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MRSPN analysis of Semi-Markovian finite source retrial queues

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Abstract In this paper, the analysis of Semi-Markovian single server retrial queues by means of Markov Regenerative Stochastic Petri Nets (*MRSPN*) is considered. We propose *MRSPN* models for the two retrial queues *M/G/1/N/N* and *M/G/1/N/N* with orbital search. By inspecting the reduced reachability graph of both *MRSPN* models, the qualitative analysis is obtained. The quantitative analysis is carried out after constructing their one step transition probability matrix and computing the steady state probability distribution of each tangible marking. As an example, the queue $M/Hypo_2/1/2/2$ is treated in order to illustrate the functionality of the *MRSPN* approach. The exact performance measures (mean number of customers in the system, mean response time, mean waiting time,...) are computed for different parameters of the two systems by an algorithm elaborated in Matlab environment.

Keywords Retrial systems · Markov Regenerative Process · Markov Regenerative Stochastic Petri Nets · Embedded Markov Chain · Steady state · Orbital search

1 Introduction

In retrial queues (queueing systems with repeated calls) an incoming customer having found the server busy does not exit the system but joins the orbit to repeat its demand after a random period (see Fig. 1). Retrial queues are widely studied by several authors: Kosten (1947), Wilkinson (1956), Cohen (1957). A survey work on the topic is written by Falin and Templeton (1997). An exhaustive bibliography is given in Artalejo (2010). These queueing

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models appear in many practical applications such as: communication systems, computer systems, telephone systems, etc.

The space heterogeneity of retrial queues is caused by the flow of the repeated calls which renders the structure of the underlying stochastic processes more complicated. Therefore, the analysis of retrial queueing systems is very difficult. In order to analyze the performance of these systems, an important number of different approximating approaches and algorithms are proposed (Abramov 2006; Artalejo and Gomez-Corral 1995; Artalejo and Pozo 2002; Berjdoudj and Aïssani 2004; Gomez-Corral 2006; Lopez-Herrero 2006; Stepanov 1983; Yang et al. 1994).

In some real applications on queueing theory, it is reasonable to assume that the rate of generation of new primary calls decreases as the number of customers in the system increases (see Falin and Artalejo 1998; Janssens 1997). This situation can be modeled by the quasi random input or by the finite source systems. Pòsafalvi and Sztrik (1987) consider the finite source queueing system with server's breakdowns. Takagi (1993) reviews the classical finite source queueing systems. Choi et al. (1994) carry out the transient and steady state analysis of *MRSPN*, as example $M/G/1/2/2$ is analyzed. Oliver and Kishor (1991) study an $M/M/1//N$ queue with vacation by means of *GSPN*. Furthermore, performance analysis of the queueing system $M/G/1//N$ with different vacation schemes is given by Ramanath and Lakshmi (2006) by using the *MRSPN* tool.

Finite source retrial queues (*FSRQ*) are introduced by Kornyshev (1969). de Kok (1984) with the regenerative process find a recursive scheme for computing the limiting probabilities of $M(\lambda_{ni})/G/1//N$ retrial system. Ohmura and Takahashi (1985) obtain the limiting distribution of $M/G/1//N$ with retrials, by applying the supplementary variable method combined with the discrete transformation. Falin and Artalejo (1998) use the previous approach to express the main characteristics in terms of server utilization of the $M/G/1//N$ retrial system. Kulkarni and Choi (1990) deal with *FSRQ* in which the server subject to breakdowns. Artalejo and Gomez-Corral (1995) present an approximative approach based on the maximum entropy for $M/G/1//N$ with different retrial policies. Li and Yang (1995) treat the *FSRQ* with vacation. During the last decade, an important number of papers on this topic have been published Almasi et al. (2005) (Homogeneous *FSRQ*), Gharbi and Ioualalen (2006), use the Markovian *GSPN* to analyze retrial systems and Zhang and Wang (2013), Wang et al. (2011), etc.

The orbital search behavior introduced in Artalejo et al. (2002) with the purpose to reduce the server's idle time. After termination of service, with probability p the server required

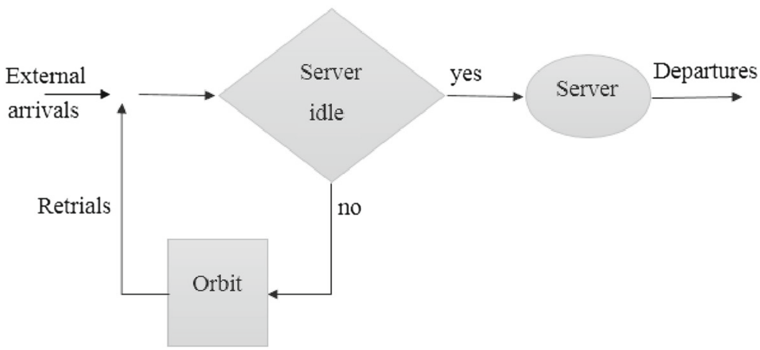


Fig. 1 Queueing system with repeated calls

to search for customer from the orbit. For the papers devoted to orbital search mechanism of infinite retrial queues see Artalejo et al. (2002), Dudin et al. (2004) and Sumitha and Chandrika (2012). In finite source retrial queues with orbital search, we found the papers Wüchner et al. (2008, 2009a, b). Particularly, in Wüchner et al. (2009a) the authors discuss the maximum response time that appears in this *FSRQ*.

In Ikhlef et al. (2014), we studied the performance evaluation of $M/G/1/2/2$ retrial queue using the *MRSPN* tool. In this work we generalized our study to modeling and analyzing the performances of the two complex retrial queues *M/G/1/N/N* and *M/G/1/N/N* with orbital search. We use the approach introduced by Choi based on the theory of Markov Regenerative Process (*MRP*). The structure of the transition probability matrix P of the Embedded Markov Chain (*EMC*) related to each systems is non homogeneous due to the arrival flow from the orbit and/or the quasi random input. Unfortunately, they are not an $M/G/1\text{-}type$ (Neuts 1989). The performance indices of the two *MRSPN* proposed for the two systems (*M/G/1/N/N* and *M/G/1/N/N* with orbital search) are computed by an algorithm elaborated in Matlab environment.

The remainder of this paper is structured as follows. In Sect. 2 we introduce some concepts related to *MRSPN*. In Sects. 3 and 4, we describe the *MRSPN* associated to the two systems *M/G/1/N/N* with retrials and *M/G/1/N/N* with orbital search. In Sect. 5, some performance measures are computed and graphical results are depicted. Finally, the Sect. 6 concludes the paper.

2 Non Markovian Stochastic Petri Nets

Stochastic Petri Nets are proposed as model for analyzing the performance and reliability of complex systems. The notion of time is added to classical Petri Nets (*PN*) by assignment, with transitions, a random variable called "firing time", these transitions are indicated by "timed transitions". The firing time expresses the delay from the enabling condition to the firing of the transition. Two large classes of Stochastic Petri Nets are defined according to the type of firing times, discrete time Stochastic Petri Nets and continuous time Stochastic Petri Nets.

