Research Article

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Asymmetric kernel method in the study of strong stability of the PH/M/1 queuing system

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Abstract: This paper proposes the nonparametric asymmetric kernel method in the study of strong stability of the PH/M/1 queuing system, after perturbation of arrival distribution to evaluate the proximity of the complex GI/M/1 system, where GI is a unknown general distribution. The class of generalized gamma (GG) kernels is considered because of its several interesting properties and flexibility. A simulation for several models illustrates the performance of the GG asymmetric kernel estimators in the study of strong stability of the PH/M/1, by computing the variation distance and the stability error.

Keywords: Asymmetric GG kernels, bandwidth parameter, GI/M/1 system, PH/M/1 queuing system, strong stability

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1 Introduction

The study of the performance of complex systems by queuing models is a research field with increasing interest. Indeed, a large class of mathematical models comes from queuing theory (computer, communication, production and manufacturing, etc.). However, various problems may be encountered during the design as well as the analysis of complex systems. The existing analytical methods are limited, and known results for many types of queues are approximate or complicated. Indeed, in most cases, we find ourselves facing systems of equations whose solutions are very difficult or which have solutions that are not easily interpretable. Therefore, one often uses approximation methods which consist in replacing the complex system by a simpler one, close to it in some sense and analytically exploitable. For example, due to the complexity of queuing systems with a general distributions, several studies have proposed the use of phase type distributions to approximate general distributions in simple queues as well as queuing networks models (see, e.g., [9, 13]).

The approximation of a non-negative distribution by a combination of exponential distributions (known as phase type (PH) distributions, e.g. Erlang, hyper-exponential, hypo-exponential, Cox, …) is desirable because this class of probability distributions allows to model a large number of random events while keeping a Markovian character, and it helps to describe the actual times in a more complex way than can be described by the exponential distribution [2, 17, 20].

The approximation of a non-negative continuous distribution by a phase type distribution given a certain precision level is possible because the class of PH distributions is dense in the set of non-negative distributions [2]. This approximation is necessary because real system are generally very complicated, so their analysis cannot lead to analytical results or it leads to complicated results which are not useful in practice.

The strong stability method (see [3]) is a powerful tool for the study of the sensitivity of stochastic models that can be described by a homogeneous Markov chain.The theory of this method is well developed and

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allows the derivation of perturbation bounds in addition to the qualitative affirmation of the robustness of the considered model. A Markov chain is said to be strongly stable when small perturbations in the inputs (transition matrix) can lead to at most a bounded deviation of the outputs (stationary vector). Under this condition, approximations (such as the use of PH distributions instead of general ones) and parameter estimation errors result in a controlled deviation in the characteristics of the system in the sense that an upper bound of this deviation can be estimated. The strong stability method has been applied to various stochastic models (queuing models [1, 7, 8, 19], queuing models with PH distribution [10], inventory models [18], risk models [6], ...). Note that, in practice, all model parameters are imprecisely known because they are obtained by means of statistical methods. In this sense, one aspect which is of interest is when a distribution governing a queuing system is unknown and we resort to nonparametric methods to estimate its density function; see for example [12, 14, 21] for asymmetric kernels families.

In this paper, we use nonparametric estimation to approximate the complex GI/M/1 system by the simpler PH/M/1 one, when the general law of arrivals GI is unknown, so its density function must be estimated by using the nonparametric kernel density method; see for example [4] on kernel density estimation in the study of the strong stability of the M/M/1 queuing system. Our analysis will focus on the stability of queuing models with PH distribution of two phases, where we propose the nonparametric asymmetric kernel method by using a class of generalized gamma (GG) kernels, because of its several interesting properties and flexibility; see [12, 14] for more details. Besides evaluating the proximity error between the corresponding service time distributions of the GI/M/1 and PH/M/1 systems and the approximation error on their stationary distributions, the norm of the deviation of the transition matrix is obtained. Our presented work is motivated by several points. First, the PH distributions are attractive in practice, in particular for queuing systems analysis. Second, the nonparametric kernel method does not impose any restriction on the unknown density GI to be estimated. As third motivation, our work can be considered as a complement to the existing literature on the study of the strong stability of queuing systems (see [4, 5, 19]).

The remainder of this paper is organized as follows. Section 2 proposes a brief recall on strong stability approach. Section 2 gives some results on strong stability of PH/M/1 queuing systems. The application of the generalized gamma kernels method in the study of strong stability of the PH/M/1 queuing system is presented in Section 3. Section 4 concludes.

2 Strong stability of PH/M/1 queuing systems

In this section, we study the strong stability of the embedded Markov chain of the queuing system PH/M/1 which is proposed as approximation to a GI/M/1 system. The perturbation considered here consists in replacing the inter-arrival distribution GI of the real system by a PH with two phases. As explained before, we aim to confirm the robustness of the PH/M/1 queuing model and then estimate the approximation error.

