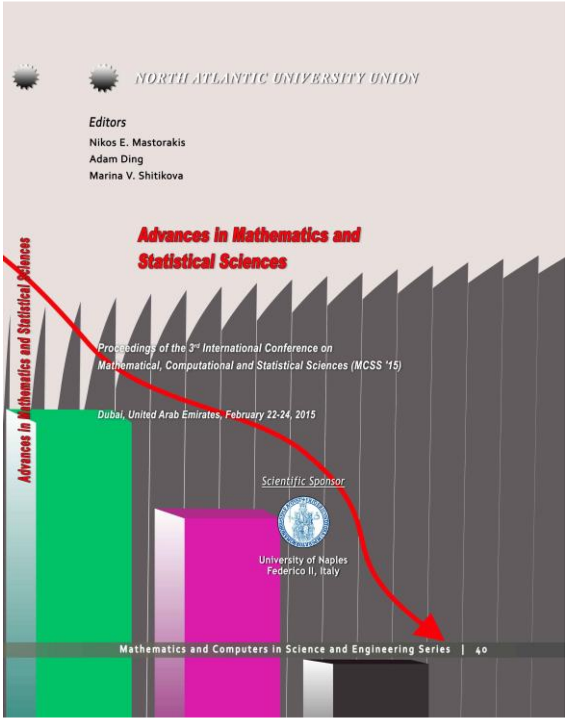

Strong Approximation for an Overflow Queueing Network

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Key–Words: Queueing, State-space truncation, Overflow model, Approximation, Algorithm

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1 Introduction

Queueing network models are among the most natural for quantitative analysis, capacity planning and buffer dimensioning of logistics, manufacturing and communication systems. In order to control and optimize a queueing network, everyone has to know its characteristics like the overall blocking or overflow probability, the average departure rate from the waiting room and the servers and the average occupation proportion of the waiting and service positions or others of special interest. However, these characteristics can only be calculated for a limited class of queueing networks and the more involved the system dynamics get, the more involved the analysis of the long run behavior usually becomes.

In this paper, we consider a general class of so called overflow queueing networks. These networks consist of two queues, where the capacity of the first queue is always finite. Customers arriving at the first queue have an overflow capability from the first to the second queue if the first queue operates at a certain fixed capacity, i.e., under certain conditions, demands arriving at the first queue are allowed to join the second queue. Due to the natural occurrence of overflow queueing problems, the related literature is vast, see for example Disney and König [4] for a broad overview. Overflow queueing models are widespread in literature. Van Doorn [24] and Parthasarathy and Sudhesh [17] study the interoverflow time distribution of a finite birth and death queue model. Koury et al. [13] and Krieger et al. [14] give reviews of iterative numerical methods for overflow queueing models. A brief discussion of numerical methods for some two-queue overflow systems and further references are given in Ching and Ng [2]. While most of these formulations are of primary interest when the focus is on numerical results. Related overflow models are studied in van Doorn [24] and Guérin, Lien [6] and the referenced literature therein using a variety of different techniques.

Despite of a growing literature on the performance analysis of this type of models, there is still no viable analytical method for predicting performances of such networks. In this paper, we propose to follow a different train of thought, and will present a directly computable perturbation bounds on the effect on the stationary behavior for state-space truncation of infinite discrete time Markov chain describing an overflow model. These perturbation bounds are obtained by using the strong stability method [12] for different truncation techniques. Indeed, we are interested in approximating stationary distributions of an infinite discrete time Markov chain describing the state of an overflow model by those corresponding of the same model after the truncation state-space of this Markov chain. More precisely, let P be the one-step transition probability matrix of the considering overflow model (with infinite waiting room), and let $_{(Q)}P$ be the northwest corner of P. Notice that $_{(Q)}P$ is not a stochastic matrix. The procedure to make $_{(Q)}P$ stochastic by adding appropriate values to its entries is called augmentation. In this paper, we are interested in determining which augmentation technique provides the best approximation in the sense that the analytic perturbation bounds derived by using the strong stability method is the minimum. This is made by numerical comparison of three different augmentation techniques. Our main contributions here are:

1. to approximate the stationary distributions of an overflow model with infinite waiting room, which has not a product form solution, by those corresponding of the same model after the truncation of its number waiting room by using the strong stability method, and

2. to show numerically comparing the obtained strong stability bounds the best augmentation technique.

This paper comprises four sections. In Section 2, we present basic definitions and tools for computing the strong stability perturbation bounds. In Section **??**, we describe the overflow network model in which the buffer size of second service station is truncated and we give the perturbation bounds corresponding to this truncation. A comparison between the obtained perturbation bounds is illustrated through numerical examples in Section 4. Eventually, we will point out directions of further research.

2 Strong Stability Approach

The main tool for our analysis is the weighted supremum norm, also called v-norm, denoted by $\|\cdot\|_v$, where v is some vector with elements $v(k,l) > 1$ for all $(k,l) \in \mathbf{S} = \{0,1\} \times \{0,\ldots,Q\}$.

Let us note $\mathfrak{B}(\mathbb{N})$, the Borel field of the natural numbers that is equipped with the discrete topology, and we consider the measurable space $(\mathbb{N}, \mathfrak{B}(\mathbb{N}))$.

Let $\mathfrak{M} = \{\mu_{(i,j)}\}$ be the space of finite measures on $\mathfrak{B}(\mathbb{N})$ and $\eta = \{f(i,j)\}$ be the space of bounded measurable functions. We associate with each transition operator P the linear mappings

$$(\mu P)_{(k,l)} = \sum_{i=0}^{1}\sum_{j=0}^{Q} \mu_{(i,j)} \ \ P_{(i,j);(k,l)}; \qquad (1)$$

$$(Pf)(k,l) = \sum_{i=0}^{1}\sum_{j=0}^{Q} f(i,j) \ \ P_{(k,l);(i,j)}. \qquad (2)$$

Introduce to $\mathfrak{M}$ the class of norms of the form

$$\|\mu\|_v = \sum_{i=0}^{1}\sum_{j=0}^{Q} v(i,j)|\mu_{(i,j)}|, \qquad (3)$$

where v is an arbitrary measurable function (not necessary finite) bounded from below by a positive constant. This norm induces in the space η the norm

$$\|f\|_v = \sup_k \sup_l \frac{|f(k,l)|}{v(k,l)}; \ \ k,l \in \{0,1\}\times\{0,\ldots,Q\}. \qquad (4)$$

Let us consider $\mathfrak{B}$, the space of bounded linear operators on the space $\{\mu \in \mathfrak{M} : \|\mu\|_v < \infty\}$, with norm

$$\begin{cases} \|Q\|_v = \sup_k \sup_l \frac{1}{v(k,l)} \sum_{i=0}^{1}\sum_{j=0}^{Q} v(i,j)|Q_{(k,l);(i,j)}|; \\ k,l \in \{0,1\} \times \{0,\ldots,Q\}. \end{cases} \qquad (5)$$

Let ν and μ be two invariant measures and suppose that these measures have finite v-norm. Then

$$\begin{cases} |\nu f - \mu f| \leq \|\nu - \mu\|_v \|f\|_v \inf_k \inf_l v(k,l); \\ k,l \in \{0,1\} \times \{0,\ldots,Q\}. \end{cases} \qquad (6)$$

for all f with $\|f\|_v$ finite.