The main extensions of the latter class are Stochastic Petri Nets (*SPN*), where exponentially distributed firing time is associated to each transition. They are defined by Molloy (1982) then extended by Marsan et al. (1984) to a class of Generalized Stochastic Petri Nets (*GSPN*) by allowing zero firing times transitions (immediate transitions). The underlying stochastic process of *SPN* or *GSPN* is a Continuous Time Markov Chain (*CTMC*). Dugan et al. (1985) define Extended Stochastic Petri Nets (*ESPN*) that allow generally distributed firing times. Under some hypothesis the nature of the underlying stochastic process of *ESPN* is a Semi-Markovian process (*SMP*). Deterministic and Stochastic Petri Nets (*DSPN*) (Marsan and Chiola 1987) are defined with the aim of combining exponential and deterministic firing times into a single model. Choi et al. (1994) introduce a new extension named Markov Regenerative Stochastic Petri Nets (*MRSPN*), where timed transitions can fire according to an exponentially or any other generally distributed firing times. When at most one generally distributed timed transition is enabled in each marking, the underlying stochastic process of *MRSPN* and *DSPN* belongs to the class of Markov Regenerative Process (*MRP*). Puliafito et al. (1998) propose Concurrent Generalized Petri Nets (*CGPN*) which constitute a generalization of all above extensions. The *CGPN* include simultaneous enabling of generally distributed timed transitions. Under the restriction that the generally distributed timed transitions are all enabled at the same instants, the underlying stochastic process of the *CGPN* is still a *MRP*.

Different approaches have been explored in the literature for dealing with-non exponentially distributed firing times: approximate analysis by phase type expansion (Cumani 1985), Markov renewal theory (Choi et al. 1994), method of supplementary variable (Cox 1955),…An analytical approach for the derivation of expression for the steady state of *MRSPN* is proved by Choi et al. (1994), where at most one generally distributed transition is enabled in each marking and its associated memory policy is of enabling type (Marsan et al. 1989). This approach is based on the observation that the underlying stochastic process $\{M(t), t \geq 0\}$ enjoys the absence of memory at certain instants of time $(t_0, t_1, t_{2,\ldots})$. These instants are referred as regeneration points. An Embedded Markov Chain (*EMC*) $\{Y_n, n \geq 0\}$ can be defined at these regeneration points. For the quantitative analysis of the *MRSPN* we need to compute:

- the matrix $K(t)$, called global kernel, given by:

$$K_{ij}(t) = P\{Y_1 = j,\ t_1 \leq t / Y_0 = i\}\ \ i, j \in \Omega. \tag{1}$$

  where Ω is the set of states of tangible markings. This matrix describes the process behavior immediately after the next Markov regenerative point. The one step transition probability matrix P of the *EMC* is derived from the global kernel $K(t)$, indeed:

$$P = K(\infty). \tag{2}$$

- the matrix $E(t)$, called the local kernel, given by:

$$E_{ij}(t) = P\{M(t) = j,\ t_1 > t / Y_0 = i\}. \tag{3}$$

  This matrix describes the behavior between two Markov regeneration points.

When the *EMC* is finite and irreducible its steady state probability vector $v = (v_1, v_2, \ldots, v_j, \ldots)$ is obtained by the solution of the linear system:

$$\begin{cases} vP = v; \\ v\mathbb{1} = 1; \end{cases} \tag{4}$$

where $\mathbb{1}$ is a column vector of ones. The steady state probabilities distributions $\pi = (\pi_1, \pi_2, \ldots, \pi_j, \ldots),\ \ j \in \Omega$ of the *MRP* can be obtained by:

$$\pi_j = \frac{\sum\limits_{k \in \Omega} v_k c_{kj}}{\sum\limits_{k \in \Omega} v_k \sum\limits_{l \in \Omega} c_{kl}}, \tag{5}$$

where $c_{ij} = \int_0^{\infty} E_{ij}(t)dt$.

3 *M/G/1/N/N* queue with classical retrial policy

In this section, we consider a single server retrial queue with finite source (there are N sources). A customer arrives from the source according to a poisson process with parameter λ. When the server is idle the customer immediately occupies the service. The service time distribution follows a general law with probability distribution function $F^g(.)$. If the server is busy, the customer joins the orbit to repeat its demand for service until it finds a free

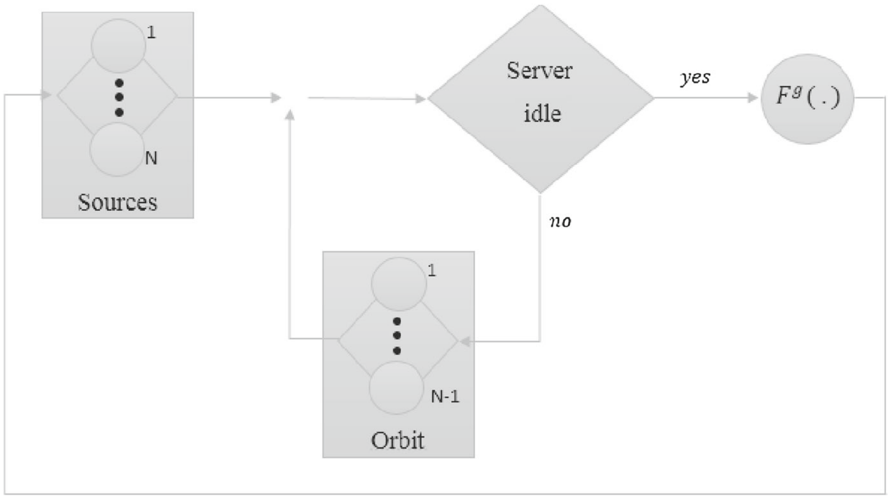


Fig. 2 *M/G/1/N/N* retrial queue with classical retrial policy

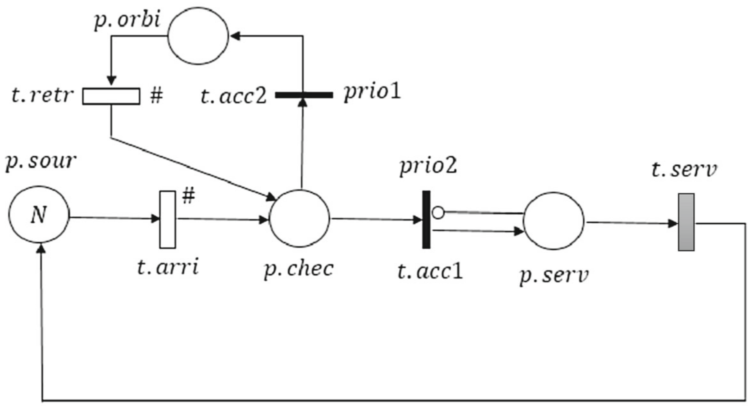


Fig. 3 *MRSPN* models the *M/G/1/N/N* retrial queue with classical retrial policy (Model I)

server. We assume that the intervals between successive repeated attempts are exponentially distributed with rate $i\gamma$ (when the orbit size is i). Figures 2 and 3 respectively shows the schematic and the *MRSPN* model describing the *M/G/1/N/N* queueing system with retrials.

