Steady state analysis of *M*/*G*/1 retrial queue with vacation and server timeout using a Petri net formalism

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Abstract: In this paper, we analyse an $M/G/1$ retrial queue with vacation and server timeout. When the system is empty, the server waits for a random time. At the expiration of this time, if there is no arrival to the system, the server takes a vacation. If there is at least one customer in the system, the server commences service. So, we propose an unbounded Markov regenerative stochastic Petri net ($MRSPN$) to model this queueing system. For the qualitative analysis of this model, we combine the Markov regenerative approach and the generating function technique. Finally, numerical illustrations are performed.

Keywords: retrial systems; Markov regenerative stochastic Petri nets; MRSPN; $M/G/1$; renewal theory; steady state; vacation policy; server timeout.

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1 Introduction

1.1 Related works on the $M/G/1$ queue with vacation and retrial

In the last decades there have been enormous contributions in the area of retrial queueing theory. To illustrate this, we can mention the work of Artalejo and Gómez-Corral (1995), the book by Falin and Templeton (1997), the survey papers of Artalejo (1999a, 1999b) and the review of Shekhar et al. (2016), ... For these systems, the service of customer is preceded and followed by the server's idle time caused by the ignorance of the status of the server and orbital customers by each other. Many researchers have tried to reduce the idle time by introducing the orbital search mechanism (see Artalejo et al., 2002; Dudin et al., 2004; Wüchner et al., 2009, etc.) or to exploit this idle time for other secondary jobs by introducing the vacations, see Doshi's (1986) survey paper, Takagi's (1991) monograph, Tian and Zhang's (2006) monograph, etc.

A wide class of policies for governing the vacation mechanism, have been discussed in the literature, namely: exhaustive service policy (Oliver and Kishor, 1991; Ramanath and Lakshmi, 2006; Binyamin et al., 2019), Bernoulli schedule vacation (Choudhuy, 2008; Arivudainambi and Gowsalya, 2018), working vacations (Servi and Finn, 2002; Ayyappan et al., 2010), server timeout (Oliver, 2015), gated service policy, limited service policy, etc. Recently, Kalita et al. (2020) established the analysis of an $M/G/1$ queue with modified vacation policy. They used the Markov regenerative process approach to obtain the explicit expressions for steady state system size probabilities. In the same year, Joshi et al. (2020) studied the steady state analysis of $D/M/1$ and $M/G/1$ models with multiple vacation queueing system.

The queueing system with vacation and server timeout is a hybrid multiple vacation scheme and single vacation scheme. One important application of the vacation scheme with server timeout is to enhance resource utilisation in the single vacation model. The notion of timeout (i.e., the maximal time to wait before retrying an action) occurs in many networking contexts. The use of timeouts is encountered especially in large-scale networks (Libman and Orda, 2002). Recently, several researchers have focused on such vacation scheme. We can cite: Oliver (2015) derived the mean waiting time of $M/G/1$ vacation queueing systems with server timeout using the stochastic decomposition property. Ramesh and Praby (2016), studied a vacation bulk queueing model with setup time and server timeout. Saritha et al. (2017) obtained an expression for expected system length of $M^X/G/1$ vacation queueing system with server timeout. Rama et al. (2018) analysed the cost function of an $M/E_K/1$ vacation queueing system with server timeout and N-policy. Satish et al. (2017) investigated the optimal strategy N-policy $M/M/1$ queueing system with server start-up and timeout. Devi et al. (2019) derived the expected system length for $M^X/G/1$ vacation queueing system with two types of repair facilities and server timeout.

Research on queueing system allowing the simultaneous presence of the server vacations and repeated attempts have gained more attention, see Artalejo (1997), Aïssani (2000), Choudhuy (2008), Kumar and Arumuganathan (2009), Boualem et al. (2011) and Varalakshmi et al. (2016). In retrial systems with vacations, customers who arrive while the server is busy or on vacation, have to join the orbit to repeat their call after a random period. Recently, interesting researches concerning this type of models have been carried out, we can mention the following works: Rajadurai et al. (2018) presented a single server retrial queueing system with balking and feedback under multiple working vacation policy, where the busy server is subjected to breakdown and repair. The probability generating functions for the numbers of customers in the system when it is free, busy, on working vacation and under repair are found by using the supplementary variable method. Gao and Zhang (2020) studied an $M/G/1$ queue with retrial customers due to server vacation. They developed the steady state analysis by the supplementary variable method and the generating function. Gao and Wang (2020) analysed a preemptive priority $M/G/1$ retrial queue with orbital search and exhaustive multiple vacations. By using embedded Markov chain (EMC) technique and the supplementary variable method, they discussed the necessary and sufficient condition for the system to be stable and the joint queue length distribution in steady state. Li et al. (2020) considered an $M/G/1$ retrial queue with general retrial times and single working vacation. They obtained the generating functions of the server state and the number of customers in the orbit by using the supplementary variable method. Rajadurai (2020) studied a single server retrial queueing system with working vacations and vacation interruption. The authors considered three different types of customers (priority customers, ordinary customers and negative customers). Using the supplementary variable technique, the steady state probability generating function of the system and its orbit were found. Revathi and Raj (2020) studied a single server retrial queueing system with optional re-service, customer search, delayed repair and vacations. The supplementary variable technique were used to calculate orbit size, server status, and performance metrics. Rajasudha et al. (2022) considered a batch arrival single server retrial queueing model under three different vacation policies (single vacation, multiple vacations, and atmost J-vacations) with impatient customers in general retrial times. The probability generating function and marginal generating function of orbit size were obtained in a steady state. Xu et al. (2022) studied an $M/G/1$ retrial queueing system with modified multiple vacations, in which a new external arrival may expel the customer being served out of the system and directly starts to be served or join the retrial orbit. By the EMC, they provide the sufficient and necessary condition of system stability. In addition, the distributions of the orbit size and the system size in steady state are derived through the supplementary variable method.

1.2 Stochastic Petri nets and queueing systems

The applicability of stochastic Petri nets for stochastic modelling and performance analysis of queueing system has received much attention from researchers. Oliver and Kishor (1991) showed how vacation queueing systems with exponentially distributed service times and finite population can be modelled by the generalised stochastic Petri nets ($GSPN$). Gharbi and Ioualalen (2010) analysed a finite source multi-server retrial queue using $GSPN$ formalism. Hakmi et al. (2017) investigated a priority queueing system. Using the Markov regenerative stochastic Petri net ($MRSPN$), Ramanath and Lakshmi (2006) studied the queue $M/G/1//N$ with different vacation schemes. Ikhlef et al. (2016) analysed an $M/G/1//N$ retrial queue with orbital search by the $MRSPN$ formalism. In the mentioned papers, the researchers analysed a queueing model with finite number of sources.