For our analysis, we will assume that $v(k,l)$ is of a particular form $v(k,l) = \alpha^k \ \beta^l$, for $\alpha > 1$ and $\beta > 1$, which implies

$$\inf_k \inf_l v(k,l) = 1; \ \ k,l \in \{0,1\} \times \{0,\ldots,Q\}. \qquad (7)$$

Hence, the bound 6 becomes

$$\begin{cases} |\nu f - \mu f| \leq \|\nu - \mu\|_v \sup_k \sup_l \frac{|f(k,l)}{v(k,l)}; \\ k,l \in \{0,1\} \times \{0,\ldots,Q\}. \end{cases} \qquad (8)$$

We say that the Markov chain X with transition kernel P verifying $\|P\|_v < \infty$ and invariant measure π is *strongly v-stable*, if every stochastic transition kernel $\widetilde{P}$ in some neighborhood $\{\widetilde{P} : \|\widetilde{P} - P\|_v < \epsilon\}$ admits a unique invariant measure

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$\widetilde{\pi}$ such that $\|\widetilde{\pi}-\pi\|_v$ tends to zero as $\|\widetilde{P}-P\|_v$ tends to zero uniformly in this neighborhood. The key criterion of strong stability of a Markov chain X is the existence of a deficient version of P defined in the following:

Let X be a Markov chain with the transition kernel P and invariant measure π. We call a deficient Markov kernel T a *residual for* P *with respect to* $\|\cdot\|_v$ if there exists a probability measure σ and a nonnegative measurable function h on $\mathbf{S}$ satisfying the following conditions:

(a) $\pi h > 0$, $\sigma \mathbf{1} = 1$, $\sigma h > 0$, and

(b) the kernel $T = P - h \circ \sigma$ is nonnegative,

(c) the v-norm of the kernel T is strictly less than one, i.e., $\|T\|_v < 1$,

(d) $\|P\|_v < \infty$,

where $\circ$ denotes the convolution between a measure and a function and $\mathbf{1}$ is the vector having all the components equal to 1.

It has been shown in [1] that a Markov chain X with the transition kernel P is strongly stable with respect to v if and only if a residual for P with respect to v exists. Although the strong stability approach originates from stability theory of Markov chains, the techniques developed for the strong stability approach allow to establish numerical algorithms for bounding $\|\widetilde{\pi}-\pi\|_v$. A bound on $\|\widetilde{\pi}-\pi\|_v$ is established in the following theorem.

Theorem 1. *([11]) Let P be strongly stable. If*

$$\|\widetilde{P}-P\|_v < \frac{1-\|T\|_v}{\|I-\Pi\|_v},$$

then, the following bound holds

$$\|\widetilde{\pi}-\pi\|_v \leq \|\pi\|_v \frac{\|I-\Pi\|_v \, \|\widetilde{P}-P\|_v}{1-\|T\|_v - \|I-\Pi\|_v \, \|\widetilde{P}-P\|_v},$$

where Π is the stationary projector of P and I is the identity matrix.

Note that the term $\|I-\Pi\|_v$ in the bound provided in Theorem 1 can be bounded by

$$\|I-\Pi\|_v \leq 1 + \|\mathbf{1}\|_v \|\pi\|_v.$$

In this case, we can also bound $\|\pi\|_v$ by

$$\frac{(\sigma v) \quad (\pi h)}{1-\rho}. \tag{9}$$

3 Analysis of the Model

3.1 Model description

Consider an overflow queueing network that consists of two queues in parallel, Q_1 and Q_2, where the first queue Q_1 has not a waiting rooms, that is, the capacity of the waiting room in first queue is 0, and the second queue Q_2 has an infinite capacity queue with First-Come, First-Served (FCFS) service discipline. Customers arriving at the first station have an overflow capability from the first to the second queue if the first server is not available, i.e., under certain conditions, demands arriving at the first service station are allowed to join the second queue. In every model, the dynamic of the first queue is or is at least similar to the famous Erlang loss systems. The services in the both stations are assumed to be exponential with parameters μ_1 and μ_2, respectively. The customers arrive according to a Poisson process with parameter λ. We assume that $\lambda < \mu_2$.

This model has no product form solution for the steady-state joint queue size distribution [9]. Furthermore, the same model can be represented as quasi birth and death processes, see for example Latouche and Ramaswami [15]. Consequently, their analysis can be carried out using a matrix-geometric approach, see Neuts [16]. Overflow queueing models are widespread in literature. Van Doorn [24] and Parthasarathy and Sudhesh [17] study the interoverflow time distribution of a finite birth and death queue model. Koury et al. [13] and Krieger et al. [14] give reviews of iterative numerical methods for overflow queueing models. A brief discussion of numerical methods for some two-queue overflow systems and further references are given in Ching and Ng [2]. van Doorn [24] and Guérin, Lien [6] and the referenced literature therein using a variety of different techniques. The overflow stream is known to be hyperexponential [24], so that the overflow station separately can be analyzed as a GI/M/s queueing system. This, however, would still require complex computational procedures for large s values [3]. Moreover, we can be interested in a performance measure that depends on both queue sizes, such as the total number of customers present, where $\mu_1 \neq \mu_2$ is allowed. van Dijk [22] establishes an explicit error bounds on state-space truncation of an overflow model. While most of these formulations are of primary interest when the focus is on numerical results, the strong stability method [12] used in the following gives a new perturbation bounds with exactly computing of the constants. This approach gives with precision the error, on the queue size stationary distribution of the considered overflow model, due to the state-space truncation.

Let (i, j) denote the number of customers at Q_1 and Q_2, respectively. $M = (\lambda + \mu_1 + \mu_2)$. Consider the discrete time Markov chain with one-step transition probabilities $(P_{(i,j);(m,n)})$ for a transition from a state (i, j) to a state (m, n) given by:

$$\begin{cases} \widetilde{P}_{(0,j);(1,j)} = \lambda/M; \\ \widetilde{P}_{(1,j);(,j+1)} = \lambda/M; \\ \widetilde{P}_{(1,j);(0,j)} = \mu_1/M; \\ \widetilde{P}_{(i,j);(i,j-1)} = \mu_2/M; \\ \widetilde{P}_{(0,j);(0,j)} = \mu_1/M; \\ \widetilde{P}_{(i,0);(i,0)} = \mu_2/M; \end{cases} \tag{10}$$

In the following we use the strong stability method to approximate the stationary distributions of an overflow model with infinite waiting room by those corresponding of the same model after the truncation of its number waiting room. This is given by considering three different types of truncation technique, and we are interested in determining which type of truncation technique provides the best approximation in the sense that the strong stability bound value is the minimum.

3.2 State-Space Truncation in the Overflow Model

In this section, for approximating the stationary distribution of an infinite Markov chain, we will establish three perturbation bounds by using the strong stability method. For that, let P be the transition probability matrix of an infinite discrete time Markov chain, describing the overflow model considered in our analysis, which has a unique stationary distribution π, and let ${}_{(Q)}P$ be the northwest corner of P. Notice that ${}_{(Q)}P$ is not a stochastic matrix. The procedure to render ${}_{(Q)}P$ stochastic by adding appropriate values to its components is called augmentation. Seneta [20] summarizes much of the literature on this. In our analysis, we will consider three different types of truncation technique: augmentation of the first column, normalization of rows and uniform augmentation. In fact, from the matrix ${}_{(Q)}P$ we construct a new stochastic matrix $M = (M_{(i,j);(m,n)})_{0\leq i,j,m,n\leq Q}$. The principle of these procedure is given as follow:

1. **Linear augmentation:** The lost probability mass during the truncation of the matrix P is redistributed on the columns of the matrix ${}_{(Q)}P$. More precisely, let

   $${}_{(Q)}A = ({}_{(Q)}A_{(i,j);(m,n)})_{0\leq i,j,m,n\leq Q}$$

   be a some stochastic matrix, for

   $$0 \leq i, j, m, n \leq Q$$

   we set:

   $${}_{(Q)}P_{(i,j);(m,n)} = P_{(i,j);(m,n)} +_{(Q)} A_{(i,j);(m,n)} \times \sum_{k>Q}\sum_{l>Q} P_{(i,k);(m,l)} \text{ for } 0 \leq i, j, m, n \leq Q.$$

   Particularly, we obtain:

   i. ***The augmentation of the first column:*** if we choose ${}_{(Q)}A_{(i,1);(m,1)} = 1$ for $0 \leq i, m \leq Q$;

   ii. ***The uniform augmentation:*** if we choose ${}_{(Q)}A_{(i,j);(m,n)} = (Q+1)^{-1}$ for $0 \leq i, j, m, n \leq Q$.