In Fig. 3 black rectangular box represents general (*GEN*) transition, white rectangular boxes represent exponential (*EXP*) transitions, thin bars represent immediate transitions. The 4-tuple ($\#p.sour$, $\#p.chec$, $\#p.serv$, $\#p.orbi$) describes all possible markings of the given *MRSPN* model, where $\#p.sour$, $\#p.chec$, $\#p.serv$, $\#p.orbi$ are respectively, the number of tokens in the places $p.sour$, $p.chec$, $p.serv$, $p.orbi$. The vector $M_0^I = (N, 0, 0, 0)$ is its initial marking (M_0^I means that the system is empty and there are N active sources).

- The *EXP* transition $t.arri$ is enabled when the place $p.sour$ contains at least one token. The firing of the *EXP* transition $t.arri$ consists to destroy a token in the place $p.sour$ and to construct a token in the place $p.chec$ (this means that a primary call is arrived). The firing rate of $t.arri$ is marking dependent and equals $(\#p.sour)\lambda$.

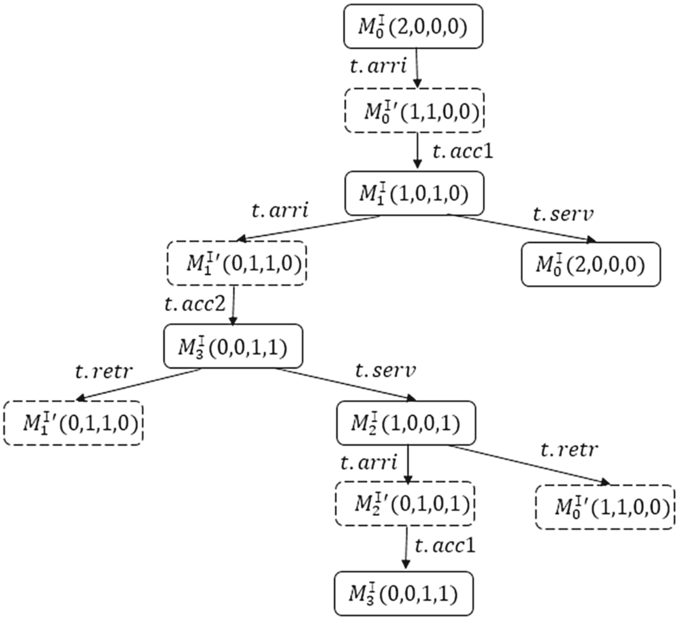
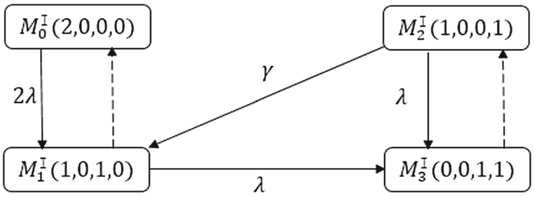


Fig. 4 Reachability tree for the *MRSPN* model of Fig. 3 ($N = 2$)

- The immediate transition $t.acc1$ is enabled when the place $p.chec$ contains a token and $p.serv$ does not contain a token. The firing of immediate transition $t.acc1$ consists to destroy a token in the place $p.chec$ and to build a token in the place $p.serv$ (this represents the fact that the customer has started its service and the server is moved from the free state to the busy state).
- The *GEN* timed transition $t.serv$ is enabled when the place $p.serv$ contains one token. The firing of the timed transition $t.serv$ consists to destroy a token in the place $p.serv$ and to construct a token in the place $p.sour$ (the costumer has completed its service and joins the source). The server is moved from the busy state to the free state. The firing policy of $t.serv$ is the race with enabling memory (Marsan et al. 1989).
- The immediate transition $t.acc2$ is enabled when the two places $p.chec$ and $p.serv$ contain a token. The firing of the immediate transition $t.acc1$ consists to destroy a token in the place $p.chec$ and to construct a token in the place $p.orbi$ (the customer joins the orbit). The immediate transition $t.acc1$ has higher priority than the immediate transition $t.acc2$.
- The *EXP* transition $t.retr$ is enabled when the place $p.orbi$ contains at least one token. The firing of the timed transition $t.retr$ consists to destroy a token in the place $p.orbi$ and to construct a token in the place $p.chec$. The firing rate of $t.retr$ is marking dependent and equals $(\#p.orbi)\gamma$.

When the place $p.sour$ contains two tokens ($N = 2$), the reachability tree which describes all possible states of *MRSPN* model starting from the initial marking $M_0^I = (2, 0, 0, 0)$ is given in Fig. 4. The boxes with rounded corners are tangible markings and the boxes with discrete rounded corners are vanishing markings.

Fig. 5 Subordinated *CTMC* for the *MRSPN* model of Fig. 3 ($N = 2$)



This reachability tree contains three vanishing markings and four tangible markings. By merging the vanishing markings into their successors tangible markings, we obtain the state transition diagram of the *MRSPN* model depicted in Fig. 5.

In this figure solid arcs indicate the firing of the *EXP* transitions $t.arri$ and $t.serv$, dotted arcs indicate the firing of the *GEN* transition $t.serv$.

Remark The conservation of the number of tokens gives the following equation:

$$\#p.sour + \#p.chec + \#p.serv + \#p.orbi = N. \tag{6}$$

According to this equation, it is clear that $\#p.sour = N - (\#p.chec + \#p.serv + \#p.orbi)$. However, the place $p.chec$ is empty because the sojourn time of a token in this place is negligible. So, $\#p.sour = N - (\#p.serv + \#p.orbi)$. Thus, it is enough to indicate the marking of our *MRSPN* by the number of tokens in the two places $p.serv$ and $p.orbi$. Thus, the system state can be described by the variables $(\#p.serv, \#p.orbi)$, which we call a micro states (Gharbi and Charabi 2012), where:

- # p.serv: is the marking of the place $p.serv$, i.e., represents the state of the server,

$$\#p.serv = \begin{cases} 1, & \text{if there is a token in the place } p.serv\text{, i.e., the server is busy;} \\ 0, & \text{if the place } p.serv \text{ is empty, i.e., the server is idle.} \end{cases}$$

- $\#p.orbi$: is the marking of the place $p.orbi$, i.e., represents the number of customers in the orbit.

Hence, having the micro states $(\#p.serv, \#p.orbi)$, the ordinary states of our *MRSPN* can be obtained by:

$$M^I = (N - (\#p.serv + \#p.orbi), 0, \#p.serv, \#p.orbi). \tag{7}$$

Thus, in the study of the model I, the markings of our $MRSPN$ will be described by the previous micro states.