Recall that the density of a phase type random variable with parameters $(\vec{\tau}, T)$, denoted by PH$(\vec{\tau}, T)$ is

$$f(x) = \vec{\tau} e^{Tx} \vec{v}, \quad x \geq 0. \tag{2.1}$$

Let X be a random variable having a PH distribution with two phases. Further,

$$\vec{\tau} = (\tau_1, \tau_2), \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where $\vec{\tau}$ is the initial probability vector, T denotes the generator transient Markov chain and $\vec{v}$ is a column vector denoting the transitions between the transient states and absorbing states. We have

$$e^{Tx} = U \begin{pmatrix} e^{D_1 x} & 0 \\ 0 & e^{D_2 x} \end{pmatrix} U^{-1},$$

where D_1 and D_2 are eigenvalues of the matrix T,

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad U^{-1} = \begin{pmatrix} \grave{a} & \grave{b} \\ \grave{c} & \grave{d} \end{pmatrix}.$$

Then

$$e^{Tx} = \begin{pmatrix} a\grave{a}e^{D_1x} + b\grave{c}e^{D_2x} & a\grave{b}e^{D_1x} + b\grave{d}e^{D_2x} \\ c\grave{a}e^{D_1x} + d\grave{c}e^{D_2x} & c\grave{b}e^{D_1x} + d\grave{d}e^{D_2x} \end{pmatrix}.$$

From equation (2.1), we obtain

$$f(x) = Be^{D_1x} + Ce^{D_2x},$$

with

$$\begin{cases} B = \tau_1 v_1 a\grave{a} + \tau_2 v_1 c\grave{a} + \tau_1 v_2 a\grave{b} + \tau_2 v_2 c\grave{b}, \\ C = \tau_1 v_1 b\grave{c} + \tau_2 v_1 d\grave{c} + \tau_1 v_2 b\grave{d} + \tau_2 v_2 d\grave{d}. \end{cases}$$

2.1 Model's description

Denote by $\tilde{\Sigma}$ the queuing system of type GI/M/1 (FIFO, ∞). Inter-arrival times are independent and identically distributed random variables with common non-negative distribution having rate λ and coefficient of variability $ca < 1$. Let G be the common cumulative function of the successive inter-arrival times. Service times are independent and identically distributed exponential random variables with rate μ and are independent of the inter-arrival times. Assume that the load of the system $\rho = \frac{\lambda}{\mu}$ is less than 1.

The sequence $\tilde{X} = \{\tilde{X}_n,\ n = 0, 1, \ldots\}$ of random variables, where $\tilde{X}_n$ represents the number of customers in the system as seen by the n-th customer at his arrival time t_n, is a time-homogeneous Markov chain [11]. Denote by $\tilde{P}$ the transition matrix of $\tilde{X}$, where the transition probabilities $\tilde{P}_{ij} = \mathbf{P}(\tilde{X}_{n+1} = j | \tilde{X}_n = i)$ are given as follows:

$$\tilde{P}_{ij} = \begin{cases} \tilde{d}_{i+1-j} = \displaystyle\int_0^{+\infty} \frac{(\mu t)^{i+1-j}}{(i+1-j)!} e^{-\mu t}\, dG(t) & \text{if } 1 \le j \le i+1, \\ 1 - \displaystyle\sum_{k=0}^{i} \tilde{d}_k & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we denote by Σ the queuing system of type PH/M/1 (FIFO, ∞) that we propose as an approximation to $\tilde{\Sigma}$. The inter-arrival times are independent and identically distributed phase type random variables having a common cumulative function PH with parameters $(\vec{\tau}, T)$. Service durations are independent and identically distributed exponential variables with rate μ and are independent of inter-arrival times.

The sequence $X = \{X_n,\ n = 0, 1 \ldots\}$ of random variables representing the number of customers in the system (Σ) as seen by the n-th arriving customer is a time-homogeneous Markov chain with transition matrix denoted by P, where

$$P_{ij} = \begin{cases} d_{i+1-j} = \mu^{i+1-j}\left(\dfrac{B}{(\mu - D_1)^{i+2-j}} + \dfrac{C}{(\mu - D_2)^{i+2-j}}\right) & \text{if } 1 \le j \le i+1, \\ 1 - \displaystyle\sum_{k=0}^{i} d_k = -\frac{B}{D_1}\left(\frac{\mu}{\mu - D_1}\right)^{i+1} - \frac{C}{D_2}\left(\frac{\mu}{\mu - D_2}\right)^{i+1} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let use suppose that the distribution of the service time of the GI/M/1 system is close to PH(t) with two phases. This proximity is then characterized by the metric

$$\mathcal{W}(G, H) = \int_0^{\infty} |G - \mathrm{PH}_2|(dt), \tag{2.2}$$

The Markov chain X is irreducible and aperiodic. It is ergodic and admits a unique stationary probability vector π. The steady-state probabilities of the ergodic Markov chain X,

$$\pi_j = \lim_{n\to\infty} \mathbb{P}[X_n = j] \quad \text{for all } j \ge 0,$$

are the solution of the following system of equations:

$$\begin{cases} \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \\ \sum_{j=0}^{\infty} \pi_j = 1. \end{cases}$$

Thus, we obtain

$$\pi_j = (1-\sigma)\sigma^j, \quad \sigma = \frac{(\mu^2 - \mu(D_1+D_2)) - \sqrt{\Delta}}{2}$$

with

$$\Delta = (\mu^2 - \mu(D_1+D_2))^2 - 4\mu^2(D_1D_2 - \mu(B+C)).$$

2.2 Strong *v*-stability of the Markov chain *X*

The first result is given by the following theorem. It confirms to us the robustness of the underlying Markov chain and hence its ability to resist to perturbations.