2. **Normalization:** We set $S_{(i,Q);(m,n)} = \sum_{j=0}^{Q}\sum_{n=0}^{N} P_{(i,j);(m,n)}$, then we choose for $0 \leq i, j, m, n \leq Q$:

   $${}_{(Q)}P_{(i,j);(m,n)} = \frac{P_{(i,j);(m,n)}}{S_{(i,Q);(m,n)}},$$

   where we assign a large value to Q in order that $S_{(i,Q);(m,n)} > O$.

3.2.1 Augmentation of the First Column

In this case, we propose the following truncation:

$$\begin{cases} P1_{(1,Q);(1,0)} = \frac{\lambda}{M}; \\ P1_{(i,j);(m,n)} = \widetilde{P}_{(i,j);(m,n)} \text{ otherwise.} \end{cases} \tag{11}$$

In order to establish strong stability bounds, we require bounds on the basic input entities such as $\overline{\pi}$ (stationary distribution of the truncated model) and $\overline{T}$ (taboo matrix corresponding to some taboo state of the matrix $\overline{P}$) and, we have to specify the test function υ that defines the v-norm. Specifically, for $\alpha > 1$ and $\beta > 1$, we will choose

$$\upsilon(k,l) = \alpha^k \beta^l. \tag{12}$$

For our analysis, we introduce the following condition:

$$1 < \frac{\mu}{\lambda} \tag{13}$$

where $\mu = \min(\mu_1, \mu_2)$. This condition corresponds to the trafic intensity condition of the infinite model.

Essential for our numerical bounds on the deviation between stationary distributions $\overline{\pi}$ (stationary distribution of the truncated model) and π (stationary distribution of the infinite model) is a bound on the deviation of the transition matrix $\overline{P}$ from P. This bound is provided in the following lemma.

Lemma 2. *If condition (13) is satisfied, then*

$$\|P1 - \widetilde{P}\| \leq \frac{\lambda}{\beta^Q \; M} = \triangle_1(\beta) \tag{14}$$

Proof. By definition, we have

$$\|P1 - \widetilde{P}\|_\upsilon =$$

$$\sup_{k=0,1} \sup_{0<l<Q} \frac{1}{\upsilon(k,l)} \sum_{i=0}^{1} \sum_{j=0}^{Q} \upsilon(i,j) |P1_{(k,l);(i,j)} - \widetilde{P}_{(k,l);(i,j)}|;$$

$$= \sup_{0 \leq i \leq Q} \sup_{0 \leq j \leq N} S(i,j),$$

where

$$S(i,j) =$$

$$\frac{1}{\upsilon(i,j)} \sum_{m=0}^{Q} \sum_{n=0}^{N} \upsilon(m,n) |\widetilde{P}_{(i,j);(m,n)} - P1_{(i,j);(m,n)}|. \tag{15}$$

- For $i = 0$

$$S(i,j) = 0. \tag{16}$$

- For $i = 1$

If $0 \leq j < Q$

$$S(i,j) = 0. \tag{17}$$

If $j = Q$

$$\begin{aligned} S(i,j) &= \frac{1}{\alpha^1 \beta^Q} \left(\alpha \beta^0 \frac{\lambda}{M} + 0 + 0 + \right). \\ &= \frac{1}{\beta^Q} \frac{\lambda}{M}. \end{aligned} \tag{18}$$

From (16), (17) and (18) we have

$$\|P1 - \widetilde{P}\|_\upsilon = \frac{1}{\beta^Q} \frac{\lambda}{M}.$$

□

Let $T1$ denote the taboo Markov kernel for taboo state $(0,0)$; more specifically, for $(i,j),(m,n)$ let:

$$T1_{(i,j)\to(m,n)} = \begin{cases} 0 \text{ if } i = j = 0, \\ P1_{(i,j);(m,n)} \text{ otherwise.} \end{cases} \tag{19}$$

In the following lemma we will identify the range for α and β that leads to verify the conditions (a) – (d). Indeed, the main work in strong stability method is finding α and β such that $\|T1\|_\upsilon < 1$ where T is defined above in (19).

Lemma 3. *Provided that condition (13) holds,and for $1 < \beta < \frac{\mu}{M}$ and $\beta < \alpha < 1 + \left(1 - \frac{1}{\beta}\right) \frac{\mu}{M}$ we have*

$$\begin{aligned} \|T1\|_\upsilon &= \max \left\{ \alpha \frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{1}{\beta} \frac{\mu_2}{M}, \alpha \frac{\lambda}{M} \right. \\ &\quad \left. + \frac{\mu_2}{M} + \frac{1}{\beta} \frac{\mu_1}{M} \right\}, \\ &= \rho 1(\alpha, \beta) < 1. \end{aligned} \tag{20}$$

Where $\mu = \min(\mu_1, \mu_2)$.

Proof. We have

$$T1\upsilon(i,j) = \sum_{m=0}^{1} \sum_{n=0}^{Q} \upsilon(m,n) T1_{(i,j)\to(m,n)}.$$

For $i = 0$

If $j = 0$

$$T1\upsilon(0,0) = 0 \tag{21}$$

If $0 < j \leq Q$

$$\begin{aligned} T1\upsilon(0,j) &= \alpha \beta^j \frac{\lambda}{M} + \alpha^0 \beta^j (1 - \frac{\lambda}{M} - \frac{\mu_2}{M}) \\ &\quad + \alpha^0 \beta^{j-1} \frac{\mu_2}{M}. \\ &= \beta^j \left(\alpha \frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{1}{\beta} \frac{\mu_2}{M} \right). \end{aligned} \tag{22}$$

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From (21) and (22) we have

$$\rho\mathbf{1}_{(i=0,0\le j\le Q)} = \alpha\frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{1}{\beta}\frac{\mu_2}{M}. \quad (23)$$

For $i = 1$

If $j = 0$

$$\begin{aligned} T1v(1,0) &= \alpha\beta\frac{\lambda}{M} + \alpha^0\beta^0\frac{\mu_1}{M} \\ &\quad +\alpha^1\beta^0(1-\frac{\lambda}{M}-\frac{\mu_1}{M}), \\ &= \alpha\left(\beta\frac{\lambda}{M} + \frac{1}{\alpha}\frac{\mu_1}{M} + \frac{\mu_2}{M}\right), \\ &\le \alpha\left(\alpha\frac{\lambda}{M} + \frac{1}{\beta}\frac{\mu_1}{M} + \frac{\mu_2}{M}\right) \quad (24) \end{aligned}$$

If $0 < j < Q$

$$\begin{aligned} T1v(1,j) &= \alpha\beta^{j+1}\frac{\lambda}{M} + \alpha^0\beta^j\frac{\mu_1}{M} \\ &\quad +\alpha\beta^{j-1}\frac{\mu_2}{M}, \\ &= \alpha\beta^j\left(\beta\frac{\lambda}{M} + \frac{1}{\beta}\frac{\mu_1}{M} + \frac{1}{\beta}\frac{\mu_2}{M}\right), \\ &\le \alpha\beta^j\left(\alpha\frac{\lambda}{M} + \frac{1}{\beta}\frac{\mu_1}{M} + \frac{\mu_2}{M}\right) \quad (25) \end{aligned}$$

If $j = Q$

$$\begin{aligned} T1v(1,Q) &= \alpha\beta^0\frac{\lambda}{M} + \alpha^0\beta^Q\frac{\mu_1}{M} \\ &\quad +\alpha^1\beta^{Q-1}(\frac{\mu_2}{M}), \\ &= \alpha\beta^Q\left(\frac{1}{\beta^Q}\frac{\lambda}{M}\right. \\ &\quad \left.+\frac{1}{\alpha}\frac{\mu_1}{M} + \frac{1}{\beta}\frac{\mu_2}{M}\right) \quad (26) \end{aligned}$$

From (24), (25) and (26) we have

$$\rho\mathbf{1}_{(i=1,0\le j\le Q)} = \alpha\frac{\lambda}{M} + \frac{1}{\beta}\frac{\mu_1}{M} + \frac{\mu_2}{M}. \quad (27)$$

From (23) and (27) we have

$$\begin{aligned} \rho1(\alpha,\beta) &= \max\left\{\alpha\frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{1}{\beta}\frac{\mu_2}{M}\right. \\ &\quad \left.,\alpha\frac{\lambda}{M} + \frac{\mu_2}{M} + \frac{1}{\beta}\frac{\mu_1}{M}\right\}. \quad (28) \end{aligned}$$

$\rho1(\alpha,\beta) < 1$ when $1 < \beta < \frac{\mu}{\lambda}$ and $\beta < \alpha < 1 + \left(1 - \frac{1}{\beta}\right)$, then we obtain

$$T1v(i,j) \le \rho1(\alpha,\beta)v(i,j) \quad (29)$$

for all $0 \le i \le 1, 0 \le j \le Q$.