The states space Ω^I of our *MRSPN* (when $N = 2$) is given by:

$$\Omega^I = \{M_0^I(0,0), M_1^I(1,0), M_2^I(0,1), M_3^I(1,1)\}.$$

Let $M_i^I, M_j^I \in \Omega^I$, the infinitesimal generator matrix $Q^I = [q^I_{M_i^I M_j^I}]$ of the subordinated *CTMC* with respect to *GEN* transition $t.serv$ is given by:

$$Q^I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

The local kernel, $E^I(t) = [E^I_{M^I_i M^I_j}(t)]$, is given by:

$$E^I(t) = \begin{pmatrix} e^{-2\lambda t} & 0 & 0 & 0 \\ 0 & e^{-\lambda t}(1 - F^g(t)) & 0 & (1 - e^{-\lambda t})(1 - F^g(t)) \\ 0 & 0 & e^{-(\gamma+\lambda)t} & 0 \\ 0 & 0 & 0 & 1 - F^g(t) \end{pmatrix}. \quad (9)$$

The global kernel, $K^I(t) = [K^I_{M^I_i M^I_j}(t)]$, is given by:

$$K^I(t) = \begin{pmatrix} 0 & 1 - e^{-2\lambda t} & 0 & 0 \\ \int_0^t e^{-\lambda x} dF^g(x) & 0 & \int_0^t [1 - e^{-\lambda x}] dF^g(x) & 0 \\ 0 & \frac{\gamma}{\gamma+\lambda}[1 - e^{-(\gamma+\lambda)t}] & 0 & \frac{\lambda}{\gamma+\lambda}[1 - e^{-(\gamma+\lambda)t}] \\ 0 & 0 & \int_0^t dF^g(x) & 0 \end{pmatrix}.$$

We suppose that the firing time density function $f^g(.)$ of *GEN* transition $t.serv$ is Hypoexponential distribution with two phases "$Hypo_2(\mu, \frac{\mu}{2})$".

The one step transition probability matrix, $P^I = [P^I_{M^I_i M^I_j}]$, is given by:

$$P^I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\mu^2}{(2\lambda+\mu)(\lambda+\mu)} & 0 & \frac{3\lambda\mu+2\lambda^2}{(2\lambda+\mu)(\lambda+\mu)} & 0 \\ 0 & \frac{\gamma}{\gamma+\lambda} & 0 & \frac{\lambda}{\gamma+\lambda} \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (10)$$

The *MRSPN* model depicted in Fig. 3 ($N = 2$) is bounded and admits $M^I_0 = (0, 0)$ like home state so it is ergodic. We calculate the steady state probabilities of *EMC* by solving the linear system $v^I P^I = v^I$ and $v^I \mathbb{1} = 1$, we obtain:

$$v^I_{(0,0)} = \frac{1}{2} \frac{\gamma\mu^2}{2\gamma\lambda^2 + 3\gamma\lambda\mu + \gamma\mu^2 + 2\lambda^3 + 3\lambda^2\mu}; \quad (11)$$

$$v^I_{(1,0)} = \frac{1}{2} \frac{\gamma(2\lambda+\mu)(\lambda+\mu)}{2\gamma\lambda^2 + 3\gamma\lambda\mu + \gamma\mu^2 + 2\lambda^3 + 3\lambda^2\mu}; \quad (12)$$

$$v^I_{(0,1)} = \frac{1}{2} \frac{\lambda(\gamma+\lambda)(3\mu+2\lambda)}{2\gamma\lambda^2 + 3\gamma\lambda\mu + \gamma\mu^2 + 2\lambda^3 + 3\lambda^2\mu}; \quad (13)$$

$$v^I_{(1,1)} = \frac{1}{2} \frac{\lambda^2(3\mu+2\lambda)}{2\gamma\lambda^2 + 3\gamma\lambda\mu + \gamma\mu^2 + 2\lambda^3 + 3\lambda^2\mu}. \quad (14)$$

The elements $c^I_{M^I_i M^I_j}$ are given by the matrix C^I:

$$C^I = \begin{pmatrix} \frac{1}{2\lambda} & 0 & 0 & 0 \\ 0 & \frac{3\mu+2\lambda}{(2\lambda+\mu)(\lambda+\mu)} & 0 & \frac{6\lambda^2+7\lambda\mu}{u(2\lambda+\mu)(\lambda+\mu)} \\ 0 & 0 & \frac{1}{\gamma+\lambda} & 0 \\ 0 & 0 & 0 & \frac{3}{\mu} \end{pmatrix}. \quad (15)$$

The steady state probabilities distributions of the *MRP* underlying of *MRSPN* model "$\pi^I = (\pi^I_{(0,0)}, \pi^I_{(1,0)}, \pi^I_{(0,1)}, \pi^I_{(1,1)})$" are given by:

$$\pi^I_{(0,0)} = \frac{\gamma\mu^3}{\gamma\mu^3 + 12\gamma\lambda^3 + 18\gamma\lambda^2\mu + 6\gamma\lambda\mu^2 + 22\lambda^3\mu + 6\lambda^2\mu^2 + 12\lambda^4}; \quad (16)$$

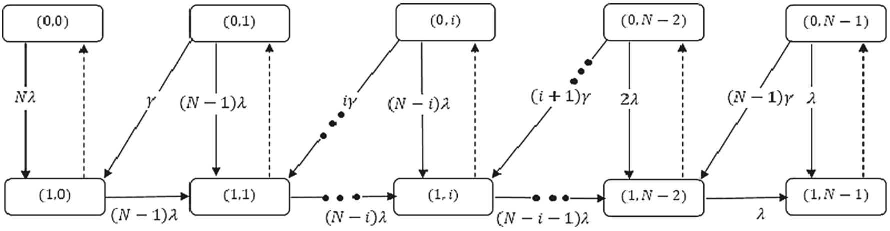


Fig. 6 Subordinated *CTMC* for the *MRSPN* of Fig. 3

$$\pi^I_{(1,0)} = \frac{2\gamma\mu\lambda(3\mu+2\lambda)}{\gamma\mu^3+12\gamma\lambda^3+18\gamma\lambda^2\mu+6\gamma\lambda\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\lambda^4}; \tag{17}$$

$$\pi^I_{(0,1)} = \frac{2\lambda^2\mu(3\mu+2\lambda)}{\gamma\mu^3+12\gamma\lambda^3+18\gamma\lambda^2\mu+6\gamma\lambda\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\lambda^4}; \tag{18}$$

$$\pi^I_{(1,1)} = \frac{2\lambda^2(6\lambda^2+9\lambda\mu+6\gamma\lambda+7\gamma\mu)}{\gamma\mu^3+12\gamma\lambda^3+18\gamma\lambda^2\mu+6\gamma\lambda\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\lambda^4}. \tag{19}$$