1.3 Our contribution

In this paper, we present an approach for modelling and analysing the $M/G/1$ retrial queue with vacation and server timeout. This approach is based on the combination of the $MRSPN$ theory with generating functions. We propose a $MRSPN$ model which allows an easier description of the behaviour of such queueing systems and an automatic generation of the closed form expression of the global kernel and the local kernel. Our $MRSPN$ is unbounded, we used the generating functions technique to obtain its quantitative analysis. We carried out the stationary analysis of this system (existence of the stationary regime, EMC, and steady state distribution). We also derived formulas for some performance measures.

1.4 Practical justification of the suggested system

Retrial queueing systems with vacation have been used to model many problems arising in telephone switching systems, telecommunication networks, computer networks and computer systems, etc. Especially, the considered queueing system can be used to model a Diff-Serv router with the assured forwarding per-hop behaviour (AF-PHB). Currently, Diff-Serv defines three PHB: such as expedited forwarding PHB (EF-PHB), assured forwarding PHB (AF-PHB), and the default PHB (Demoor et al., 2011). PHB may be specified in terms of their resources (e.g., buffer size, bandwidth), priority relative to other PHB, or in terms of their relative observable traffic characteristics (e.g., delay, loss) to provide a better quality of service (QoS). The AF-PHB group specifies a forwarding behaviour in which packets see a very small amount of loss. Queuing approach is one of the vital mechanisms in traffic management system. For this reason, it is important to implement a queuing discipline that governs the buffering mechanism of packets (voice, data, ...) while they are waiting to be transmitted. We assume that the packets of the AF-PHB is accommodated in waiting room with infinite capacity (orbit) when the server is occupied and the vacations represent services of packets (default PHB). We assume that the orbit has infinite capacity. In general, this assumption does not considerably alter the results as the studied queues of AF-PHB have a sufficiently large capacity in practice. The arrival process of AF packets can be modelled by a Poisson process. The total time spend in the server by packets (voice, data, ...) can have any general distribution. We also assume that the packets cannot leave the system without getting served. With these assumptions, it seems realistic to proposed $M/G/1$ retrial queue with vacation and server timeout as a model of Diff-Serv router with AF-PHB.

2 Steady state of the MRSPN

2.1 MRSPN's underlying stochastic process

The $MRSPN$ class contains transitions with zero firing times, exponentially distributed or generally distributed firing times. Choi et al. (1994) have shown that, under the restriction the general transitions are mutually exclusive, the underlying stochastic process $M(t)_{t\geq 0}$ of this class is a Markov regenerative process. This class noted as the $MRSPN^*$ class. The process $M(t)_{t\geq 0}$ has a sequence of embedded time points $t_0, t_1, ...$ such that the states $Y_0, Y_1, ...$ of the process at these time points satisfy the Markov property. These time points are indicated by the Markov regeneration instants (regenerative time points, RTP) (see Cinlar, 1975; Ciardo et al., 1993). Note that the stochastic process $(Y_n)_{n\geq 0}$ is an EMC.

2.2 Quantitative analysis of $MRSPN$ using Markov renewal method

The steady state solution of the $MRSPN$, based on the Markov renewal method, can be summarised in the following steps:

1 Constructing the two kernels: global kernel $K(t) = [K_{ij}(t)]$ and local kernel $E(t) = [E_{ij}(t)]$, where:

 a $K_{ij}(t) = \mathbb{P}(Y_1 = j, T_1 \leq t | Y_0 = i), i, j \in \Omega$, is the probability that the system transits from state i to state j upon firing of a general transition, or upon firing of an exponential transition which is computed with a general transition.

 b $E_{ij}(t) = \mathbb{P}(M(t) = j, T_1 > t | Y_0 = i), i, j \in \Omega' \subset \Omega$, is the probability that the system transits from state i to state j while the general transition and the exponential that is competing with a general transition do not firing. The matrix $E(t)$ describes the behaviour of the marking process between two regeneration instants.

2 Obtaining the one step transition probability matrix $P = [P_{ij}] = K(\infty)$ of the EMC defined at RTP, $\{t_n, n \geq 1\}$, and the matrix $\alpha = [\alpha_{ij}]$. Here the elements α_{ij} represent the mean time that the $MRSPN$ spends in state j between two

successive regeneration instants, given that it started in state i after the last regeneration. They are given by:

$$\alpha_{ij} = \int_0^\infty E_{ij}(t)dt. \tag{1}$$

3 Computing υ the steady state probability vector of the EMC by solving the linear system:

$$\begin{cases} \upsilon P = \upsilon, \\ \upsilon \mathbb{1} = 1. \end{cases} \tag{2}$$

where $\mathbb{1}$ is a column vector of ones.

4 Computing the steady state probability distribution vector π of the $MRSPN$, by the formula:

$$\pi = \frac{1}{\upsilon \alpha \mathbb{1}} \upsilon \alpha. \tag{3}$$

3 M/G/1 retrial queue with vacation and timeout

3.1 System description

We consider an $M/G/1$ retrial queueing system with vacation and server timeout. The primary customers arrive according to a Poisson process with mean arrival rate λ. An arriving customer first checks whether the server is idle or on vacation. Therefore, if the server is idle or is not on vacation, the customer obtains service immediately. Otherwise, the customer joins a virtual room called orbit (which is of infinite capacity) for retrial. The secondary customers (customers in the orbit) try independently of each other for service and keep making retrials until they obtain service. The intervals between two successive repeated attempts are exponentially distributed with rate $j\gamma$ (when there are j customers in the orbit). When the system is empty (the sever is idle and the orbit is empty), the server waits for fixed time c (is called server timeout). At the expiration of this time, the server commences another vacation if no customer has arrived; otherwise, it serves exhaustively before commencing another vacation (the server immediately takes a vacation when the system becomes empty). We assume that the vacation time is arbitrarily distributed with cumulative distribution function $F_1(t)$ (probability density function $f_1(t)$). The customer is served according to a general cumulative distribution function $F_2(t)$ (probability density function $f_2(t)$). We assume that all the random variables defined above are independent and both the vacation time and the service time distribution have finite first two moments: $\beta_k^{(1)}$, $\beta_k^{(2)}$, $k = 1, 2$. We also denote $\varphi_i(s) = \int_0^\infty e^{-st} dF_i(t)$, $i = 1, 2$ the Laplace-Stieltjes transform.