Theorem 1. *The Markov chain $X = \{X_n,\ n = 0, 1 \ldots\}$, where X_n is the number of customers in the system as seen by the n-th arrival of customer, is v-strongly stable for the test function $v(k) = \beta^k$ for all β such that $1 < \beta < \beta_0$, where*

$$\beta_0 = \frac{(1 - R_1 - R_2) + \sqrt{\Delta_0}}{2(R_3 + R_4 + R_1R_2)},$$

with

$$\Delta_0 = (1 - R_1 - R_2)^2 - 4(R_3 + R_4 + R_1R_2)$$

and

$$R_1 = \frac{D_1}{\mu}, \quad R_2 = \frac{D_2}{\mu}, \quad R_3 = \frac{B}{\mu}, \quad R_4 = \frac{C}{\mu}.$$

Proof. To prove the strong v-stability of the embedded Markov chain X for the function $v(k) = \beta^k$, with $\beta > 1$, we check the conditions of the theorem given in [3, 15].

Therefore, we consider the measurable function $h(i) = P_{i0}$ and the measure

$$\alpha_j = \mathbf{1}_{j=0} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Obviously,

$$\pi h = \sum_{i\geq 0} \pi_i h_i = \sum_{i\geq 0} \pi_i P_{i0} > 0,$$
$$\alpha \mathbf{1} = \sum_{j\geq 0} \alpha_j = \alpha_0 + \sum_{j\geq 1} \alpha_j = \alpha_0 = 1,$$
$$\alpha h = \sum_{i\geq 0} \alpha_i h_i = \alpha_0 h_0 + \sum_{i\geq 1} \alpha_i h_i = h_0 = P_{00} > 0.$$

Moreover,

$$T_{ij} = P_{ij} - h_i \alpha_j = \begin{cases} 0 & \text{if } j = 0, \\ P_{ij} & \text{if } j > 0. \end{cases}$$

Hence, T is non-negative.

Presently, we aim to show that there exists a positive constant $\rho < 1$ such that $Tv(i) \leq \rho v(i)$ for all i. Indeed, we have

$$\begin{aligned} Tv(i) &= \sum_{j\geq 0} v(j) T_{ij} = v(0) T_{i0} + \sum_{j=1}^{i+1} \beta^j d_{i+1-j} \\ &= \sum_{j=1}^{i+1} \beta^j \mu^{i+1-j} \left(\frac{B}{(\mu - D_1)^{i+2-j}} + \frac{C}{(\mu - D_2)^{i+2-j}} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{B\beta^{i+1}}{\mu - D_1}\sum_{j=1}^{i+1}\Big(\frac{\mu}{\beta(\mu - D_1)}\Big)^{i+1-j} + \frac{C\beta^{i+1}}{\mu - D_2}\sum_{j=1}^{i+1}\Big(\frac{\mu}{\beta(\mu - D_2)}\Big)^{i+1-j}\\
&= \frac{B\beta^{i+1}}{\mu - D_1}\left(\frac{1-\big(\frac{\mu}{\beta(\mu-D_1)}\big)^{i+1}}{1-\big(\frac{\mu}{\beta(\mu-D_1)}\big)}\right) + \frac{C\beta^{i+1}}{\mu - D_2}\left(\frac{1-\big(\frac{\mu}{\beta(\mu-D_2)}\big)^{i+1}}{1-\big(\frac{\mu}{\beta(\mu-D_2)}\big)}\right)\\
&= \frac{B\beta^{i+1}\big(1-\big(\frac{\mu}{\beta(\mu-D_1)}\big)^{i+1}\big)}{\mu - D_1 - \frac{\mu}{\beta}} + \frac{C\beta^{i+1}\big(1-\big(\frac{\mu}{\beta(\mu-D_2)}\big)^{i+1}\big)}{\mu - D_2 - \frac{\mu}{\beta}}.
\end{aligned}
$$

Let us put $R_1 = \frac{D_1}{\mu}$, $R_2 = \frac{D_2}{\mu}$, $R_3 = \frac{B}{\mu}$, $R_4 = \frac{C}{\mu}$,

$$
\begin{aligned}
Tv(i) &= \beta^{i+1}\left[\left(R_3\frac{1-\big(\frac{1}{\beta^{i+1}(1-R_1)^{i+1}}\big)}{1-R_1-\frac{1}{\beta}}\right) + \left(R_4\frac{1-\big(\frac{1}{\beta^{i+1}(1-R_2)^{i+1}}\big)}{1-R_2-\frac{1}{\beta}}\right)\right],\\
Tv(i) &= \beta^{i+1}\left[\frac{R_3}{1-R_1-\frac{1}{\beta}} + \frac{R_4}{1-R_2-\frac{1}{\beta}} + C\right],
\end{aligned}
$$

where

$$
C = \frac{-1}{\beta^{i+1}}\left[\frac{R_3}{(1-R_1)^{i+1}}\frac{1}{1-R_1-\frac{1}{\beta}} + \frac{R_4}{(1-R_2)^{i+1}}\frac{1}{1-R_2-\frac{1}{\beta}}\right].
$$

We have $R_1R_2 = -R_3R_2 - R_4R_1$, $R_1 < 1 - \frac{1}{\beta}$ and $R_2 < 1 - \frac{1}{\beta}$, so $C < 0$. Therefore,

$$
Tv(i) \le \beta^{i+1}\left[\frac{R_3}{1-R_1-\frac{1}{\beta}} + \frac{R_4}{1-R_2-\frac{1}{\beta}}\right],
$$

i.e., $Tv(i) \le \rho v(i)$ with

$$
\rho = \frac{R_3\beta}{1-R_1-\frac{1}{\beta}} + \frac{R_4\beta}{1-R_2-\frac{1}{\beta}}.
$$