For all N we obtain the state transition diagram (see Fig. 6) of the *MRSPN* depicted in Fig. 3 with states space Ω^I such that:

$$\Omega^I = \{M^I_{2i} = (0,i), M^I_{2i+1} = (1,i) : 0 \le i \le N-1\} \; and \; \mid \Omega^I \mid = 2N.$$

The regeneration points are defined as follows:

- Let $t_0 = 0$, the marking of the *MRSPN* depicted in Fig. 3 is in the state $M^I_0 = (0,0)$, the next instant t_1 corresponds to the firing of $t.arri$.
- If at the nth regeneration points t_n, the marking of the *MRSPN* depicted in Fig. 3 is in the state $M^I_{2i} = (0,i), i \ge 1$, the regeneration instant t_{n+1} corresponds to the firing of $t.retr$ or $t.arri$.
- If at the nth regeneration points t_n, the marking of the *MRSPN* depicted in Fig. 3 is in the state $M^I_{2i+1} = (1,i), i \ge 0$, the regeneration instant t_{n+1} corresponds to the firing of $t.serv$.

Let $M^I_i, M^I_j \in \Omega^I$, the infinitesimal generator matrix $Q^I = [q^I_{M^I_i M^I_j}]$ of the subordinated *CTMC* with respect to *GEN* transition $t.serv$ is given by:

$$q^I_{M^I_i M^I_j} = \begin{cases} -\left(\frac{2N-i-1}{2}\right)\lambda, & \textit{if } 1 \le i < 2N-1, \; i \text{ odd and } j = i; \\ \left(\frac{2N-i-1}{2}\right)\lambda, & \textit{if } 1 \le i < 2N-1, \; i \text{ odd and } j = i+2; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

The local Kernel $E^I(t) = [E^I_{M_i^I M_j^I}(t)]$ is given by:

$$E^I_{M_i^I M_j^I}(t) = \begin{cases} e^{-N\lambda t}, \quad \textit{if} \;\; i = j = 0; \\ C_{N-\frac{i+1}{2}}^{\frac{j-i}{2}} (1 - e^{-\lambda t})^{\frac{j-i}{2}} (e^{-\lambda t})^{N-\frac{j+1}{2}} \; (1 - F^g(t)), \\ \textit{if} \;\; 1 \le i < 2N, \; i \text{ odd and } i \le j < 2N, \; j \text{ odd}; \\ e^{-\left[\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda\right]t}, \quad \textit{if} \;\; 2 \le i < 2N, i \text{ even and } j = i; \\ 0, \qquad \text{otherwise.} \end{cases} \tag{21}$$

The global kernel $K^I(t) = [K^I_{M_i^I M_j^I}(t)]$ is given by:

$$K^I_{M_i^I M_j^I}(t) = \begin{cases} 1 - e^{-N\lambda t}, \quad \textit{if}\; i = 0 \text{ and } j = 1 \\ \int_0^t C_{N-\frac{i+1}{2}}^{\frac{j-i+1}{2}} (1 - e^{-\lambda x})^{\frac{j-i+1}{2}} (e^{-\lambda x})^{N-\frac{j+2}{2}} dF^g(x), \\ \textit{if} \;\; 1 \le i \le 2N-1, \quad i \; \textit{odd} \text{ and } i - 1 \le j \le 2N-2, \; j \text{ even}; \\ \frac{\frac{i}{2}\gamma}{\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda} \left(1 - e^{-\left[\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda\right]t}\right), \\ \textit{if} \;\; 2 \le i \le 2N-2, i \text{ even and } j = i - 1; \\ \frac{(N-\frac{i}{2})\lambda}{\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda} \left(1 - e^{-\left[\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda\right]t}\right), \\ \textit{if} \;\; 2 \le i \le 2N-2, i \text{ even and } j = i + 1; \\ 0, \qquad \text{otherwise.} \end{cases} \tag{22}$$

The one step transition probability matrix $P^I = [P^I_{M_i^I M_j^I}]$ is given by:

$$P^I_{M_i^I M_j^I} = \begin{cases} 1, \quad \textit{if} \;\; i = 0, j = 1; \\ \int_0^\infty C_{N-\frac{i+1}{2}}^{\frac{j-i+1}{2}} (1 - e^{-\lambda x})^{\frac{j-i+1}{2}} (e^{-\lambda x})^{N-\frac{j+2}{2}} dF^g(x), \\ \textit{if} \;\; 1 \le i \le 2N-1, i \text{ odd and } i - 1 \le j \le 2N-2, \; j \text{ even}; \\ \frac{\frac{i}{2}\gamma}{\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda}, \; \textit{if} \;\; 2 \le i \le 2N-2, \quad i \; \textit{even} \text{ and } j = i - 1; \\ \frac{(N-\frac{i}{2})\lambda}{\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda}, \; \textit{if} \;\; 2 \le i \le 2N-2, \quad i \; \textit{even} \text{ and } j = i + 1; \\ 0, \qquad \text{otherwise.} \end{cases} \tag{23}$$

The matrix $C^I = [c^I_{M_i^I M_j^I}]$ is given by:

$$c^I_{M_i^I M_j^I} = \begin{cases} \frac{1}{N\lambda}, & \textit{if } i = j = 0 \\ \int_0^\infty C_{N-\frac{i+1}{2}}^{\frac{j-i}{2}} (1-e^{-\lambda t})^{\frac{j-i}{2}} (e^{-\lambda t})^{N-\frac{j+1}{2}} \; (1-F^g(t))dt, & \\ \textit{if } 1 \le i < 2N, i \text{ odd and } i \le j < 2N, j \text{ odd}; & \\ \frac{1}{\frac{i}{2}\gamma + (N-\frac{i}{2})\lambda}, & \textit{if } 2 \le i < 2N, i \text{ even and } j = i; \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

After computing the steady state probabilities distributions:

$$\pi^I = (\pi^I_{(0,0)}, \pi^I_{(1,0)}, \ldots, \pi^I_{(0,i)}, \pi^I_{(1,i)}, \ldots, \pi^I_{(0,N-1)}, \pi^I_{(1,N-1)});$$

various performance characteristics of *M/G/1/N/N* with retrials can be derived:

- The mean number of customers in the orbit (n_o):

$$n_o^I = \sum_{i=1}^{N-1} i \; [\pi^I_{(0,i)} + \pi^I_{(1,i)}]. \tag{25}$$

- The mean number of customers in the service or in the orbit (n_s):

$$n_s^I = \sum_{i=1}^{N-1} i \; [\pi^I_{(0,i)} + \pi^I_{(1,i)}] + \sum_{i=0}^{N-1} \pi^I_{(1,i)}. \tag{26}$$