3.2 MRSPN model suggested

The $MRSPN$ that describes this systems is depicted in Figure 1. Our $MRSPN$ model contains six places (noted by cercles) and eight transitions (noted by rectangular boxes and thin bars), where:

- the place $p.check$ contains the primary or repeated calls ready for service
- the place $p.orbit$ represents the orbit
- the place $p.service$ contains the customer in service
- the place $p.state$ is the state of the server (free, occupied or vacation)
- the place $p.counter$ is used as a counter
- the place $p.vacation$ represents that the server is on vacation
- the white rectangular boxes represent the exponential transitions ($t.arrival$ and $t.retrial$)
- the black rectangular boxes represent the general transitions ($t.service$ and $t.startup$)
- the thin bars represent the immediate transitions ($t.access1$, $t.access2$ and $t.access3$)
- the dash box represents the deterministic transition ($t.timeout$).

4 Analysis of the steady state distribution

4.1 The closed form expression for the two matrices P and α

The markings (micro-states) of our $MRSPN$ are given by:

$$M_i = (\#p.service, \#p.orbit, \#p.vacation).$$

The process $M(t)_{t\geq 0}$ underlying our $MRSPN$ is a Markov regenerative process with state space:

$$\Omega = \{(i, j, k)/i \in \{0, 1\}, j \in \mathbb{N}, k \in \{0, 1\}\}.$$

We consider the sequence of epochs $\{t_n; n \geq 0\}$ at which the process is observed. The RTP of our model are defined as follows:

- If, at the n^{th} RTP T_n, the system is in the state $(0, j, 1)$ with $j \geq 0$, the T_{n+1} is the firing of the transition $t.startup$.
- If, at the n^{th} RTP T_n, the system is in the state $(0, 0, 0)$, the T_{n+1} is the firing of the transition $t.timeout$ or the firing of the transition $t.arrival$.
- If, at the n^{th} RTP T_n, the system is in the state $(0, j, 0)$ with $j \geq 1$, the T_{n+1} is the firing of the transition $t.retrial$ or $t.arrival$.
- If, at the n^{th} RTP T_n, the system is in the state $(1, j, 0)$ with $j \geq 1$, the T_{n+1} is the firing of the transition $t.service$.

So, the sequence of random variables $(Y_n)_{n\geq 0}$ forms an EMC with state space:

$$\Omega' = \{(0, 0, 1), (i, j, 0) : i \in \{0, 1\} \text{ and } j \in \mathbb{N}\}.$$

The one step transition probability matrix $P = [P_{(i,j,k)(l,m,n)}]$ of this EMC is given by:

$$P_{(i,j,k)(l,m,n)} = \begin{cases} a = e^{-\lambda c} & \text{if } i = j = k = 0 \\ & \text{and } l = m = 0, n = 1; \\ b = 1 - e^{-\lambda c} & \text{if } i = j = k = 0 \\ & \text{and } l = 1, m = n = 0; \\ a_l^{(1)} = \int_0^\infty \frac{(\lambda x)^l}{l!} dF_1(x) & \text{if } i = j = 0, k = 1, \\ & \text{and } l = n = 0, m \geq 0; \\ a_0^{(2)} = \int_0^\infty e^{-\lambda x} dF_2(x) & \text{if } i = 1, j = k = 0, \\ & \text{and } l = m = 0, n = 1; \\ a_l^{(2)} = \int_0^\infty \frac{(\lambda x)^l e^{-\lambda x}}{l!} dF_2(x) & \text{if } i = 1, j = k = 0, \\ & \text{and } l = n = 0, m \geq 1; \\ a_l^{(2)} = \int_0^\infty \frac{(\lambda x)^l e^{-\lambda x}}{l!} dF_2(x) & \text{if } i = 1, j \geq 1, k = 0, \\ & \text{and } l = 0, m \geq j, n = 0; \\ c_{l+1}^{(1)} = \frac{(l+1)\gamma}{\lambda + (l+1)\gamma} & \text{if } i = k = 0, j \geq 1, \\ & \text{and } l = 1, m = j - 1, n = 0; \\ c_l^{(2)} = \frac{\lambda}{\lambda + l\gamma} & \text{if } i = k = 0, j \geq 1, \\ & \text{and } l = 1, m = j, n = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The matrix $\alpha = [\alpha_(i,j,k)(l,m,n)]$ is given by:

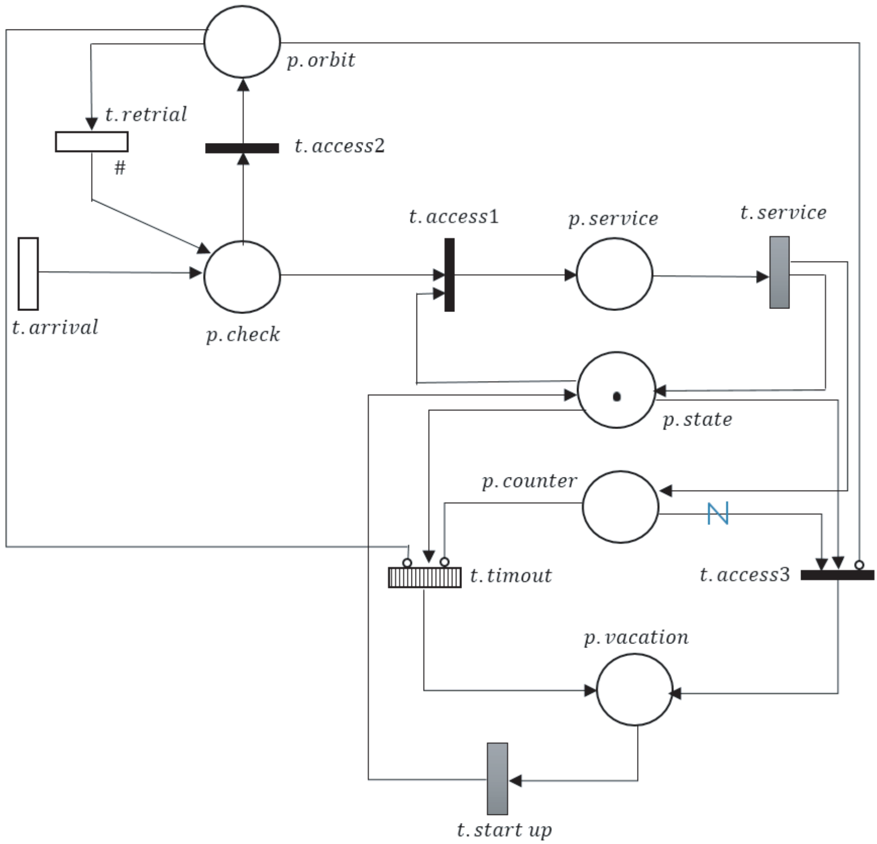
$$\alpha_{(i,j,k)(l,m,n)} = \begin{cases} q_0 = \frac{1}{\lambda}(1 - e^{-\lambda c}) & \text{if } i = j = k = 0 \\ & \text{and } l = m = n = 0; \\ \alpha_{(l)}^{(1)} = \int_0^\infty \frac{(\lambda x)^l e^{-\lambda x}}{l!} [1 & \text{if } i = j = 0, k = 1, \\ -F_1(x)]dx & \text{and } l = 0, m \geq j, n = 1; \\ \alpha_l^{(2)} = \int_0^\infty \frac{(\lambda x)^l e^{-\lambda x}}{l!} [1 & \text{if } i = 1, j \geq 0, k = 0, \\ -F_2(x)]dx & \text{and } l = 1, m \geq j, n = 0; \\ q_l = \frac{1}{\lambda + l\gamma} & \text{if } i = j = 0, j \geq 1, k = 0, \\ & \text{and } l = 0, m = j, n = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

4.2 Stability condition

The first point to be investigated is the ergodicity of the EMC.

Theorem 1: The EMC Y_n is ergodic if and only if: $\lambda\beta_2^{(1)} < 1$.

Figure 1 Unbounded $MRSPN$ models the $M/G/1$ retrial queue with vacation and timeout (see online version for colours)



Proof 1: It can be shown from the results in Neuts (1989) that the steady state probability vector exists and is positive if the EMC is ergodic. Using the lexicographical sequence for the states, the one step transition probability matrix P of our EMC can be written as the following block matrix:

$$P = \begin{pmatrix} W_0 & W_1 & W_2 & W_3 & W_4 & W_5 & \cdots \\ A_0 & C_1 & D_1 & A_3 & A_4 & A_5 & \cdots \\ & A_0 & C_2 & D_2 & A_3 & A_4 & \cdots \\ & & A_0 & C_3 & D_3 & A_3 & \cdots \\ & & & \ddots & \ddots & \ddots & \end{pmatrix},$$

where

$$W_0 = \begin{pmatrix} 0 & a \\ a_0^{(1)} & 0 \end{pmatrix}, \; W_1 = \begin{pmatrix} b & 0 \\ 0 & a_1^{(1)} \end{pmatrix}, \; W_i = \begin{pmatrix} 0 & 0 \\ 0 & a_i^{(2)} \end{pmatrix}_{i \geq 2}, \; A_0 = \begin{pmatrix} 0 & a_0^{(2)} \\ 0 & 0 \end{pmatrix},$$

$$C_i = \begin{pmatrix} 0 & a_0^{(2)} \\ c_i^{(1)} & 0 \end{pmatrix}_{i \geq 1}, \; D_i = \begin{pmatrix} 0 & a_2^{(2)} \\ c_i^{(2)} & 0 \end{pmatrix}_{i \geq 1} \text{ and } A_i = \begin{pmatrix} 0 & a_i^{(2)} \\ 0 & 0 \end{pmatrix}_{i \geq 3}.$$

We have: $A = A_0 + C_i + D_i + \sum_{i=3}^{\infty} A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the invariant probability vector of matrix A is $\tau = (1/2, 1/2)$. The vector β is defined by $\beta = C_i\mathbb{1} + 2D_i\mathbb{1} + \sum_{i=3}^{\infty} A_i\mathbb{1}$, explicitly is given by $\beta = (2 - c_i^{(1)}, \lambda\beta_2^{(1)})$. The EMC Y_n is positive recurrent if and only if: $\tau\beta < 1 \Leftrightarrow \lambda\beta_2^{(1)} < \lim_{i\to\infty} \frac{i\gamma}{\lambda + i\gamma}$.

4.3 Generating functions of the server's state and system size at the RTP

The second point to be investigated is to compute the generating functions of the server's state and system size at the regeneration instants.

Theorem 2: Under the stability condition, the generating functions of the server's state and system size at the regeneration instants are given by:

$$\begin{aligned} V_0(z) &= \lambda S(z)\frac{A_2(z)(z-1)}{z - A_2(z)} + av_{(0,0)}\frac{z\big[A_1(z) - A_2(z)\big]}{z - A_2(z)} \\ &\quad + \varphi_2(\lambda)v_{(1,0)}\frac{z\big[A_1(z) - 1\big]}{z - A_2(z)}, \end{aligned} \tag{6}$$

$$\begin{aligned} V_1(z) &= z\Bigg[\lambda S(z)\frac{z-1}{z - A_2(z)} + av_{(0,0)}\frac{\big[A_1(z) - z\big]}{z - A_2(z)} \\ &\quad + \varphi_2(\lambda)v_{(1,0)}\frac{\big[A_1(z) - 1\big]}{z - A_2(z)}\Bigg], \end{aligned} \tag{7}$$

$$\begin{aligned} V(z) &= \lambda S(z)\frac{(z-1)\big[A_2(z) + z\big]}{z - A_2(z)} + av_{(0,0)}\left[1 + \frac{z[2A_1(z) - A_2(z) - z]}{z - A_2(z)}\right] \\ &\quad + \varphi_2(\lambda)v_{(1,0)}\left[1 + \frac{2z[A_1(z) - 1]}{z - A_2(z)}\right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} S(z) &= e^{\frac{\lambda}{\gamma}\int_1^z \frac{A_2(u)-1}{u - A_2(u)}du}\Bigg[S(1) + \int_1^z \left(\frac{av_{(0,0)}}{\gamma}\frac{A_1(t) - A_2(t)}{t - A_2(t)}\right. \\ &\quad \left. + \frac{\varphi_2(\lambda)v_{(1,0)}}{\gamma}\frac{A_1(t) - 1}{t - A_2(t)}\right) \times e^{\frac{-\lambda}{\gamma}\int_1^t \frac{A_2(u)-1}{u - A_2(u)}du} dt\Bigg], \end{aligned} \tag{9}$$