And it follows that the v-norm of $T1$ is equal to $\rho1(\alpha,\beta)$ which proves the claim. □

In the following lemma we will identify the range for α and β that leads to finite v-norm of $P1$. For that, we choose the measurable function

$$h1(i,j) = \mathbf{1}_{\{i=0,j=0\}} = \begin{cases} 1 & \text{for } i=j=0 \\ 0 & \text{otherwise} \end{cases} \quad (30)$$

and the probability measure

$$\sigma1_{(i,j)} = P_{(0,0)\to(i,j)}. \quad (31)$$

Lemma 4. *Provided that (13) holds, the v-norm of* $\pi1$ *is bounded by*

$$\begin{aligned} \|\pi1\|_v &= \frac{\pi1_{(0,0)}}{1-\rho1(\alpha,\beta)}\left(1+(\alpha-1)\frac{\lambda}{M}\right) \quad (32) \\ &= C_0(\alpha,\beta) < \infty, \quad (33) \end{aligned}$$

Where $\rho1(\alpha,\beta)$ *was defined in (28)*

Proof. According to equation (9), we have

$$\|\pi1\|_v \le \frac{(\sigma1v)(\pi h)}{1-\rho1}.$$

By definition

$$\sigma1v = \sum_{i=0}^{1}\sum_{j=0}^{Q}\sigma1_{(i,j)}v(i,j) = 1+(\alpha-1)\frac{\lambda}{M}. \quad (34)$$

and

$$\pi1h1 = \sum_{i=0}^{1}\sum_{j=0}^{Q}\pi1(i,j)h1(i,j) = \pi1(0,0) > 0. \quad (35)$$

Hence

$$\|\pi1\|_v = \frac{\pi1_{(0,0)}}{1-\rho1(\beta)}\left(1+(\alpha-1)\frac{\lambda}{M}\right) = C_0(\alpha,\beta).$$

□

Let $\beta_0 = \sup\{\beta : \rho1(\alpha,\beta) < 1\}$ and $\alpha_0 = \sup\{\alpha : \rho1(\alpha,\beta) < 1\}$

Theorem 5. *For all* β *such that* $1 < \beta < \beta_0$ *the discrete time Markov chain describing the overflow queue with finite buffers is v-strongly stable for the test function* $v(k,l) = \alpha^k\beta^l$.

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Proof. We have $\pi 1 h1 = \pi 1(0,0)$, $\sigma 1\mathbf{1} = 1$, and

$$\sigma 1 h1 = \sigma 1_{(0,0)} = 1 - \frac{\lambda}{M} > 0.$$

$$T1_{(i,j)\to(m,n)} = \begin{cases} 0 \text{ if } i = j = 0, \\ P1_{(i,j);(m,n)} \text{ otherwise.} \end{cases}$$

Hence, the Kernel $T1$ is non negative.

We verify that $\|P1\|_\upsilon < \infty$. We have $T1 = P1 - h1 \circ \sigma 1$ then $P1 = T1 + h1 \circ \sigma 1$.

$$\|P1\|_\upsilon \leq \|T1\|_\upsilon + \|h1\|_\upsilon \cdot \|\sigma 1\|_\upsilon.$$

Or, according to equation (29)

$$\|T1\|_\upsilon \leq \rho 1(\alpha,\beta) < 1. \tag{36}$$

According to equations (4) and (3), we have

$$\|h1\|_\upsilon = \sup_{i=0}^{1} \sup_{j=0}^{Q} \frac{|h1(i,j)|}{\upsilon(i,j)} = 1,$$

and

$$\begin{aligned} \|\sigma 1\|_\upsilon &= \sum_{i=0}^{1}\sum_{j=0}^{Q} \upsilon(i,j)|\sigma 1_{(i,j)}|, \\ &= 1 + (\alpha - 1)\frac{\lambda}{M}, \\ &\leq 1 + (\alpha_0 - 1)\frac{\lambda}{M} < \infty. \end{aligned}$$

where $\alpha_0 = \sup\{\alpha : \rho 1(\alpha,\beta) < 1\}$.
Then

$$\|P1\|_\upsilon < \infty.$$

□

By Theorem 5, the general bound provided Theorem 1 can be applied to the kernels $\widetilde{P}$ and $P1$ for our overflow model. Specifically, we will insert the individual bounds provided in Lemma 2, Lemma 3 and Lemma 4, which yields the following result.

Theorem 6. *Let $\widetilde{P}$ and $P1$ be the steady state joint queue size distributions of discrete time Markov chains in the overflow model with finite capacity and the overflow model with infinite capacity respectively.*
For all $1 < \beta < \beta_0$ and $\alpha_0 = \sup\{\alpha : \rho 1(\alpha,\beta) < 1\}$, and under the condition

$$\triangle_1(\alpha,\beta) < \frac{1 - \rho 1(\alpha,\beta)}{C_{01}(\alpha,\beta)},$$

We have the following result:

$$\begin{aligned} \|\pi 1 - \widetilde{\pi}\|_\upsilon &\leq \frac{C_{01}(\alpha,\beta) C1(\alpha,\beta)\triangle_1(\alpha,\beta)}{1 - \rho 1(\alpha,\beta) - C1(\alpha,\beta)\triangle_1(\alpha,\beta)}, \\ &= \mathbf{SSB_1}(\alpha,\beta). \end{aligned} \tag{37}$$

Where $C1(\alpha,\beta) = 1 + C_{01}(\alpha,\beta)$.

Proof. Note that if $\beta \in]1,\beta_0[$ and $\alpha \in]\beta,\alpha_0[$ already implies $C_{01}(\alpha,\beta) < \infty$ and $\rho 1(\alpha,\beta) < 1$. Hence lemma 2 and lemma 4 apply.

□

3.2.2 Normalization of Rows

in this method, We set

$$R(i,Q) = \sum_{j=1}^{Q} P(i,j),$$

we choose for $1 \leq i,j \leq Q$:

$$P_Q = \frac{P(i,j)}{R(i,Q)}.$$

So, we propose the following truncation

$$\begin{cases} P2_{(1,Q)\to(1,Q-1)} = \frac{\mu_2}{\mu_1+\mu_2}; \\ P2_{(1,Q)\to(0,Q)} = \frac{\mu_1}{\mu_1+\mu_2}; \\ P2_{(i,j)\to(m,n)} = \widetilde{P}_{(i,j)\to(m,n)} \text{ otherwise.} \end{cases} \tag{38}$$

In the following we establish the bounds for the normalization of rows' truncation technique. For this end, it's sufficient to proceed by following the same sketch of proof used in the first case of the truncation.