- The mean number of active sources (a_s):

$$a_s^I = N - n_s^I. \tag{27}$$

- The mean generation rate of primary calls ($\overline{\lambda}$):

$$\overline{\lambda}^I = \lambda \left[\sum_{i=0}^{N-1} (N-i) \; \pi^I_{(0,i)} + \sum_{i=0}^{N-2} (N-i-1) \; \pi^I_{(1,i)} \right]. \tag{28}$$

- The mean generation rate of repeated calls ($\overline{\gamma}$):

$$\overline{\gamma}^I = \gamma \sum_{i=1}^{N-1} i \; [\pi^I_{(0,i)} + \pi^I_{(1,i)}]. \tag{29}$$

- The probability that the server is busy (B_s):

$$B_s^I = \sum_{i=0}^{N-1} \pi^I_{(1,i)}. \tag{30}$$

- The blocking probability of a primary calls (B_p):

$$B_p^I = \frac{\lambda \sum_{i=0}^{N-2} (N-i-1) \; \pi^I_{(1,i)}}{\overline{\lambda}^I}. \tag{31}$$

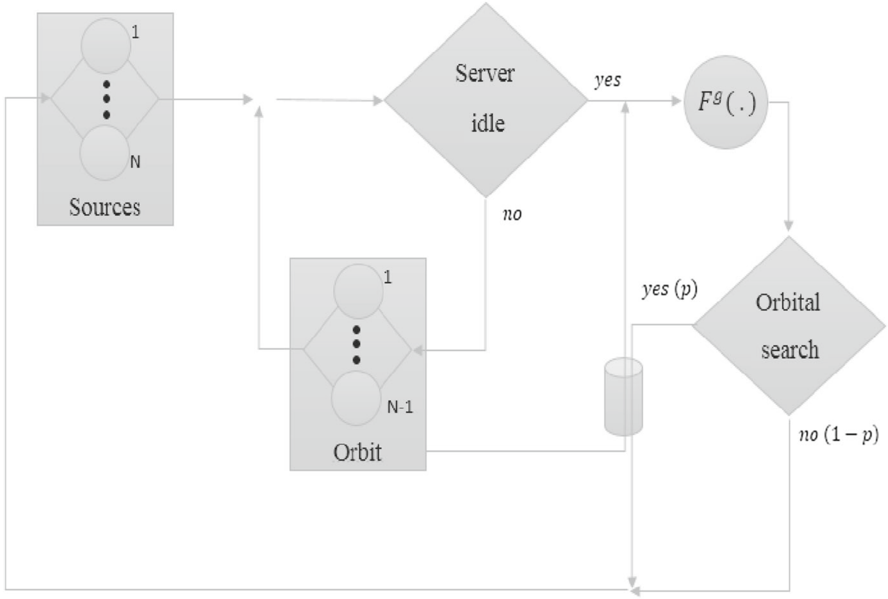


Fig. 7 *M/G/1/N/N* retrial queue with orbital search

- The blocking probability of a repeated calls (B_r):

$$B_r^I = \frac{\gamma \sum_{i=1}^{N-1} i \, \pi_{(1,i)}^I}{\overline{\gamma}^I}. \tag{32}$$

- The mean waiting time ($\overline{\omega}$), from Little's law:

$$\overline{w}^I = \frac{n_o^I}{\overline{\lambda}^I}. \tag{33}$$

- The mean response time (ϖ), from Little's law:

$$\varpi^I = \frac{n_s^I}{\overline{\lambda}^I}. \tag{34}$$

4 *M/G/1/N/N* Retrial queue with orbital search

We study the retrial system *M/G/1/N/N* with orbital search mechanism. After a service completion, with probability p the server takes a customer from the orbit for service and with probability $(1-p)$ the server becomes idle until a new arrival captures the server. We assume that the search time is negligible. Figures 7 and 8 respectively shows the schematic and the *MRSPN* model describing the retrial queue *M/G/1/N/N* with orbital search.

The 5-tuple $(\#p.sour, \#p.chec, \#p.serv, \#p.orbi, \#p.sear)$ describes all possible markings of the given *MRSPN* model. The vector $M_0^{II} = (N, 0, 0, 0, 0)$ is its initial marking.

At the difference of the model I, this model considers the orbital search mechanism. So, we keep the same interpretation as that of the model I for which we add the changes made

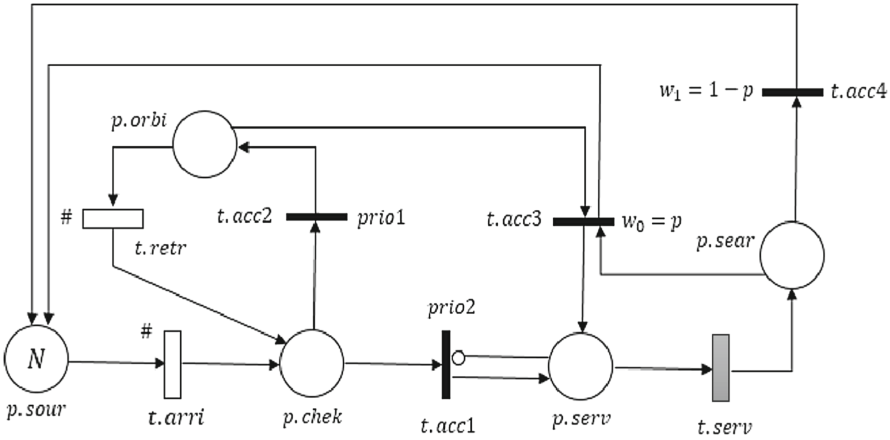


Fig. 8 *MRSPN* models the *M/G/1/N/N* retrial queue with orbital search (Model *II*)

by this mechanism. Thus, the sub-net modeling this mechanism contains two immediate transitions $t.acc3$, $t.acc4$ and one place $p.sear$.

- The firing of the *GEN* transition $t.serv$ consists to destroy a token in the place $p.serv$ and to construct a token in the place $p.sear$. The presence of a token in the place $p.sear$ means that the customer has completed its service and he is ready to join the source.
- The immediate transition $t.acc3$ is enabled when the place $p.sear$ contains a token and the place $p.orbi$ contains at least one token. The instantly firing of the immediate transition $t.acc3$, with weight p, consists to destroy a token in each of the two places $p.sear$, $p.orbi$ and to construct a token in each of the two places $p.serv$, $p.sour$, this means that the server searches a customer in the orbit and the customer, who has finished its service, joins the source.
- The immediate transition $t.acc4$ is enabled when the place $p.sear$ contains a token. The instantly firing of the immediate transition $t.acc4$, with weight $(1-p)$, consists to destroy a token in the place $p.sear$ and to construct a token in the place $p.sour$, this means that the customer joins the source and the server remains idle.

By reasoning in the same way as in the previous model I, the ordinary states of our *MRSPN* given by $M^{II} = (\#p.sour, \#p.chec, \#p.serv, \#p.orbi, \#p.sear)$, can be described by the micro states $M^{II} = (\#p.serv, \#p.orbi)$.