$$\begin{aligned} S(1) &= \frac{1 - \lambda\beta_2^{(1)}}{2\lambda} - \frac{2a\lambda(\beta_1^{(1)} - \beta_2^{(1)})}{2\lambda}v_{(0,0)} \\ &\quad - \frac{\varphi_2(\lambda)(2\lambda\beta_1^{(1)} - \lambda\beta_2^{(1)} + 1)}{2\lambda}v_{(1,0)}, \end{aligned} \tag{10}$$

$$v_{(0,0)} = \frac{1-\lambda\beta_2^{(1)}}{2}\left[e^{\frac{\lambda}{\gamma}\int_0^1 \frac{A_2(u)-1}{u-A_2(u)}du}\right.$$
$$+\frac{(2\lambda\beta_1^{(1)}-\lambda\beta_2^{(1)}+1)-a\varphi_1(\lambda)(1+\lambda\beta_2^{(1)})}{2\varphi_1(\lambda)}$$
$$+\frac{\lambda a}{\gamma}\int_0^1 \frac{A_1(t)-A_2(t)}{t-A_2(t)}e^{\frac{\lambda}{\gamma}\int_t^1 \frac{A_2(u)-1}{u-A_2(u)}du}dt$$
$$\left.+\frac{\lambda}{\gamma}\frac{1-a\varphi_1(\lambda)}{\varphi_1(\lambda)}\int_0^1 \frac{A_1(t)-1}{t-A_2(t)}e^{\frac{\lambda}{\gamma}\int_t^1 \frac{A_2(u)-1}{u-A_2(u)}du}dt\right]^{-1}, \tag{11}$$

$$v_{(1,0)} = \frac{1-a\varphi_1(\lambda)}{\varphi_2(\lambda)\varphi_1(\lambda)}v_{(0,0)}, \tag{12}$$

and $A_k(z) = \varphi_k(\lambda-\lambda z)$, $k \in \{1,2\}$.

Proof 2: From the formula (2) and by introducing the following generating functions:

$$A_k(z) = \sum_{i=0}^{\infty} z^i a_i^k, k \in \{1,2\},\ V_k(z) = \sum_{i=0}^{\infty} z^i v_{(k,i)}, k \in \{0,1\},$$

$$S(z) = \sum_{i=0}^{\infty} z^i v_{(0,i)}\frac{1}{\lambda+i\gamma},\ V(z) = \sum_{i=0}^{\infty} z^i\big[v_{(0,i)}+v_{(1,i)}+v_{(2,i)}\big],$$

we obtain:

$$V_1(z) = z\Big(\lambda S(z)+\gamma S'(z)-av_{(0,0)}\Big), \tag{13}$$

$$V_0(z) = \frac{A_2(z)V_1(z)}{z}+A_1(z)av_{(0,0)}+\big[A_1(z)-1\big]\varphi_2(\lambda)v_{(1,0)}, \tag{14}$$

and

$$V(z) = V_0(z)+V_1(z)+av_{(0,0)}+\varphi_2(\lambda)v_{(1,0)}. \tag{15}$$

On the other hand, we have the formula:

$$V_0(z) = \lambda S(z)+\gamma zS'(z). \tag{16}$$

By combining equations (13), (14) and (16) we obtain the following equation for the generating function $S(z)$:

$$S'(z) = \frac{\lambda}{\gamma}\frac{A_2(z)-1}{z-A_2(z)}S(z)+\frac{av_{(0,0)}}{\gamma}\frac{A_1(z)-A_2(z)}{z-A_2(z)}$$
$$+\frac{\varphi_1(\lambda)v_{(1,0)}}{\gamma}\frac{A_1(z)-1}{z-A_2(z)}. \tag{17}$$

By inserting the formula (17) into equations (13), (14) and (15) we obtain the three equations (6), (7) and (8).

The equation (17) is a first order differential equation for $S(z)$ with non-constant coefficients. Thus, it can be solved by standard methods which lead to the formula (9). With the help of the Hôpital's rule [in equation (17)] and the initial condition $V(1) = 1$, we obtain:

$$S'(1) = \frac{\lambda}{\gamma}\frac{\lambda\beta_2^{(1)}}{1-\lambda\beta_2^{(1)}}S(1) + \frac{av_{(0,0)}}{\gamma}\frac{\lambda(\beta_1^{(1)}-\beta_2^{(1)})}{1-\lambda\beta_2^{(1)}} + \frac{\varphi_1(\lambda)v_{(1,0)}}{\gamma}\frac{\lambda\beta_1^{(1)}}{1-\lambda\beta_2^{(1)}}, \tag{18}$$

and $S(1)$ given in the equation (10).

The two equations (11) and (12) are obtained by computing $V_0(0) = v_{(0,0)}$ and $V_1'(0) = v_{(1,0)}$.

4.4 *Generating functions of the server's state and system size at arbitrary instants*

Finally, the third point to be investigated is to compute the generating functions of the server's state and system size at arbitrary instants.