We aim now to clarify the conditions under which $\rho < 1$,

$$
\begin{aligned}
\frac{R_3\beta}{1-R_1-\frac{1}{\beta}} + \frac{R_4\beta}{1-R_2-\frac{1}{\beta}} < 1 \implies & \frac{R_3\beta(1-R_2-\frac{1}{\beta}) + R_4\beta(1-R_1-\frac{1}{\beta})}{(1-R_1-\frac{1}{\beta})(1-R_2-\frac{1}{\beta})} < 1\\
\implies & \frac{\beta^2(R_3+R_4+R_1R_2) - \beta(1-R_1-R_2) + 1}{1-\frac{1}{\beta}} < 0\\
\implies & \beta^2(R_3+R_4+R_1R_2) - \beta(1-R_1-R_2) + 1 < 0.
\end{aligned}
$$

The left-hand side polynomial in β admits two positive roots, and since β is assumed > 1, we conclude that

$$
\rho < 1 \quad \text{if} \quad 1 < \beta < \beta_0,
$$

where

$$
\beta_0 = \frac{(1-R_1-R_2) + \sqrt{\Delta_0}}{2(R_3+R_4+R_1R_2)} \quad \text{and} \quad \Delta_0 = (1-R_1-R_2)^2 - 4(R_3+R_4+R_1R_2).
$$

Finally, we have

$$
\rho = \frac{R_3\beta}{1-R_1-\frac{1}{\beta}} + \frac{R_4\beta}{1-R_2-\frac{1}{\beta}} < 1,
$$

where $Tv(i) \le \rho\beta^k$, under the condition

$$
1 < \beta < \frac{(1-R_1-R_2) + \sqrt{(1-R_1-R_2)^2 - 4(R_3+R_4+R_1R_2)}}{2(R_3+R_4+R_1R_2)}.
$$

Let us now check that $\|P\|_v < \infty$. We have

$$
T = P - h\circ\alpha \implies P = T + h\circ\alpha \implies \|P\|_v \le \|T\|_v + \|h\|_v\|\alpha\|_v,
$$

whereas

$$\|T\|_v = \sup_{i\geq 0} \frac{1}{\beta^i} \sum_{j\geq 0} \beta^j |T_{ij}| \leq \sup_{i\geq 0} \frac{1}{\beta^i} \rho\beta^i \leq \rho < 1,$$
$$\|h\|_v = \sup_{i\geq 0} \frac{1}{v(i)} |h(i)| = \sup_{i\geq 0} \frac{1}{\beta^i} P_{i0} < 1,$$
$$\|\alpha\|_v = \sum_{j\geq 0} v(j)\alpha_j = \sum_{j\geq 0} \beta^j \alpha_j = \alpha_0 + \sum_{j\geq 1} \beta^j \alpha_j = 1.$$

Thus, $\|P\|_v < \infty$. Hence, all the conditions are satisfied. □

Theorem 2. *Let π and $\tilde{\pi}$ be the stationary distributions of the embedded Markov chains of the system (Σ) and $(\tilde{\Sigma})$, respectively. Then, for all $1 < \beta < \beta_0$ and under the condition*

$$\mathcal{W}(G, H) < \frac{(1-\rho)(1-\beta\sigma)}{(1+\beta)(2-\sigma(1+\beta))},$$

we have

$$\|\pi - \tilde{\pi}\|_v \leq \frac{(1+\beta)(1-\sigma)(2-\sigma(1+\beta))}{(1-\beta\sigma)^2(1-\rho) - (1-\beta\sigma)(2-\sigma(1+\beta))(1+\beta)\mathcal{W}(G,H)} \mathcal{W}(G, H), \tag{2.3}$$

where

$$\rho = \frac{R_3\beta}{1 - R_1 - \frac{1}{\beta}} + \frac{R_4\beta}{1 - R_2 - \frac{1}{\beta}} < 1.$$

Proof. Firstly, we compute $\|\mathbf{1}\|_v$ and $\|\pi\|_v$ as follows:

$$\|\mathbf{1}\|_v = \sup_{k\geq 0} \frac{1}{\beta^k} = 1, \quad \|\pi\|_v = \sum_{j\geq 0} \beta^j \pi_j = \sum_{j\geq 0} \beta^j (1-\sigma)\sigma^j = (1-\sigma) \sum_{j\geq 0} (\beta\sigma)^j = \frac{1-\sigma}{1-\beta\sigma}.$$

Therefore,

$$C = 1 + \|\mathbf{1}\|_v \|\pi\|_v = \frac{2-\sigma(1+\beta)}{1-\beta\sigma}.$$

Using the results of [3, 16], we obtain

$$\|\pi - \tilde{\pi}\|_v \leq \frac{(1+\beta)(1-\sigma)(2-\sigma(1+\beta))\mathcal{W}(G,H)}{(1-\beta\sigma)^2(1-\rho) - (1-\beta\sigma)(2-\sigma(1+\beta))(1+\beta)\mathcal{W}(G,H)}.$$ □

3 Generalized gamma kernels in the study of strong stability of the PH/M/1 queuing system

This section investigates the strong stability of the PH/M/1 queuing system using the asymmetric GG kernel method, by computing the variation distance $\mathcal{W}$ and the error given by equations (2.2) and (2.3), respectively, using simulation study. The Monte Carlo simulation study is realized on seven models given as follows:

(D1) the gamma model with parameters $(a, b) = (0.75, 1.25)$;

(D2) the Weibull model with parameters $(a, b) = (1.5, 25)$;

(D3) the PH_2 model with parameters

$$\left\{(1, 0), \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}\right\};$$

(D4) the PH_2 model with parameters

$$\left\{(1, 0), \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix}\right\};$$

(D5) the PH_2 model with parameters

$$\left\{(1, 0), \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}\right\};$$

Kernel	Explicit form	Parameters
MG	$K_{\mathrm{MG}}(y;x,h)=\dfrac{y^{\alpha-1}\exp\{-y/(\beta/\alpha)\}}{(\beta/\alpha)^{\alpha}\Gamma[\alpha]},\quad y\geq 0$	$(\alpha,\beta,\gamma)=\begin{cases}(\frac{x}{h},x,1) & \text{for } x\geq 2h,\\ (\frac{x^2}{4h^2}+1,\frac{x^2}{4h}+h,1) & \text{for } x\in[0,2h)\end{cases}$
NK	$K_{\mathrm{NK}}(y;x,h)=\dfrac{2(\alpha/2)^{\alpha/2}}{[(\alpha/2)[\beta\Gamma(\alpha/2)/\Gamma((\alpha+1)/2)]^2]^{\alpha/2}\Gamma(\alpha/2)}y^{2(\alpha/2)-1}\times\exp[-\dfrac{\alpha/2}{(\alpha/2)[\beta\Gamma(\alpha/2)/\Gamma((\alpha+1)/2)]^2]}y^2],\quad y\geq 0$	$(\alpha,\beta,\gamma)=\begin{cases}(\frac{x}{h},x,2) & \text{for } x\geq 2h,\\ (\frac{x^2}{4h^2}+1,\frac{x^2}{4h}+h,2) & \text{for } x\in[0,2h)\end{cases}$
W	$K_{\mathrm{W}}(y;x,h)=\dfrac{\alpha}{\beta/\Gamma(1+1/\alpha)}\left[\dfrac{y}{\beta/\Gamma(1+1/\alpha)}\right]^{\alpha-1}\times\left[-[\beta/\Gamma(1+1/\alpha)]^{\alpha}\right],\quad y\geq 0$	$(\alpha,\beta,\gamma)=\begin{cases}(\sqrt{\frac{x}{h}},x,\sqrt{\frac{x}{h}}) & \text{for } x\geq 2h,\\ (\frac{x}{2h}+1,\frac{x^2}{4h}+h,\frac{x}{2h}+1) & \text{for } x\in[0,2h)\end{cases}$

Table 1: Generalized gamma (MG, NK and W) kernels.

Kernel	C	V
MG	1	$1/(2\sqrt{\pi})$
W	$\pi^2/12$	$1/(2\sqrt{2})$
NK	1/2	$1/(\sqrt{2\pi})$

Table 2: Values of C and V for MG, NK and W kernels.

(D6) the PH_3 model with parameters

$$\left\{(1,0,0),\begin{pmatrix}-2 & 2 & 0\\ 0 & -6 & 6\\ 0 & 0 & -1/2\end{pmatrix}\right\};$$

(D7) the PH_3 model with parameters

$$\left\{(1,0,0),\begin{pmatrix}-2 & 1 & 0\\ 0 & -3 & 3/2\\ 0 & 0 & -4\end{pmatrix}\right\}.$$

For each model, 500 data sets of sample sizes $n=100$, 250, 500 and 1000 are generated. To estimate the PDF of the considered models, we used the GG kernel estimator given by (see [14])

$$\hat{f}_{\mathrm{GG}}(x)=\frac{1}{n}\sum_{i=1}^{n}K_{\mathrm{GG}}(X_i;x,h)=\frac{1}{n}\sum_{i=1}^{n}\frac{\gamma X_i^{\alpha-1}\exp\left[-\left\{\frac{X_i}{\beta\Gamma(\alpha/\gamma)/\Gamma((\alpha+1)/\gamma)}\right\}^{\gamma}\right]}{\{\beta\Gamma(\alpha/\gamma)/\Gamma((\alpha+1)/\gamma)\}^{\alpha}\Gamma(\alpha/\gamma)},\quad x>0,$$

where $\{X_i\}_{i=1}^n$ is drawn from a density f, $(\alpha,\beta,\gamma)=(\alpha_h(x),\beta_h(x),\gamma_h(x))$ are continuous functions of the design point x and the bandwidth h, which satisfies some conditions (see [14] for more details). We consider three cases of GG kernels (MG, NK and W kernels) given in Table 1 in the study of strong stability of the PH/M/1 queuing system because of its several interesting properties and flexibility.