We obtain the state transition diagram (see Fig. 9) of the *MRSPN* model depicted in Fig. 8, its state space:

$$\Omega^{II} = \{M_{2i}^{II} = (0, i),\ M_{2i+1}^{II} = (1, i) : 0 \leq i \leq N-1\} \text{ and } \mid \Omega^{II} \mid = 2N.$$

We suppose that $N = 2$ and the firing time density function $f^g(.)$ of *GEN* transition $t.serv$ is given by Hypoexponential distribution with two phases "$Hypo_2(\mu, \frac{\mu}{2})''$.

The one step transition probability matrix $P^{II} = [P^{II}_{M_i^{II} M_j^{II}}]$ is given by:

$$P^{II} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\mu^2}{(2\lambda+\mu)(\lambda+\mu)} & \frac{p(3\lambda\mu+2\lambda^2)}{(2\lambda+\mu)(\lambda+\mu)} & \frac{(1-p)(3\lambda\mu+2\lambda^2)}{(2\lambda+\mu)(\lambda+\mu)} & 0 \\ 0 & \frac{\gamma}{\gamma+\lambda} & 0 & \frac{\lambda}{\gamma+\lambda} \\ 0 & p & 1-p & 0 \end{pmatrix}. \quad (35)$$

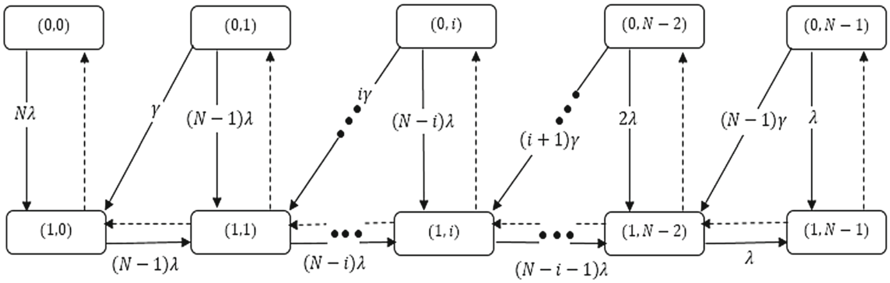


Fig. 9 Subordinated *CTMC* for the *MRSPN* model of Fig. 8

The *MRSPN* model depicted in Fig. 8 ($N = 2$) is bounded and admits $M_0^{II} = (0,0)$ like home state so it is ergodic. We compute the steady state probability vector v^{II} of the *EMC* by solving the linear system $v^{II}P^{II} = v^{II}$ and $v^{II}\mathbb{1} = 1$, we find:

$$v^{II}_{(0,0)} = \frac{\mu^2(p\lambda+\gamma)}{4\lambda^3+6\lambda^2\mu+2p\lambda\mu^2-2p\gamma\lambda^2-3p\lambda\gamma\mu+4\gamma\lambda^2+6\lambda\gamma\mu-2p\lambda^3-3p\lambda^2\mu+2\gamma\mu^2};$$
$$v^{II}_{(1,0)} = \frac{(2\lambda+\mu)(\lambda+\mu)(p\lambda+\gamma)}{4\lambda^3+6\lambda^2\mu+2p\lambda\mu^2-2p\gamma\lambda^2-3p\lambda\gamma\mu+4\gamma\lambda^2+6\lambda\gamma\mu-2p\lambda^3-3p\lambda^2\mu+2\gamma\mu^2};$$
$$v^{II}_{(0,1)} = \frac{\lambda(\lambda+\gamma)(2\lambda+3\mu-2p\lambda-3p\mu)}{4\lambda^3+6\lambda^2\mu+2p\lambda\mu^2-2p\gamma\lambda^2-3p\lambda\gamma\mu+4\gamma\lambda^2+6\lambda\gamma\mu-2p\lambda^3-3p\lambda^2\mu+2\gamma\mu^2};$$
$$v^{II}_{(1,1)} = \frac{\lambda^2(2\lambda+3\mu-2p\lambda-3p\mu)}{4\lambda^3+6\lambda^2\mu+2p\lambda\mu^2-2p\gamma\lambda^2-3p\lambda\gamma\mu+4\gamma\lambda^2+6\lambda\gamma\mu-2p\lambda^3-3p\lambda^2\mu+2\gamma\mu^2}.$$

The steady state probabilities distributions:

$$\pi^{II} = \left(\pi^{II}_{(0,0)}, \pi^{II}_{(1,0)}, \pi^{II}_{(0,1)}, \pi^{II}_{(1,1)}\right)$$

of the *MRP* underlying to the *MRSPN* model, are given by:

$$\pi^{II}_{(0,0)} = \frac{\mu^3(p\lambda+\gamma)}{p\lambda\mu^3+\gamma\mu^3-4p\lambda^3\mu+18\gamma\lambda^2+6\lambda\gamma\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\gamma\lambda^3+12\lambda^4};$$
$$\pi^{II}_{(1,0)} = \frac{2\lambda\mu(p\lambda+\gamma)(2\lambda+3\mu)}{p\lambda\mu^3+\gamma\mu^3-4p\lambda^3\mu+18\gamma\lambda^2+6\lambda\gamma\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\gamma\lambda^3+12\lambda^4};$$
$$\pi^{II}_{(0,1)} = \frac{2\lambda^2\mu(2\lambda+3\mu-2p\lambda-3p\mu)}{p\lambda\mu^3+\gamma\mu^3-4p\lambda^3\mu+18\gamma\lambda^2+6\lambda\gamma\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\gamma\lambda^3+12\lambda^4};$$
$$\pi^{II}_{(1,1)} = \frac{2\lambda^2(6\gamma\lambda+7\gamma\mu+6\lambda^2+9\lambda\mu-2p\lambda\mu)}{p\lambda\mu^3+\gamma\mu^3-4p\lambda^3\mu+18\gamma\lambda^2+6\lambda\gamma\mu^2+22\lambda^3\mu+6\lambda^2\mu^2+12\gamma\lambda^3+12\lambda^4}.$$

If we put $p = 0$ in the above steady state probabilities distributions we constat that the model I is a particular case of the model II.