Theorem 3: For our $MRSPN$ model, we obtain the generating functions of the server's state and system size at the arbitrary instants:

$$\pi_0(z) = \frac{2\lambda}{1+2(\lambda q_0-1)v_{(0,0)}-\varphi_2(\lambda)v_{(1,0)}}\left[\left(q_0-\frac{1}{\lambda}\right)v_{(0,0)}+S(z)\right], \tag{19}$$

$$\pi_1(z) = \frac{2\lambda}{1+2(\lambda q_0-1)v_{(0,0)}-\varphi_2(\lambda)v_{(1,0)}}\left\{z\left[S(z)\frac{[A_2(z)-1]}{z-A_2(z)} + \frac{av_{(0,0)}}{\lambda}\frac{[A_1(z)-z][A_2(z)-1]}{[z-A_2(z)](z-1)} + \frac{\varphi_2(\lambda)v_{(1,0)}}{\lambda}\frac{[A_1(z)-1][A_2(z)-1]}{[z-A_2(z)](z-1)}\right]\right\}, \tag{20}$$

$$\pi_2(z) = \frac{2\lambda}{1+2(\lambda q_0-1)v_{(0,0)}-\varphi_2(\lambda)v_{(1,0)}}\frac{[1-A_1(z)](av_{(0,0)}+\varphi_2(\lambda)v_{(1,0)})}{\lambda(1-z)}. \tag{21}$$

$$\pi(z) = \frac{2\lambda}{1+[2(\lambda q_0-1)-\frac{1-\varphi_1(\lambda)a}{\varphi_1(\lambda)}]v_{(0,0)}}\left\{S(z)\frac{A_2(z)[z-1]}{[z-A_2(z)]} + \frac{v_{(0,0)}}{\lambda}\left[\lambda q_0 - 1 + \frac{1}{\varphi_1(\lambda)}\frac{A_2(z)[A_1(z)-1]}{z-A_2(z)} + a\frac{z[1-A_2(z)]}{z-A_2(z)}\right]\right\}. \tag{22}$$

Proof 3: We have: $\alpha\mathbb{1} = \left(q_0, \beta_1^{(1)}, \beta_2^{(1)}, \frac{1}{\lambda+\gamma}, \beta_2^{(1)}, \frac{1}{\lambda+2\gamma}, ...\right)^T$, thus: $\upsilon\alpha\mathbb{1} = \frac{1+2(\lambda q_0 - 1)v_{(0,0)} - \varphi_1(\lambda)v_{(1,0)}}{2\lambda}$.

By introducing the probability generating functions:

$$B_k(z) = \sum_{i=0}^{\infty} z^i \alpha_i^k, k \in \{1,2\},\ \pi_k(z) = \sum_{i=0}^{\infty} z^i \pi_{k,i}, k \in \{0,1,2\}$$

to the formula (3), we obtain:

$$\pi_0(z) = \frac{2\lambda\left[\left(q_0 - \frac{1}{\lambda}\right)v_{(0,0)} + S(z)\right]}{1+2(\lambda q_0 - 1)v_{(0,0)} - \varphi_1(\lambda)v_{(1,0)}}, \tag{23}$$

$$\pi_1(z) = \frac{2\lambda\left[B_2(z)V_1(z)\right]}{1+2(\lambda q_0 - 1)v_{(0,0)} - \varphi_1(\lambda)v_{(1,0)}}, \tag{24}$$

$$\pi_2(z) = \frac{2\lambda\left[B_1(z)\big(av_{(0,0)} + \varphi_2(\lambda)v_{(1,0)}\big)\right]}{1+2(\lambda q_0 - 1)v_{(0,0)} - \varphi_1(\lambda)v_{(1,0)}}, \tag{25}$$

and

$$\pi(z) = \frac{2\lambda\left[\left(q_0 - \frac{1}{\lambda}\right)v_{(0,0)} + S(z) + B_1(z)\big(av_{(0,0)} + \varphi_2(\lambda)v_{(1,0)}\big) + B_2(z)V_1(z)\right]}{1+2(\lambda q_0 - 1)v_{(0,0)} - \varphi_1(\lambda)v_{(1,0)}}. \tag{26}$$

We replace $V_1(z)$, $B_1(z) = \frac{1-A_1(z)}{\lambda - \lambda z}$ and $B_2(z) = \frac{1-A_2(z)}{\lambda - \lambda z}$ in equations (24), (25) and (26) we found the formulas (20), (21), (22) and (23).

5 Performance measures

In this section, we obtain some performance measures of the system considered.

- The mean number of customers in the system ($n_s = \pi'(1)$):

$$\begin{aligned} n_s = {} & \frac{2\lambda}{1 + \left[2(\lambda q_0 - 1) - \frac{1-\varphi_1(\lambda)a}{\varphi_1(\lambda)}\right]v_{(0,0)}}\left[\frac{s'(1)}{1-\lambda\beta_2^{(1)}}\right. \\ & + \frac{2\lambda\beta_2^{(1)}(1-\lambda\beta_2^{(1)}) + \lambda^2\beta_2^{(2)}}{2(\lambda\beta_2^{(1)} - 1)^2}s(1) \\ & + \left(\frac{\lambda^3\beta_2^{(2)}\beta_1^{(1)} + \lambda^2\beta_1^{(2)}(1-\lambda\beta_2^{(1)}) + 2\lambda^2\beta_1^{(1)}\beta_2^{(1)}(1-\lambda\beta_2^{(1)})}{2\varphi_1(\lambda)(\lambda\beta_2^{(1)} - 1)^2}\right. \\ & \left.\left. - \frac{a(2\lambda\beta_2^{(1)}(1-\lambda\beta_2^{(1)}) + \lambda^2\beta_{(2)}^2)}{2(\lambda\beta_2^{(1)} - 1)^2}\right)\frac{v_{(0,0)}}{\lambda}\right]. \end{aligned} \tag{27}$$

- The probability that server is under vacation (p_v):

$$p_v = \pi_2(1) = \frac{2\lambda}{1 + \left[2(\lambda q_0 - 1) - \dfrac{1 - \varphi_1(\lambda)a}{\varphi_1(\lambda)}\right] v_{(0,0)}} \frac{\beta_1^{(1)} v_{(0,0)}}{\varphi_1(\lambda)}. \tag{28}$$

- The mean number of customers in the orbit (n_o):

$$n_o = n_s - \lambda \beta_2^{(1)}. \tag{29}$$

- The mean waiting time ($\overline{\omega}$), from Little's law:

$$\overline{\omega} = \frac{n_o}{\lambda}. \tag{30}$$

- The mean response time (ϖ), from Little's law:

$$\varpi = \frac{n_s}{\lambda}. \tag{31}$$

- The overall rate of retrials at which the orbiting customers request service is given as:

$$\bar{\gamma} = \gamma n_o. \tag{32}$$

Remark 1: The model considered has been analysed with deterministic timeout c, the analysis can be extended to the case where the timeout is a random variable with distribution $H(t)$. In this case only a slight modification is required in the results. Specifically, we replace the three factors a, b, q_0 by: $a = \int_0^\infty e^{-\lambda x} dH(x)$, $b = \int_0^\infty [1 - e^{-\lambda x}] dH(x)$ and $q_0 = \int_0^\infty e^{-\lambda x}[1 - H(x)]dx$.