For bandwidth selection, we used the rule of thumb (RT) and the unbiased cross validation (UCV) methods for comparison. The rule of thumb selector is obtained (see [14]) by replacing the unknown density f by a known reference gamma model with parameters $\alpha>0$ and $\beta>0$,

$$\hat{h}_{\mathrm{RT}}=\left\{\frac{4^{\alpha-1}V\beta^{5/2}\Gamma(\alpha)\Gamma(\alpha+5/2)}{C^2C_\alpha\Gamma(2\alpha)}\right\}^{2/5}n^{-2/5},$$

where

$$C_\alpha=\frac{1}{4}(\alpha-2)^2(\alpha-1)^2-\alpha(\alpha-2)(\alpha-1)^2+\frac{1}{2}\alpha(3\alpha-4)(\alpha-1)\left(\alpha+\frac{1}{2}\right)$$
$$-\alpha(\alpha-1)\left(\alpha+\frac{1}{2}\right)(\alpha+1)+\frac{1}{4}\alpha\left(\alpha+\frac{1}{2}\right)(\alpha+1)\left(\alpha+\frac{3}{2}\right)$$

and the constants C and V are given in Table 2. Note that the parameters α and β are replaced by the corresponding estimators $\hat{\alpha}$ and $\hat{\beta}$, which can be obtained by using the method of moments.

The UCV technique can be adapted for GG kernel estimators. For a given GG (MG, NK and W) kernel with target $x > 0$ and bandwidth $h > 0$, the optimal bandwidth h_{ucv} is obtained by

$$h_{\mathrm{ucv}} = \arg\min_h \mathrm{UCV}(h),$$

where

$$\begin{aligned}\mathrm{UCV}(h) &= \int_0^\infty \{\hat{f}_{\mathrm{GG}}(x)\}^2\,dx - \frac{2}{n}\sum_{i=1}^n \hat{f}_{\mathrm{GG}}^{(-i)}(X_i)\\ &= \int_0^\infty \left\{\frac{1}{n}\sum_{i=1}^n K_{\mathrm{GG}}(X_i; x, h)\right\}^2 dx - \frac{2}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}^n K_{\mathrm{GG}}(X_j; X_i, h).\end{aligned}$$

Tables 3, 4, 5 report the deviation and stability errors for GG kernel estimators for considered models as general arrival distribution. Note that the deviation and stability errors are also given using the theoretical density f. From these tables, we can observe clearly that

(1) the deviation distance and the stability errors based on 500 replications decrease as sample size n increases for the all estimators based on MG, NK and W kernels;
(2) for all models, the GG kernel estimators combined with RT bandwidth selector perform better than the GG estimators combined with UCV bandwidth;
(3) in general, the deviation distance and the stability errors obtained using the GG kernel estimators with RT selector are close to those given when using the theoretical density f;
(4) the stability errors using the GG kernel estimators combined with UCV bandwidth selector are not obtained for some situations because the condition stability does not hold.

The comparison is also given in Figures 1 and 2. Figure 1 shows the stability error behavior versus β (domain of the feasible values of beta). We observe that the error decreases for the values of beta in the neighborhood of the lower bound ($1.0428 < \beta < 1.8965$) for (D1) (resp. $1.2296 < \beta < 2.9296$ for (D2)). This can be explained by the way that it represent the frontier the stability domain (critical region). We also notice that it increases in the neighborhood of the upper bound ($1.9328 < \beta < 2.3705$ for (D1)) (resp. $3.0668 < \beta < 10.7817$ for (D2)). On the other hand, the error increases in a reasonable way with the values of beta (favorable region). Nevertheless, it is interesting to consider the minimum error which corresponds in our case to $\beta = 1.3503$ for (D1) (resp.$\beta = 1.2296$ for (D2)).

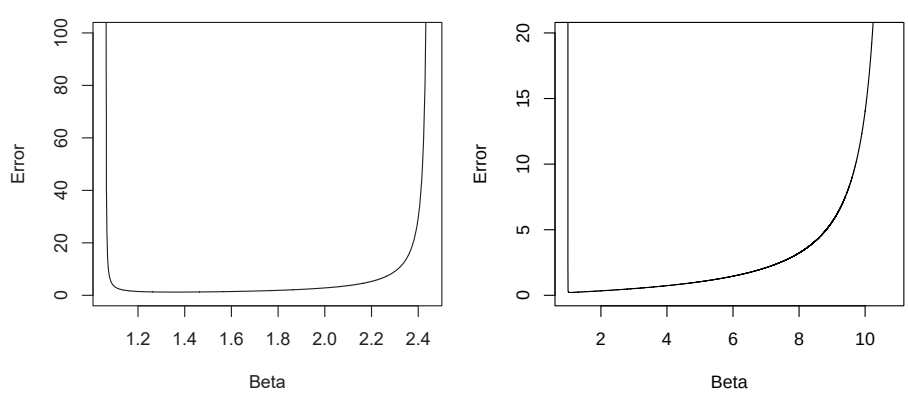

Figure 1: Error variation versus β obtained using Weibull kernel combined with RT bandwidth for (D1) (in left) (D2) (in right) models with sample size $n = 500$.

