics of the real system with those of a classical method (Bareche and Aïssani 2008, 2011, 2014a,b).

9.4. Conclusion and further directions

The chapter reviews the evolution of the applications of the strong stability method to queueing systems and their networks. Many types of queues and networks have been already studied and both qualitative (stability) and quantitative (perturbation bounds) are obtained in most cases. However, queueing systems may contain a large number of parameters and the perturbation of each might have a completely different effect from the others. Therefore, it is necessary to extend the stability analysis to those unexplored models and perturbation types, for instance, perturbation of the collusion probability in the $M/M/1$ retrial queue with collisions and transmission errors (modeling the access method in the channel of IEEE 802.11 (Lakaour *et al.* 2018)), as well as the perturbation of the parameter characterizing the impatient customers in the $M/G/c/k$ queueing systems with impatient customers (modeling the Cloud Data Center (Outamazirt *et al.* 2018)).

The application to general queueing networks might be challenging because of the high dimension of the embedded Markov chain and the complex dynamics of those models. Additionally, perturbations in one node result in the loss of the Markov property in many other (connected) nodes or in the whole network. However, particular networks (e.g. tandem networks) can be studied.

The theory of strong stability is rich in results. Conditions for the stability and quantitative estimates might be obtained with various techniques. For instance, one might use generalized inverses, ergodicity coefficients, eigenvalues, etc. Applying other strong stability results to queues is essential in order to further sharpen the perturbation bounds.

Finally, comparison with other methods demonstrates the usefulness of the theory (see, e.g., Abbas and Heidergott (2010)).

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Contents

Preface

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Queueing theory is a huge and very rapidly developing branch of science that originated a long time ago from the pioneering works by Erlang (1909) on the analysis of the models for telephone communication.

Now, it is growing in various directions including a theoretical analysis of queueing models and networks of rather complicated structure using rather sophisticated mathematical models and various types of stochastic processes. It also includes very wide areas of applications: computing and telecommunication networks, traffic engineering, mobile telecommunications, etc.

The aim of this book is to reflect the current state-of-the-art and some contemporary directions of the analysis of queueing models and networks including some applications.

The first volume of the book consists of 10 chapters written by world-known experts in these areas. These chapters cover a large spectrum of theoretical and asymptotic results for various types of queueing models, including different applications.

Chapter 1 is devoted to the investigation of some theoretical problems for non-classical queueing models including the analysis of queues with inter-dependent arrival and service times.

Chapter 2 deals with the analysis of some characteristics of fluid queues including busy period, congestion analysis and loss probability.

Queueing Theory 1,
coordinated by Vladimir ANISIMOV, Nikolaos LIMNIOS. © ISTE Editions 2020.

Some contemporary tendencies in the asymptotic analysis of queues are reflected in the following three survey Chapters 3, 7 and 10.

Chapter 3 includes the results on the average and diffusion approximation of Markov queueing systems and networks with a small series parameter ε including applications to some Markov state-dependent queueing models and some other type of models, in particular, repairman problem, superposition of Markov processes and semi-Markov type queueing systems.

Diffusion and Gaussian limits for multi-channel queueing networks with rather general time-dependent input flow and under heavy traffic conditions including some applications to networks with semi-Markov or renewal type input and Markov service are considered in Chapter 7.

Chapter 10 is devoted to the asymptotic analysis of time-varying queues using the large deviations principle for two-time-scale non-homogeneous Markov chains including the analysis of the queue length process and some characterizations of the quality and the efficiency of the system.

The analysis of so-called retrial queueing models is reflected in two Chapters – 4 and 8.

In Chapter 4, two models that provide some modifications of "First-Come First-Served" retrial queueing system introduced by Laszlo Lakatos are investigated.

Chapter 8 gives a review of recent results on single server finite-source retrial queueing systems with random breakdowns and repairs and collisions of the customers.

The analysis of transient behavior of the infinite-server queueing models with a mixed arrival process and Coxian service times and of the Markov-modulated infinite-server queue with general service times is considered in Chapter 5.

Chapter 6 deals with the applications of fast simulation methods used in queueing theory to solve some high-dimension combinatorial problems in case the other approaches fail.

A survey on the analysis of a strong stability method and its applications to queueing systems and networks and some perspectives are considered in Chapter 9.

The second volume of the book will include the additional chapters devoted to some other contemporary directions of the analysis of queueing models.

The volumes will be useful for graduate and PhD students, lecturers, and also the researchers and developers working in mathematical and stochastic modelling and various applications in computer and communication networks, science and engineering in the departments of Mathematics & Applied Mathematics, Statistics, or Operations Research at universities and in various research and applied centres.

Dedication to Igor Mykolayovych Kovalenko who died shortly after the writing of this book

Ukrainian and world science is mourning the loss of a brilliant scientist, Professor Igor Mykolayovych Kovalenko, who died on October 19, 2019, after a difficult fight with heart disease.

Prof. Igor Kovalenko was a prominent Ukrainian mathematician in the field of probability theory and its practical applications, a disciple and associate of Boris Gnedenko and Vladimir Korolyuk. He became famous worldwide for his book *Introduction to Queueing Theory*, written together with Gnedenko. He founded a scientific school in the theory of reliability, queueing theory and cryptography, well known in Ukraine and all over the world.

Igor Kovalenko was born on March 16, 1935 in Kyiv, Ukraine. After graduating from the Faculty of Mechanics and Mathematics of Kyiv Taras Shevchenko University, he worked at the Computing Centre of the Academy of Sciences of Ukraine. From 1962 till 1971, Kovalenko worked in Moscow, where he headed a laboratory at the Moscow Institute of Electronic Engineering, and together with other Gnedenko's disciples, was the head of the seminar on queueing theory at Moscow State University. Many leading scientists of the former Soviet Union and foreign countries attended this seminar.

Based on the model of piecewise linear Markov processes developed by him, Kovalenko built a mathematical model of a complex defence system reliability and developed numerical algorithms for its implementation based on the method of a small parameter.

In 1964, Igor Kovalenko became a Doctor of Technical Sciences. He formulated the principle of monotonous failures, which, while maintaining high accuracy, significantly simplified the calculations of system reliability. In 1970, Kovalenko was awarded the degree of Doctor of Physics and Mathematics for another thesis on the probabilistic theory of systems of random Boolean equations. Being a doctor twice over is a very rare practice in the scientific world.

After returning to Kyiv in 1971, Prof. Kovalenko founded and headed the Department of Mathematical Methods of the Theory of Complex Systems Reliability at the V.M. Glushkov Institute of Cybernetics. Two areas of research formed the mainstream of investigations: approximate combined analytical and statistical methods of reliability analysis, and theoretical and applied cryptography, systems and methods for data protection. Under his guidance, the first national standard in the field of cryptographic information security was developed in Ukraine.

Prof. Kovalenko is the author of 25 monographs and more than 200 articles. He was elected as an Academician of the National Academy of Sciences of Ukraine in 1978 (Corresponding Member since 1972). He was an extremely hard-working, honest and sincere person, a competent manager and, thanks to his human qualities, professional experience and knowledge, highly respected among his colleagues.

Prof. Igor Kovalenko left many disciples, among them there are many professors and associate professors. All of them preserve in their memory the unforgettable days of joining the science and independent creativity under the guidance of a Great Scientist and Teacher, hours of direct communication with a person of great erudition and high culture.