6 Numerical results of the performance measures

We use the performance measures derived previously to obtain numerical results. Specifically, we present some tables and curves which illustrate the effect of the values of the system parameters such us: timeout, arrival rate, retrial rate, vacation time distribution and service time distribution on the system performance measures. Under the stability condition, the computations are done by developing a program in MATLAB software.

6.1 Interpretation of the results

From Tables 1–3, we observe that all the considered characteristics monotonically decrease as the timeout c increases for all three types of service time distributions and vacation time distributions. Furthermore, at high values of c, these characteristics become constants.

Figure 2 The effect of the arrival rate λ on the mean response time ϖ for different values of retrial rate γ; 'c = 0.8, λ = 1, ..., 7, $f_1(t)$ = two-stage hypoexponential(5, 7), $f_2(t)$ = two-stage Erlang(9)' (see online version for colours)

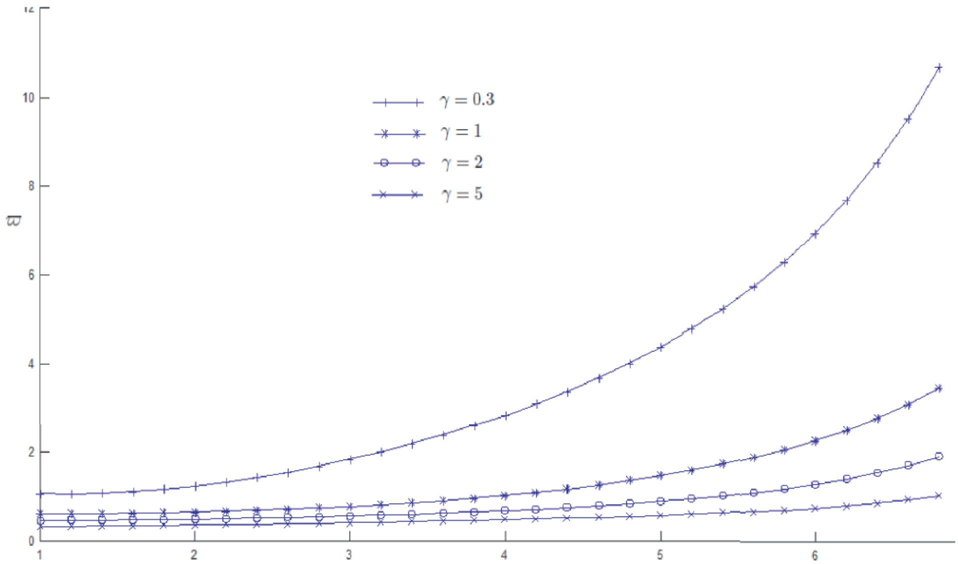
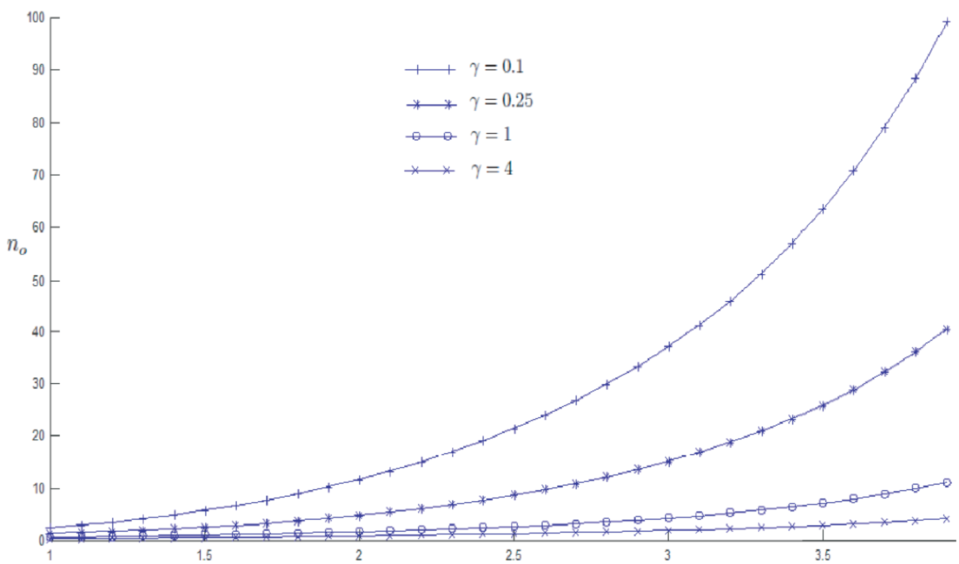


Figure 3 The effect of arrival rate λ on the mean number of customers in the orbite n_o for different values of retrial rate γ; 'c = 2, λ = 1, ..., 4, $f_1(t)$ = two-stage hyperexponential(0.8, 2, 6), $f_2(t)$ = two-stage hypoexponential(12, 10)' (see online version for colours)



We show in Figure 2, the influence of the parameters arrival rate λ and retrial rate γ on the mean response time ϖ. We notice that ϖ increases for decreasing values of λ and decreases when γ increases. The arrival rate λ has a significant effect on ϖ, especially when γ is small.

Table 1 The effect of the values of c on system performance measures, λ = 0.6, $f_1(t)$ = exponential(2), $f_2(t)$ = exponential(4), γ = 0.4

	c = 0	*c = 2*	*c = 5*	*c = 10*	*c = 100*
n_s	1.1021	0.8005	0.7498	0.7418	0.7414
p_v	0.3121	0.1697	0.1457	0.1420	0.1418
n_o	0.9521	0.6505	0.5998	0.5918	0.5914
$\overline{\omega}$	1.5868	1.0841	0.9996	0.9863	0.9857
ϖ	1.8368	1.3341	1.2496	1.2363	1.2357
$\bar{\gamma}$	0.3808	0.2602	0.2399	0.2367	0.2366

Table 2 The effect of the values of c on system performance measures, λ = 1.5, $f_1(t)$ = two-stage hypoexponential (12, 9), $f_2(t)$ = two-stage hyperexponential (1/3, 2.5, 5), γ = 0.5

	c = 0	*c = 2*	*c = 5*	*c = 10*	*c = 100*
n_s	3.0027	2.8479	2.8441	2.8441	2.8441
p_v	0.0564	0.0276	0.0269	0.0268	0.0268
n_o	2.6027	2.4479	2.4441	2.4441	2.4441
$\overline{\omega}$	1.7351	1.6320	1.6294	1.6294	1.6294
ϖ	2.0018	1.8961	1.8960	1.8960	1.8960
$\bar{\gamma}$	1.3013	1.2240	1.2220	1.2220	1.2220