		$n = 100$		$n = 250$		$n = 500$		$n = 1000$	
		$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error
					Gamma(0.75, 1.25)				
$f(x)$		0.08376	3.38279	0.06016	1.94906	0.04581	1.01734	0.04233	0.81570
MG	UCV	0.17777	—	0.13373	—	0.09683	—	0.08134	6.68873
	PI	0.11836	—	0.08140	8.51085	0.05862	1.80445	0.05347	1.31874
W	UCV	0.15412	—	0.13006	—	0.09683	—	0.08456	9.52077
	PI	0.11391	—	0.07185	3.75351	0.05471	1.50655	0.05031	1.14955
NK	UCV	0.11941	—	0.07937	6.84214	0.05793	1.74688	0.05665	1.51848
	PI	0.11583	—	0.07270	3.97493	0.05503	1.52855	0.05084	1.17617
					Weibull(1.5, 25)				
$f(x)$		0.04591	0.23110	0.03471	0.16545	0.03338	0.15820	0.02940	0.13690
MG	UCV	0.08419	0.52495	0.06496	0.36181	0.06010	0.32613	0.05451	0.28744
	PI	0.07593	0.45025	0.05938	0.32076	0.05051	0.26053	0.04385	0.21892
W	UCV	0.08419	0.52495	0.06483	0.36082	0.05732	0.30643	0.04932	0.25315
	PI	0.07593	0.45025	0.06157	0.33657	0.05150	0.26700	0.04233	0.20974
NK	UCV	0.08575	0.53993	0.06867	0.39054	0.05729	0.30622	0.05108	0.26457
	PI	0.07274	0.42333	0.05909	0.31870	0.04729	0.23994	0.04262	0.21148

Table 3: Deviation and stability errors for considered models in simulation study with different estimators for (D1) and (D2) models.

		$n = 100$		$n = 250$		$n = 500$		$n = 1000$	
		$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error
					$\mathrm{PH}_2\left\{(1,0), \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}\right\}$				
$f(x)$		0.03756	0.31939	0.03964	0.35537	0.02815	0.21602	0.02259	0.16644
MG	UCV	0.10272	1.93763	0.10398	2.16075	0.07957	1.00149	0.05559	0.53929
	PI	0.09069	1.89332	0.090670	0.92007	0.07267	0.84230	0.05284	0.49943
W	UCV	0.14461	12.57867	0.14593	21.52266	0.10867	2.14463	0.08149	1.05197
	PI	0.08206	1.11667	0.08362	1.22628	0.06637	0.71763	0.04818	0.04638
NK	UCV	0.13026	5.06489	0.13139	6.22444	0.10926	2.18140	0.07787	0.96083
	PI	0.08367	1.16379	0.08367	1.22790	0.06699	0.72899	0.04786	0.43224
					$\mathrm{PH}_2\left\{(1,0), \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix}\right\}$				
$f(x)$		0.05887	0.76374	0.03943	0.36020	0.03257	0.28332	0.02069	0.16975
MG	UCV	0.13543	—	0.09023	151803	0.06890	0.86994	0.05330	0.59311
	PI	0.11095	4.39953	0.08294	124502	0.06422	0.76629	0.04345	0.43653
W	UCV	0.18524	—	0.13147	6.97975	0.10114	2.13036	0.08624	1.49900
	PI	0.09927	2.69468	0.07433	0.98980	0.06052	0.69207	0.04014	0.39052
NK	UCV	0.17521	—	0.14113	15.13252	0.10385	2.31761	0.09065	1.70384
	PI	0.10097	2.87272	0.07552	1.02159	0.05869	0.65760	0.03984	0.38649
					$\mathrm{PH}_2\left\{(1,0), \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}\right\}$				
$f(x)$		0.24498	—	0.24673	—	0.22656	—	0.22616	—
MG	UCV	0.13614	—	0.09182	2.93552	0.07447	1.41468	0.05561	0.97237
	PI	0.09995	11.87899	0.06942	1.29010	0.05457	0.75067	0.03979	0.54535
W	UCV	0.14845	—	0.11760	21.98498	0.08727	2.19588	0.07667	2.11835
	PI	0.09542	7.30759	0.06806	1.23347	0.05055	0.65867	0.03901	0.52902
NK	UCV	0.14370	—	0.10147	4.70361	0.08577	2.07906	0.07210	1.76934
	PI	0.09894	10.47059	0.06948	1.29267	0.05266	0.70572	0.03792	0.50677

Table 4: Deviation and stability errors for considered models in simulation study with different estimators for (D3), (D4) and (D5) models.