For our bounds, we require bounds on the basic input entities such as $\pi 2$ and $T2$.
In order to establish bounds, we have to specify υ. Specifically, for $\beta > 1$ and $\alpha > 1$, we will choose

$$\upsilon(k,l) = \alpha^k \beta^l. \tag{39}$$

as our norm-defining mapping.
We introduce the following condition:

$$1 < \frac{\mu}{\lambda} \tag{40}$$

Where $\mu = \min(\mu_1,\mu_2)$, essential for our numerical bound on the deviation between stationary distribution $\pi 2$ and $\widetilde{\pi}$ and a bound on the deviation of the transition kernel $\widetilde{P}$ from $P2$. This bound is provided in the following lemma.

Lemma 7. *If condition (40) is satisfied, then*

$$\begin{aligned}\|P2-\widetilde{P}\| &\leq \frac{1}{\beta}\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\mu_2}{M}\right)\\ &+\frac{1}{\alpha}\left(\frac{\mu_1}{\mu_1+\mu_2}-\frac{\mu_1}{M}\right),\\ &= \triangle_2(\alpha,\beta). \qquad (41)\end{aligned}$$

Proof. By definition, we have

$$\|P2-\widetilde{P}\|_v = \sup_{k=0,1}\sup_{0<l<Q}\frac{1}{v(k,l)}\times \sum_{i=0}^{1}\sum_{j=0}^{Q} v(i,j)|P2_{(k,l);(i,j)}-\widetilde{P}_{(k,l);(i,j)}|,$$

$$= \sup_{0\leq i\leq Q}\sup_{0\leq j\leq N} S'(i,j),$$

where

$$S'(i,j) = \frac{1}{v(i,j)}\times \sum_{m=0}^{Q}\sum_{n=0}^{N} v(m,n)|\widetilde{P}_{(i,j);(m,n)}-P2_{(i,j);(m,n)}|. \qquad (42)$$

For $i=0$

$$S'(i,j)=0, \qquad (43)$$

For $i=1$

if $0\leq j<Q$

$$S'(i,j)=0, \qquad (44)$$

if $j=Q$

$$\begin{aligned}S'(i,j) &= \frac{1}{\alpha^1\beta^Q}\left(\alpha^1\beta^{Q-1}\left|\frac{\mu_2}{M}-\frac{\mu_2}{\mu_1+\mu_2}\right| +\alpha^0\beta^Q\left|\frac{\mu_1}{M}-\frac{\mu_1}{\mu_1+\mu_2}\right|\right),\\ &\leq \frac{1}{\beta}\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\mu_2}{M}\right) +\frac{1}{\alpha}\left(\frac{\mu_1}{\mu_1+\mu_2}-\frac{\mu_1}{M}\right). \qquad (45)\end{aligned}$$

From (43), (44) and (45), we have

$$\begin{aligned}\|P2-\widetilde{P}\| &\leq \frac{1}{\beta}\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\mu_2}{M}\right)+\frac{1}{\alpha}\left(\frac{\mu_1}{\mu_1+\mu_2}-\frac{\mu_1}{M}\right),\\ &\leq \triangle_2(\alpha,\beta).\end{aligned}$$

□

Let $T2$ denote the taboo Markov kernel for taboo state $(0,0)$; more, for $(i,j),(m,n)$, we have

$$T2_{(i,j);(m,n)} = \begin{cases} 0 \text{ if } i=j=0,\\ P_{(i,j);(m,n)} \text{ otherwise.}\end{cases} \qquad (46)$$

Lemma 8. *Provided that (40) holds, we have*

$$\|T2\|_v = \max\left\{\alpha\frac{\lambda}{M}+\frac{1}{\beta}\frac{\mu_2}{M}+\frac{\mu_1}{M}, \frac{1}{\beta}\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\mu_2}{M}\right)+\frac{1}{\alpha}\left(\frac{\mu_1}{\mu_1+\mu_2}-\frac{\mu_1}{M}\right)\right\},$$

$$=\rho2(\alpha,\beta)<1. \qquad (47)$$

Proof. We have

$$T2v(i,j)=\sum_{m=0}^{1}\sum_{n=0}^{Q} v(m,n)T2_{(i,j);(m,n)}.$$

For $i=0$

If $j=0$

$$T2v(0,0)=0 \qquad (48)$$

If $0<j\leq Q$

$$\begin{aligned}T2v(0,j) &= \alpha^1\beta^j\frac{\lambda}{M}+\alpha^0\beta^{j-1}\frac{\mu_2}{M} +\alpha^0\beta^j\frac{\mu_1}{M},\\ &= \beta^j\left(\alpha\frac{\lambda}{M}+\frac{1}{\beta}\frac{\mu_2}{M}+\frac{\mu_1}{M}\right)\\ &= \beta^j\rho_1. \qquad (49)\end{aligned}$$

For $i=1$

If $j=0$

$$\begin{aligned}T2v(1,0) &= \alpha^1\beta^1\frac{\lambda}{M}+\alpha^0\beta^0\frac{\mu_1}{M} +\alpha^1\beta^0\frac{\mu_2}{M},\\ &= \alpha\left(\beta\frac{\lambda}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\frac{\mu_2}{M}\right)\\ &= \alpha\rho_2. \qquad (50)\end{aligned}$$

If $0<j<Q$

$$\begin{aligned}T2v(1,j) &= \alpha^1\beta^j\frac{\lambda}{M}+\alpha^0\beta^j\frac{\mu_1}{M} +\alpha^1\beta^{j-1}\frac{\mu_2}{M},\\ &= \alpha\beta^j\left(\beta\frac{\lambda}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\frac{\mu_2}{M}\right)\\ &= \alpha\beta^j\rho_3. \qquad (51)\end{aligned}$$

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If $j = Q$

$$\begin{aligned} T2\upsilon(1,Q) &= \alpha^1\beta^{Q-1}\left(\frac{\mu_2}{\mu_1+\mu_2}\right) \\ &+\alpha^0\beta^Q\left(\frac{\mu_1}{\mu_1+\mu_2}\right), \\ &= \alpha^1\beta^Q\left\{\frac{1}{\beta}\left(\frac{\mu_2}{\mu_1+\mu_2}-\frac{\mu_2}{M}\right)\right. \\ &\left.+\frac{1}{\alpha}\left(\frac{\mu_1}{\mu_1+\mu_2}-\frac{\mu_1}{M}\right)\right\} \\ &= \alpha^1\beta^Q\rho_5. \end{aligned} \quad (52)$$

From (48), (49), (50) and (51), we obtain

$$T2\upsilon(i,j) \leq \rho2(\alpha,\beta)\upsilon(i,j)$$

where

$$\rho2(\alpha,\beta) = \max(\rho_1,\rho_2,\rho_3,\rho_4).$$

If $\beta > 1$ and $\beta < \alpha < 1 + (1-\frac{1}{\beta})\frac{\mu}{\lambda}$ with $\mu = \min(\mu_1,\mu_2)$, then $\rho2(\alpha,\beta) < 1$
And the υ-norm of $T2$ is equal to $\rho2(\alpha,\beta)$ which proves the claim. □

To proof the v-stabiliry of the Markov chain P, we choose the measurable function

$$h2(i,j) = \mathbf{1}_{\{i=0,j=0\}} = \left\{\begin{array}{c} 1 \text{ for } i=j=0 \\ 0 \text{ otherwise} \end{array}\right. \quad (53)$$

and the measure

$$\sigma2_{(i,j)} = P_{(0,0)\to(i,j)}. \quad (54)$$

Lemma 9. *Provided that (40) holds, the υ-norm of $\pi2$ is bounded by*

$$\begin{aligned} \|\pi2\|_\upsilon &= \frac{\pi2_{(0,0)}}{1-\rho2(\alpha,\beta)}\left(\alpha\frac{\lambda}{M}+\frac{\mu_1}{M}+\frac{\mu_2}{M}\right) \quad (55) \\ &= C_{02}(\alpha,\beta) < \infty. \quad (56) \end{aligned}$$