Table 3 The effect of the values of c on system performance measures, 'λ = 0.5, $f_1(t)$ = two-stage Erlang(6), $f_2(t)$ = exponential(5), γ = 0.25'

	c = 0	*c = 2*	*c = 5*	*c = 10*	*c = 100*
n_s	0.9420	0.5049	0.4629	0.4550	0.4544
p_v	0.2656	0.0749	0.0565	0.0531	0.0528
n_o	0.8420	0.4049	0.3629	0.3550	0.3544
$\overline{\omega}$	1.6841	0.8099	0.7100	0.7087	0.7087
ϖ	1.8841	1.0099	0.9258	0.9100	0.9087
$\bar{\gamma}$	0.2105	0.1012	0.0907	0.0886	0.0886

In Figure 3, we provide the mean number of customers in the orbit n_o with a change of arrival rate λ and retrial rate γ. We see that n_o grows rapidly for increasing values of λ and decreases with increasing values of γ.

In Figure 4, we plot the curve of the probability that the server is on vacation, p_v, versus arrival rate λ and vacation rate α. We see that, p_v is a decreasing function of both the arrival rate λ and retrial rate γ. Furthermore we observe that all curves tend to approach each other for large value of the arrival rate λ.

In Figure 5, we study the impact of the mean number of customers in the system n_s for various values of the service rate μ and the arrival rate λ. We observe that the mean number of customers in the system n_s decreases with increasing values of μ and increases with increasing values of λ.

Figure 4 The effect of arrival rate λ on the probability that server is under vacation p_v for different values of vacation rate α; 'c = 1, λ = 1.5, ..., 3, $f_1(t)$ = two-stage Erlang(α), $f_2(t)$ = exponential(8), γ = 0.4' (see online version for colours)

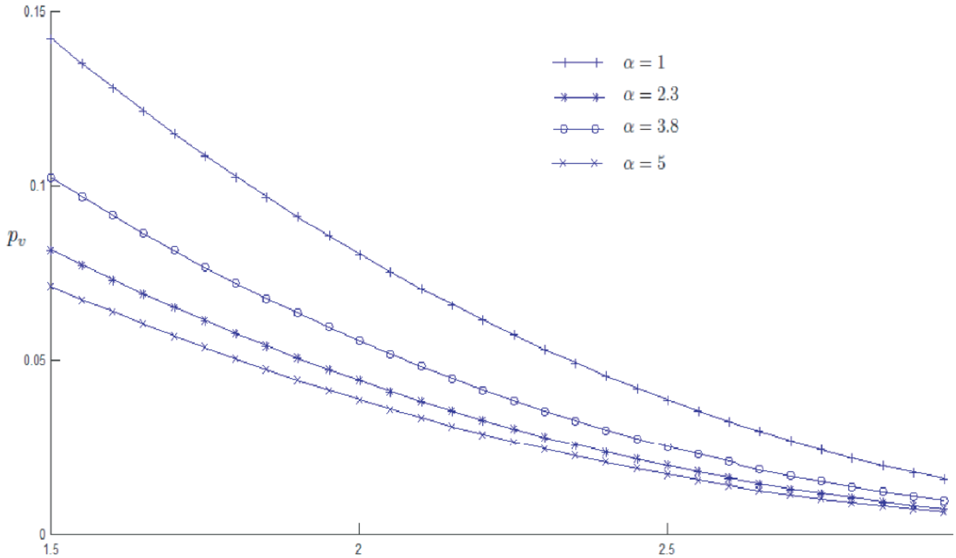
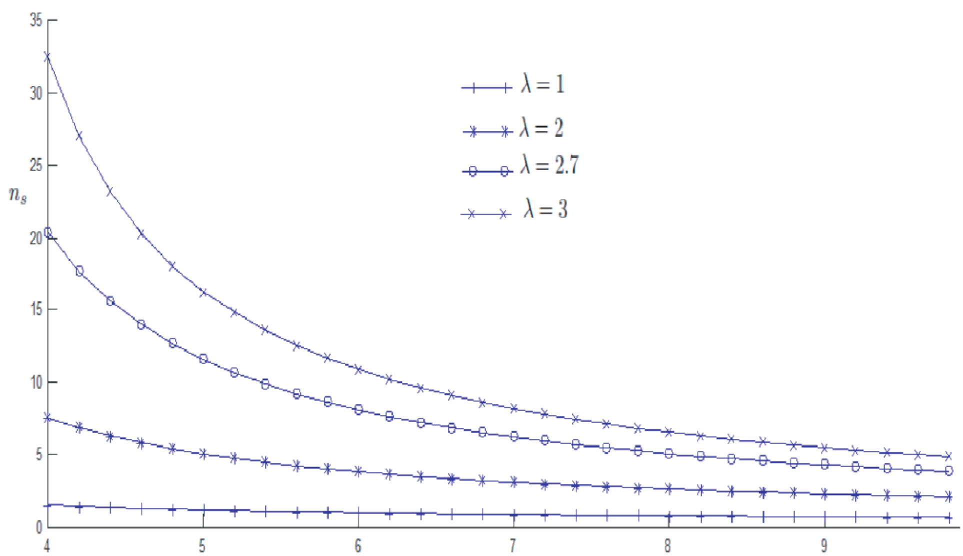


Figure 5 The effect of service rate μ on the mean number of customers in the system for different values of arrival rate λ; 'c = 1.6, $f_2(t)$ = two-stage Erlang(μ), μ = 4, ..., 10, $f_1(t)$ = two-stage hyperexponential(0.2, 8, 11), γ = 0.3' (see online version for colours)



7 Conclusions

We studied the non markovien retrial queue $M/G/1$ with vacation and server timeout by the $MRSPN$ tool. We have obtained the steady state probability generating function of the server's state and system size. Moreover, the performance measures are computed. The effects of the various parameters on the system performances are given numerically. For a future study, we will focus on the cost optimisation analysis and the transient analysis of the model considered. We can also extend this model to multi-server retrial queue.

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