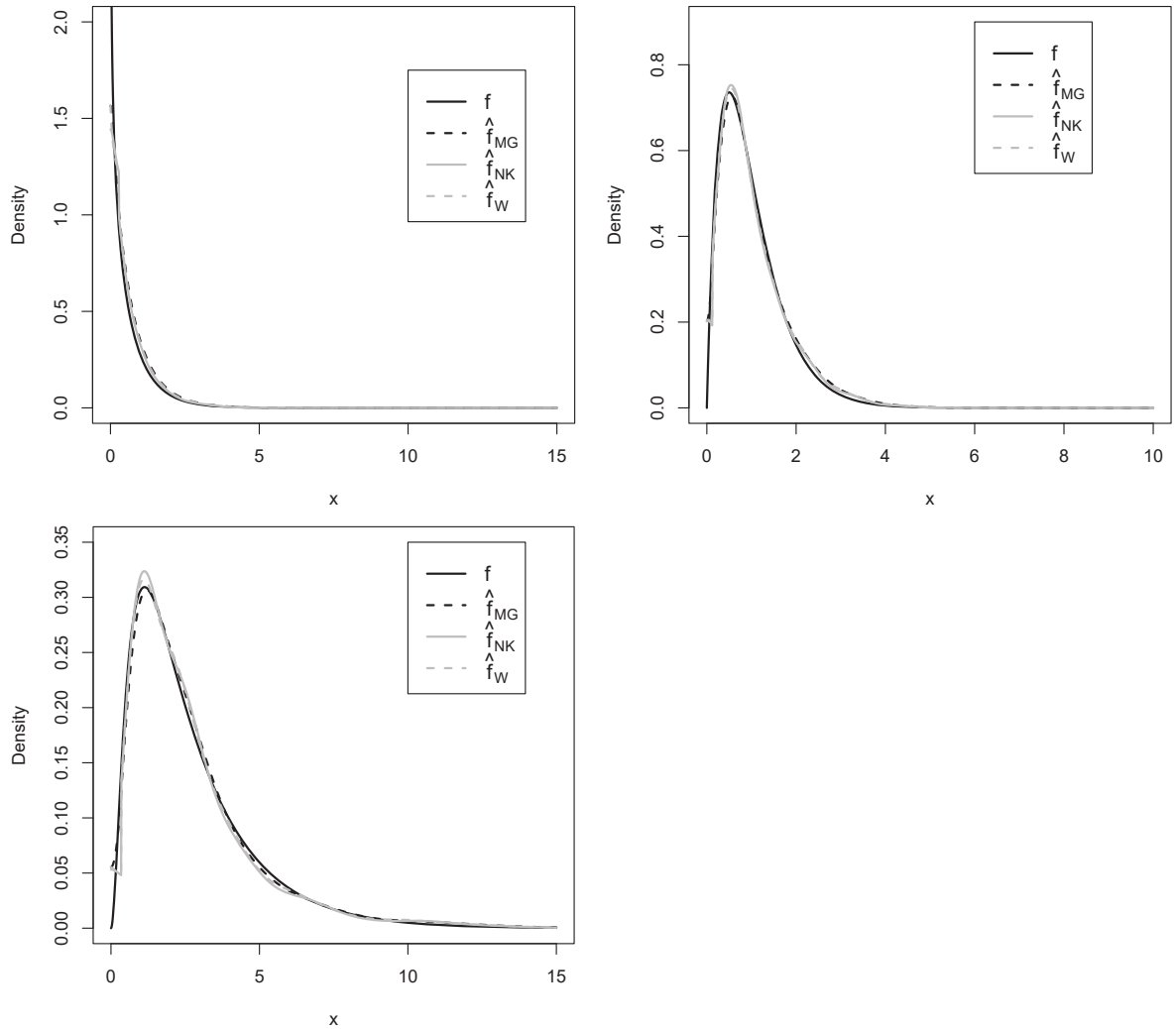

Figure 2: True PDF and GG (MG, NK and W) kernel estimators for (D1), (D3) and (D6) models with $n = 500$ using RT bandwidth. The (D1) model (in left) and the (D3) model (in right) in first row and the (D6) model in second row.

4 Conclusion

In this paper, we show the interest of using the nonparametric asymmetric kernel method in the study of strong stability of the PH/M/1 queuing system. It has been established that PH/M/1 queuing systems with phase type arrival distributions are strongly stable (robust) with respect to the considered perturbation and hence they can resist it to some extent. In our study, we estimated the general law of arrivals using a class of generalized gamma (GG) kernels where the smoothing parameter is obtained with plug-in and UCV techniques. Then an upper bound to the resultant deviation of the stationary distribution are computed. The obtained results in the simulation study show the efficiency of the proposed approach.

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		$n = 100$		$n = 250$		$n = 500$		$n = 1000$	
		$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error	$\mathcal{W}$	Error
				$\mathrm{PH}_3\left\{(1,0,0), \begin{pmatrix} -2 & 2 & 0 \\ 0 & -6 & 6 \\ 0 & 0 & -1/2 \end{pmatrix}\right\}$					
$f(x)$		0.09997	1.00130	0.07325	0.59788	0.06221	0.45447	0.05593	0.39022
MG	UCV	0.14266	2.49382	0.09191	0.88469	0.07875	0.65425	0.06381	0.47106
	PI	0.11767	1.43216	0.08315	0.73823	0.06548	0.49005	0.05389	0.37072
W	UCV	0.14635	2.73448	0.10447	1.14360	0.07990	0.67020	0.06244	0.45634
	PI	0.11613	1.38744	0.08720	0.803021	0.06889	0.52906	0.05689	0.39959
NK	UCV	0.13874	2.26995	0.09624	0.96658	0.08033	0.67624	0.07016	0.54338
	PI	0.12250	1.58393	0.09216	0.88923	0.07344	0.58439	0.06161	0.44756
				$\mathrm{PH}_3\left\{(1,0,0), \begin{pmatrix} -2 & 1 & 0 \\ 0 & -3 & 3/2 \\ 0 & 0 & -4 \end{pmatrix}\right\}$					
$f(x)$		0.07456	2.10431	0.05716	1.15051	0.03783	0.48675	0.03114	0.38288
MG	UCV	0.13423	—	0.11073	—	0.07679	2.01722	0.06185	1.24017
	PI	0.09915	9.73142	0.08163	3.33054	0.06005	1.08977	0.05042	0.81879
W	UCV	0.15224	—	0.11732	—	0.09162	3.98754	0.07274	1.87860
	PI	0.09151	5.0752	0.08117	3.24906	0.05692	0.97651	0.04886	0.77338
NK	UCV	0.14201	—	0.10862	—	0.08872	3.42185	0.07549	2.10259
	PI	0.09278	5.54673	0.08251	3.49553	0.05893	1.04777	0.04915	0.78165

Table 5: Deviation and stability errors for considered models in simulation study with different estimators for (D6) and (D7) models.

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