Proof. We have [1]

$$\|\pi2\|_\upsilon \leq \frac{(\sigma2\upsilon)(\pi2h)}{1-\rho2(\alpha,\beta)}.$$

By definition

$$\begin{aligned} \sigma2\upsilon &= \sum_{i=0}^{1}\sum_{j=0}^{Q}\sigma2_{(i,j)}h2(i,j), \\ &= \alpha\frac{\lambda}{M}+\frac{\mu_1}{M}+\frac{\mu_2}{M}. \end{aligned} \quad (57)$$

and

$$\pi2h2 = \sum_{i=0}^{1}\sum_{j=0}^{Q}\pi(i,j)h(i,j) = \pi(0,0) > 0. \quad (58)$$

Hence

$$\begin{aligned} \|\pi2\|_\upsilon &= \frac{\pi_{(0,0)}}{1-\rho2(\alpha,\beta)}\left(\alpha\frac{\lambda}{M}+\frac{\mu_1}{M}+\frac{\mu_2}{M}\right) \quad (59) \\ &= C_{02}(\alpha,\beta). \quad (60) \end{aligned}$$

□

Let

$$\beta_0 = \sup\{\beta : \rho2(\alpha,\beta) < 1\},$$

and

$$\alpha_0 = \sup\{\alpha : \rho2(\alpha,\beta) < 1\}.$$

Theorem 10. *For all α and β such that $1 < \beta < \beta_0$, $\beta < \alpha < \alpha_0$ the discrete time Markov chain describing the overflow queue with finite buffers is υ-strongly stable for the test function $\upsilon(k,l) = \alpha^k\beta^l$.*

Proof. We have $\pi2h2 = \pi2(0,0)$, $\sigma2\mathbf{1} = 1$, and

$$\sigma2h2 = \sigma2_{(0,0)} = 1 - \frac{\lambda}{M} > 0.$$

$$T2_{(i,j);(m,n)} = \left\{\begin{array}{c} 0 \text{ if } i=j=0, \\ P2_{(i,j);(m,n)} \text{ otherwise.} \end{array}\right.$$

Hence, the Kernel $T2$ is non negative.
We verify that $\|P2\|_\upsilon < \infty$. We have
$T2 = P2 - h2\circ\sigma2$ then $P = T2 + h2\circ\sigma2$.

$$\|P2\|_\upsilon \leq \|T2\|_\upsilon + \|h2\|_\upsilon\cdot\|\sigma2\|_\upsilon.$$

Or, according to equation (47)

$$\|T2\|_\upsilon \leq \rho2(\alpha,\beta) < 1. \quad (61)$$

According to equations (4) and (3), we have

$$\|h2\|_\upsilon = \sup_{i=0}^{1}\sup_{j=0}^{Q}\frac{|h2(i,j)|}{\upsilon(i,j)} = 1,$$

and

$$\begin{aligned} \|\sigma2\|_\upsilon &= \sum_{i=0}^{1}\sum_{j=0}^{Q}\upsilon(i,j)|\sigma2_{(i,j)}|, \\ &= 1 + (\alpha-1)\frac{\lambda}{M}, \\ &\leq 1 + (\alpha_0-1)\frac{\lambda}{M} < \infty. \end{aligned}$$

where $\alpha_0 = \sup\{\alpha : \rho2(\alpha,\beta) < 1\}$.
Then

$$\|P2\|_\upsilon < \infty.$$

□

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By this theorem, the general bound provided by Kartachov [1] can be used to the Kernel $\widetilde{P}$ and $P2$ for our overflow model.

Theorem 11. *Let $\widetilde{P}$ and $P2$ be the steady state joint queue size distributions of discrete time Markov chains in the overflow model with finite capacity and the overflow model with infinite capacity respectively.*
For all $1 < \beta < \beta_0$ and $\alpha_0 = \sup\{\alpha : \rho 2(\alpha,\beta) < 1\}$, and under the condition

$$\triangle_2(\alpha,\beta) < \frac{1-\rho 2(\alpha,\beta)}{C_{02}(\alpha,\beta)},$$

We have the following result:

$$\begin{aligned}\|\pi 2 - \widetilde{\pi}\|_\upsilon &\leq \frac{C_{02}(\alpha,\beta)C2(\alpha,\beta)\triangle_2(\alpha,\beta)}{1-\rho 2(\alpha,\beta)-C2(\alpha,\beta)\triangle_2(\alpha,\beta)} \\ &= \mathbf{SSB_2}(\alpha,\beta). \qquad (62)\end{aligned}$$

Where $C2\alpha,\beta) = 1 + C_{02}(\alpha,\beta)$.

Proof. Note that if $\beta \in]1,\beta_0[$ and $\alpha \in]\beta,\alpha_0[$ already implies $C_{02}(\alpha,\beta) < \infty$ and $\rho 2(\alpha,\beta) < 1$. Hence lemma 7 and lemma 9 apply. □

3.2.3 Uniform Augmentation

Let

$$\theta_{((i,j),Q)} = \sum_{m=0}^{1}\sum_{n=Q+1}^{\infty} P_{(i,j);(m,n)}$$

For $i=0$

$$\theta_{((0,j),Q)} = 0$$

For $i=1$

If $j<Q$

$$\theta_{((1,j),Q)} = 0$$

If $j=Q$

$$\begin{aligned}\theta_{((1,Q),Q)} &= \sum_{n=Q+1}^{\infty} P_{(1,Q);(0,n)} + \sum_{n=Q+1}^{\infty} P_{(1,Q);(1,n)} \\ &= \frac{\lambda}{M}\end{aligned}$$

We propose the following truncation

$$\begin{cases} P3_{(1,Q);(1,Q-1)} = \frac{\mu_2}{M} + \frac{1}{2(Q+1)}\frac{\lambda}{M}; \\ P3_{(1,Q);(0,Q)} = \frac{\mu_1}{M} + \frac{1}{2(Q+1)}\frac{\lambda}{M}; \\ P3_{(1,Q);(i,j)} = \frac{1}{2(Q+1)}\frac{\lambda}{M}; \\ P3_{(i,j);(m,n)} = \widetilde{P}_{(i,j);(m,n)} \text{ otherwise.} \end{cases} \qquad (63)$$

For our bounds, we require bounds on the basic input entities such as $\pi 3$ and $T3$.
In order to establish bounds, we have to specify υ.
Specifically, for $\beta > 1$ and $\alpha > 1$, we will choose

$$\upsilon(k,l) = \alpha^k\beta^l. \qquad (64)$$

as our norm-defining mapping.
We introduce the following condition:

$$1 < \frac{\mu}{\lambda} \qquad (65)$$

Where $\mu = \min(\mu_1,\mu_2)$, essential for our numerical bound on the deviation between stationary distribution $\pi 3$ and $\widetilde{\pi}$ and a bound on the deviation of the transition kernel $\widetilde{P}$ from $P3$. This bound is provided in the following lemma.

Lemma 12. *If condition (65) is satisfied, then*

$$\begin{aligned}\|P3-\widetilde{P}\| &\leq \frac{1}{(\beta-1)}\frac{1}{(Q+1)}\frac{\lambda}{M} \\ &= \triangle_3(\alpha,\beta). \qquad (66)\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}\|P3-\widetilde{P}\|_\upsilon &= \sup_{k=0,1}\sup_{0<l<Q}\frac{1}{\upsilon(k,l)}\times \\ &\quad \sum_{i=0}^{1}\sum_{j=0}^{Q}\upsilon(i,j)\left|P3_{(k,l);(i,j)} - \widetilde{P}_{(k,l);(i,j)}\right|, \\ &= \sup_{0\leq i\leq Q}\sup_{0\leq j\leq N} S"(i,j),\end{aligned}$$

where

$$S"(i,j) =$$

$$\frac{1}{\upsilon(i,j)}\sum_{m=0}^{Q}\sum_{n=0}^{N}\upsilon(m,n)\left|\widetilde{P}_{(i,j);(m,n)} - P3_{(i,j);(m,n)}\right|. \qquad (67)$$

For $i=0$

$$S"(i,j) = 0, \qquad (68)$$

For $i=1$

if $0\leq j<Q$

$$S"(i,j) = 0, \qquad (69)$$

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if $j = Q$

$$\begin{aligned} S"(i,j) &= \frac{1}{\alpha\beta^Q}\left\{\alpha\beta^{Q-1}\frac{1}{2(Q+1)}\frac{\lambda}{M}\right. \\ &+\alpha^0\beta^Q\frac{1}{2(Q+1)}\frac{\lambda}{M} \\ &+\sum_{j=0}^{Q-1}\alpha^0\beta^j\frac{1}{2(Q+1)}\frac{\lambda}{M} \\ &\left.+\sum_{j=0,j\neq Q-1}^{Q}\alpha^1\beta^j\frac{1}{2(Q+1)}\frac{\lambda}{M}\right\}, \end{aligned}$$

$$\leq \left(1+\frac{1}{\alpha}\right)\frac{1}{2(Q+1)}\frac{\lambda}{M}\left(1+\frac{1}{\beta-1}\right).(70)$$

From (68), (69) and (70), we have

$$\begin{aligned} \|P3-\widetilde{P}\| &\leq \left(1+\frac{1}{\alpha}\right)\frac{1}{2(Q+1)}\frac{\lambda}{M}\left(1+\frac{1}{\beta-1}\right) \\ &= \triangle_3(\alpha,\beta). \end{aligned}$$

□

Let $T3$ denote the taboo Markov kernel for taboo state $(0,0)$; more, for $(i,j),(m,n)$, we have

$$T3_{(i,j);(m,n)} = \begin{cases} 0 \text{ if } i=j=0, \\ P3_{(i,j);(m,n)} \text{ otherwise.} \end{cases} \quad (71)$$

Lemma 13. *Provided that (65) holds, we have*

$$\|T3\|_v = \rho 3(\alpha,\beta) < 1. \quad (72)$$

Proof. We have

$$T3v(i,j) = \sum_{m=0}^{1}\sum_{n=0}^{Q} v(m,n)T2_{(i,j);(m,n)}.$$

For $i = 0$

If $j = 0$

$$T3v(0,0) = 0 \quad (73)$$

If $0 < j \leq Q$

$$\begin{aligned} T3v(0,j) &= \alpha^1\beta^j\frac{\lambda}{M}+\alpha^0\beta^{j-1}\frac{\mu_2}{M}+\alpha^0\beta^j\frac{\mu_1}{M}, \\ &= \beta^j\left(\alpha\frac{\lambda}{M}+\frac{1}{\beta}\frac{\mu_2}{M}+\frac{\mu_1}{M}\right) \\ &= \beta^j\rho_1. \quad (74) \end{aligned}$$

For $i = 1$

If $j = 0$

$$\begin{aligned} T3v(1,0) &= \alpha^1\beta^1\frac{\lambda}{M}+\alpha^0\beta^0\frac{\mu_1}{M} \\ &+\alpha^1\beta^0\frac{\mu_2}{M}, \\ &= \alpha\left(\beta\frac{\lambda}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\frac{\mu_2}{M}\right) \\ &= \alpha\rho_2. \quad (75) \end{aligned}$$

If $0 < j < Q$

$$\begin{aligned} T3v(1,j) &= \alpha^1\beta^j\frac{\lambda}{M}+\alpha^0\beta^j\frac{\mu_1}{M} \\ &+\alpha^1\beta^{j-1}\frac{\mu_2}{M}, \\ &= \alpha\beta^j\left(\beta\frac{\lambda}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\frac{\mu_2}{M}\right) \\ &= \alpha\beta^j\rho_3. \quad (76) \end{aligned}$$

If $j = Q$

$$\begin{aligned} T3v(1,Q) &= \alpha\beta^{Q-1}\left(\frac{\mu_2}{M}+\frac{1}{2(Q+1)}\frac{\lambda}{M}\right) \\ &+\alpha^0\beta^Q\left(\frac{\mu_1}{M}+\frac{1}{2(Q+1)}\frac{\lambda}{M}\right) \\ &+\sum_{j=0}^{Q-1}\alpha^0\beta^j\frac{1}{2(Q+1)}\frac{\lambda}{M} \\ &+\sum_{j=0,j\neq Q-1}^{Q}\alpha^1\beta^j\frac{1}{2(Q+1)}\frac{\lambda}{M}, \end{aligned}$$

$$\begin{aligned} &\leq \alpha^1\beta^Q\left\{\frac{1}{\beta}\frac{\mu_2}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\left(\frac{1}{\alpha}+1\right)\times\right. \\ &\left.\left(1+\frac{1}{(\beta-1)}\right)\frac{1}{2(Q+1)}\frac{\lambda}{M}\right\}, \\ &\leq \alpha^1\beta^Q\left\{\frac{1}{\beta}\frac{\mu_2}{M}+\frac{1}{\alpha}\frac{\mu_1}{M}+\left(\frac{1}{\alpha}+1\right)\times\right. \\ &\left.\left(1+\frac{1}{(\beta-1)}\right)\frac{1}{4}\frac{\lambda}{M}\right\}, \\ &\leq \alpha^1\beta^Q\rho_5. \quad (77) \end{aligned}$$

If $\alpha > \beta$ then

$$\rho 3(\alpha,\beta) = \max(\rho_1,\rho_5).$$

It is easy to proof that for $1 < \beta < \frac{\mu}{\lambda}$ and $\beta < \alpha < 1+\left(1-\frac{1}{\beta}\right)\frac{\mu}{\lambda}$ we have

$$\rho 3(\alpha,\beta) < 1$$

□

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To proof the υ-stabiliry of the Markov chain $P3$, we choose the measurable function

$$h3(i,j) = \mathbf{1}_{\{i=0,j=0\}} = \begin{cases} 1 & \text{for } i=j=0 \\ 0 & \text{otherwise} \end{cases} \quad (78)$$

and the measure

$$\sigma 3_{(i,j)} = P3_{(0,0)\to(i,j)}. \quad (79)$$

Lemma 14. *Provided that (65) holds, and for* $1 < \beta < \frac{\mu}{\lambda}$ *and* $\beta < \alpha < 1 + \left(1 - \frac{1}{\beta}\right)\frac{\mu}{\lambda}$ *the* υ-*norm of* $\pi 3$ *is bounded by*

$$\begin{aligned} \|\pi 3\|_\upsilon &= \frac{\pi 3_{(0,0)}}{1-\rho 3(\alpha,\beta)}\left(\alpha\frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{\mu_2}{M}\right), \\ &= C03(\alpha,\beta). \end{aligned} \quad (80)$$

Proof. We have [1]

$$\|\pi 3\|_\upsilon \le \frac{(\sigma 3\upsilon)(\pi 3h)}{1-\rho 3(\alpha,\beta)}.$$

By definition

$$\begin{aligned} \sigma 3\upsilon &= \sum_{i=0}^{1}\sum_{j=0}^{Q} \sigma 3_{(i,j)} h3(i,j), \\ &= \frac{\lambda}{M}\alpha\beta^0 + \left(\frac{\mu_1}{M} + \frac{\mu_2}{M}\right)\alpha^0\beta^0, \\ &= \alpha\frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{\mu_2}{M}. \end{aligned} \quad (81)$$

and

$$\pi 3h = \sum_{i=0}^{1}\sum_{j=0}^{Q} \pi 3(i,j)h(i,j) = \pi 3(0,0) > 0. \quad (82)$$

Hence

$$\begin{aligned} \|\pi\|_\upsilon &= \frac{\pi 3_{(0,0)}}{1-\rho 3(\alpha,\beta)}\left(\alpha\frac{\lambda}{M} + \frac{\mu_1}{M} + \frac{\mu_2}{M}\right) \\ &= C_{03}(\alpha,\beta). \end{aligned}$$

□

Let

$$\beta_0 = \sup\{\beta : \rho 3(\alpha,\beta) < 1\},$$

and

$$\alpha_0 = \sup\{\alpha : \rho 3(\alpha,\beta) < 1\}.$$

Theorem 15. *For all* α *and* β *such that* $1 < \beta < \beta_0$, $\beta < \alpha < \alpha_0$ *the discrete time Markov chain describing the overflow queue with finite buffers is* υ-*strongly stable for the test function* $v(k,l) = \alpha^k\beta^l$.

Proof. We have $\pi 3h3 = \pi 3(0,0)$, $\sigma 3\mathbf{1} = 1$, and

$$\sigma 3h3 = \sigma 3_{(0,0)} = 1 - \frac{\lambda}{M} > 0.$$

$$T3_{(i,j);(m,n)} = \begin{cases} 0 & \text{if } i=j=0, \\ P3_{(i,j);(m,n)} & \text{otherwise.} \end{cases}$$

Hence, the Kernel $T3$ is non negative.
We verify that $\|P3\|_\upsilon < \infty$. We have $T3 = P3 - h3\circ\sigma 3$ then $P3 = T3 + h3\circ\sigma 3$.

$$\|P3\|_\upsilon \le \|T3\|_\upsilon + \|h3\|_\upsilon \cdot \|\sigma 3\|_\upsilon.$$

Or, according to equation (72)

$$\|T3\|_\upsilon \le \rho 3(\alpha,\beta) < 1. \quad (83)$$

According to equations (4) and (3), we have

$$\|h3\|_\upsilon = \sup_{i=0}^{1}\sup_{j=0}^{Q}\frac{|h3(i,j)|}{\upsilon(i,j)} = 1,$$

and

$$\begin{aligned} \|\sigma 3\|_\upsilon &= \sum_{i=0}^{1}\sum_{j=0}^{Q}\upsilon(i,j)|\sigma 3_{(i,j)}|, \\ &= 1 + (\alpha-1)\frac{\lambda}{M}, \\ &\le 1 + (\alpha_0-1)\frac{\lambda}{M} < \infty. \end{aligned}$$

where $\alpha_0 = \sup\{\alpha : \rho 3(\alpha,\beta) < 1\}$.
Then

$$\|P3\|_\upsilon < \infty.$$

□

By this theorem, the general bound provided by Kartachov [1] can be used to the Kernel $\widetilde{P}$ and $P3$ for our overflow model.

Theorem 16. *Let* $\widetilde{P}$ *and* $P3$ *be the steady state joint queue size distributions of discrete time Markov chains in the overflow model with finite capacity and the overflow model with infinite capacity respectively.*
For all $1 < \beta < \beta_0$ *and* $\alpha_0 = \sup\{\alpha : \rho 3(\alpha,\beta) < 1\}$, *and under the condition*

$$\triangle_3(\alpha,\beta) < \frac{1-\rho 3(\alpha,\beta)}{C_{03}(\alpha,\beta)},$$

We have the following result:

$$\begin{aligned} \|\pi 3 - \widetilde{\pi}\|_\upsilon &\le \frac{C_{03}(\alpha,\beta)C3(\alpha,\beta)\triangle_3(\alpha,\beta)}{1-\rho 3(\alpha,\beta) - C3(\alpha,\beta)\triangle_3(\alpha,\beta)}, \\ &= \mathbf{SSB_3}(\alpha,\beta). \end{aligned} \quad (84)$$

Where $C3(\alpha,\beta) = 1 + C_{03}(\alpha,\beta)$.

Proof. Note that if $\beta \in]1, \beta_0[$ and $\alpha \in]\beta, \alpha_0[$ already implies $C_{03}(\alpha, \beta) < \infty$ and $\rho(\alpha, \beta) < 1$. Hence lemma 5.1 and lemma 5.3 apply. □

4 Numerical Examples

In this section we will apply our bounds put forward in Theorem 6, Theorem 11 and Theorem 16. Below we give the numerical results of the computing of the three bounds $\mathbf{SSB}_1$, $\mathbf{SSB}_2$ and $\mathbf{SSB}_3$, where we set $\lambda = 0.1$, $\mu_1 = 2.5$, $\mu_2 = 2$, $\alpha = 6.8$ and $\beta = 6.7$. Table 1 (see also the figure) shows the numerical values of the three computed bounds for the used techniques of truncation, which are:

$\mathbf{SSB}_1$: Augmentation of the first column;

$\mathbf{SSB}_2$: Normalization of rows;

$\mathbf{SSB}_3$: Uniform Augmentation.

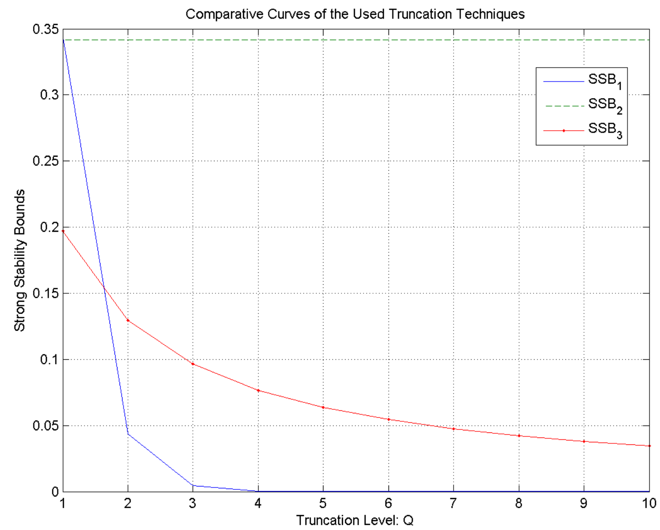

Figure 1: Les deux bornes obtenues par l'augmentation uniforme

Q	$\mathbf{SSB}_1$	$\mathbf{SSB}_2$	$\mathbf{SSB}_3$
1	0.344803530673826	0.341674708676110	0.197024072883162
2	0.043597387957457	0.341641102156400	0.129427358770771
3	0.004641753432986	0.341640279388323	0.096367343570474
4	$4.288049412137165e-004$	0.341640260200825	0.076760384611717
5	$3.581751061599958e-005$	0.341640259761924	0.063783050422334
6	$2.781737851868289e-006$	0.341640259751964	0.054559125816011
7	$2.045226117079339e-007$	0.341640259751739	0.047665958926824
8	$1.441282369176394e-008$	0.341640259751734	0.042319219279720
9	$9.813867379097345e-010$	0.341640259751734	0.038050999520226
10	$6.500052949288499e-011$	0.341640259751734	0.034564866103792

Table 1: Numerical results of the used truncation techniques

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From these numerical results, it is easy to see that, the values of our bounds $\mathbf{SSB}_1$, $\mathbf{SSB}_2$ and $\mathbf{SSB}_3$ decrease as the value of level truncation Q increases and, for the fixed parameters of our model, the technique of the augmentation of the first column provides the best approximation to π while that of the normalization of rows provides the worst.

5 Further Research

Analytical solutions for multi-server queues have been obtained for a few special cases and, many approximation techniques of truncation have been developed on the performance analysis of this kind of queueing models. The error bound results are essentially based on the strong stability approach. This approach is also applicable to other performance measures and to more general queueing networks such that retrial multi-server queues. Further research in this direction is thus recommended.

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Abstract: Queueing network models are among the most natural for quantitative analysis. However most models have no product form solutions for the steady state distribution. Besides, when we compute the solutions for infinite state space of this kind of models, the state-space has to be truncated, in some way, into a finite one. Many truncation techniques are used in the order to approximate the steady state distribution of the infinite state space of these models by that of the truncated one. In this paper, we show numerically comparing some obtained strong stability perturbation bounds that the augmentation of the first column provides the best truncation technique to approximate the steady state distribution of an overflow model.

Key–Words: Queueing, State-space truncation, Overflow model, Approximation, Algorithm