For all N, the global kernel $K^{II}(t) = [K^{II}_{M_i^{II}M_j^{II}}(t)]$ is given by:

$$K^{II}_{M_i^{II},M_j^{II}}(t)=\begin{cases}1-e^{-N\lambda t}, & \textit{if } i=0 \text{ and } j=1;\\ \int_0^t e^{-\lambda(N-1)x}dF^g(x), & \textit{if } i=1 \text{ and } j=0;\\ \int_0^t C_{N-1}^{\frac{j+1}{2}}(1-e^{-\lambda x})^{\frac{j+1}{2}}(e^{-\lambda x})^{\frac{2N-j-3}{2}}dF^g(x)p, & \\ \textit{if } \ i=1 \text{ and } 1\le j<2N-1,\ j \text{ odd}; & \\ \int_0^t C_{N-1}^{\frac{j}{2}}(1-e^{-\lambda x})^{\frac{j}{2}}(e^{-\lambda x})^{\frac{2N-j-2}{2}}dF^g(x)(1-p), & \\ \textit{if } \ i=1, 2\le j\le 2N-2 \text{ and } j\text{even}; & \\ \frac{\frac{i}{2}\gamma}{\frac{i}{2}\gamma+(N-\frac{i}{2})\lambda}(1-e^{-[\frac{i}{2}\gamma+(N-\frac{i}{2})\lambda]t}), & \\ \textit{if } \ 2\le i\le 2N-2, i \text{ even and } j=i-1; & \\ \frac{(N-\frac{i}{2})\lambda}{\frac{i}{2}\gamma+(N-\frac{i}{2})\lambda}(1-e^{-[\frac{i}{2}\gamma+(N-\frac{i}{2})\lambda]t}), & \\ \textit{if } \ 2\le i\le 2N-2\ , i \text{ even and } j=i+1; & \\ \int_0^t C_{\frac{2N-i-1}{2}}^{\frac{j-i+2}{2}}(1-e^{-\lambda x})^{\frac{j-i+2}{2}}(e^{-\lambda x})^{\frac{2N-j-3}{2}}dF^g(x)p, & \\ \textit{if } \ 3\le i\le 2N-1, i \text{ odd and } i-2\le j<2N-1,\ j \text{ odd}; & \\ \int_0^t C_{\frac{2N-i-1}{2}}^{\frac{j-i+1}{2}}(1-e^{-\lambda x})^{\frac{j-i+1}{2}}(e^{-\lambda x})^{\frac{2N-j-2}{2}}dF^g(x)(1-p), & \\ \textit{if } \ 3\le i\le 2N-1, i \text{ odd and } i-1\le j\le 2N-2,\ j \text{ even}; & \\ 0, & \text{otherwise.}\end{cases} \tag{36}$$

The model II characteristics can be obtained by replacing, in formulas (25)–(34), the steady state probability distribution vector π^I by π^{II}. The model II has the same local kernel with the model I, its one step transition probability matrix $P^{II}=[P^{II}_{M_i^{II}M_j^{II}}]$ is given by $P^{II}=K^{II}(\infty)$.

Remark The given models I and II can be extended easily to queues with constant retrial policy just by omitting in model I (Fig. 3) and model II (Fig. 8) the symbol # for the *EXP* transition "$t.retr$" and changing some elements in the global kernel $K(t)$ and in the local kernel $E(t)$ like this:

$$K_{M_iM_j}(t)=\frac{\gamma\ (1-e^{-[\gamma+(N-\frac{i}{2})\lambda]t})}{\gamma+(N-\frac{i}{2})\lambda},\ \ \textit{if } 2\le i\le 2N-2,\ i \textit{ even and } j=i-1;$$

$$K_{M_iM_j}(t)=\frac{(N-\frac{i}{2})\lambda\ (1-e^{-[\gamma+(N-\frac{i}{2})\lambda]t})}{\gamma+(N-\frac{i}{2})\lambda},\ \ \textit{if } 2\le i\le 2N-2,\ i \textit{ even and } j=i+1;$$

$$E_{M_iM_j}(t)=e^{-[\gamma+(N-\frac{i}{2})\lambda]t},\ \ \textit{if } 2\le i<2N,\ i \textit{ even and } j=i.$$

Table 1 Comparison of stationary distributions of the model $M/H_2/1/N/N$ retrial queue given in Artalejo and Gomez-Corral (1995), with the *MRSPN* (Model I)

i	$p_{(0,i)}$	$p_{(1,i)}$	$\pi^I_{(0,i)}$	$\pi^I_{(1,i)}$
0	0.50585	0,20421	0.5058559	0.2042149
1	0.09927	0.10799	0.0992711	0.1079912
2	0.02249	0.04007	0.0224982	0.0400719
3	0.00463	0.01151	0.0046380	0.0115193
4	0.00079	0.00255	0.0008000	0.0025573
5	0.00010	0.00041	0.0001066	0.0004180
6	$0,96 \times 10^{-5}$	0.00004	$0,97 \times 10^{-5}$	0,0000451
7	$0,44 \times 10^{-6}$	$0,24 \times 10^{-5}$	$0,44 \times 10^{-6}$	$0,24 \times 10^{-5}$

Table 2 Comparison of some characteristics of the model $M/M/1/N/N$ with orbital search given in Wüchner et al. (2009a), with the *MRSPN* (Model II)

Measures	M/M/1/3/3	*MRSPN* (Model II)
n_o^{II}	0.3648981538	0.3648981
n_s^{II}	0.6044528670	0.6044528
a_s^{II}	2.395547133	2.3955472
$\overline{\lambda}^{II}$	0.2395547133	0.2395547
$\overline{w}^{II}$	1.523235126	1.5232351
ϖ^{II}	2.523235126	2.5232351

5 Numerical results

We show how the *MRSPN* model given in Sects. 3 and 4 may be used to obtain the performance measures of the two retrial queueing systems *M/G/1/N/N* (Model I) and *M/G/1/N/N* with orbital search (Model II). The numerical results are established using the algorithm elaborated in Matlab. Graphical results are depicted to investigate the influence of arrival rate, retrial rate and orbital search probability on the mean response time, the mean number of customers in the orbit, and the server utilization.

In Table 1, the model I proposed for the queue *M/G/1/N/N* with classical retrial policy, is validated by the exact numerical results given in Artalejo and Gomez-Corral (1995) for the parameters "$N = 8$, $\lambda = 0.5$, $\gamma = 7.2$, $q = 0.35$, $\mu_1 = 12$, $\mu_2 = 9$". We see that the performance indices corresponding the *MRSPN* associated to *M/G/1/N/N* queue with retrials are similar to those obtained in Artalejo and Gomez-Corral (1995).

In Table 2, the model II proposed to *M/G/1/N/N* retrial queue with orbital search is validated by the exact numerical results given in Wüchner et al. (2009a) for the parameters "$N = 3$, $\lambda = 0.1$, $\gamma = 0.0025$, $\mu = 1$, $p = 0.5$". We see that the performance indices corresponding to the *MRSPN* associated to retrial queue *M/G/1/N/N* with orbital search are similar to those obtained in Wüchner et al. (2009a).

In Tables 3, 4, and 5, we present numerical results for different service time distributions with two phases (Hypoexponential "$Hypo_2$: $f^g(x) = \frac{\mu_1\mu_2}{\mu_1-\mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x})$" with the parameters "$\lambda = 0.8$, $N = 7$, $\gamma = 5$, $\mu_1 = 21$, $\mu_2 = 14$, $p = 0.5$", Hyperexponential "H_2:$f^g(x) = q\mu_1 e^{-\mu_1 x} + (1-q)\mu_2 e^{-\mu_2 x}$" with the parameters "$\lambda = 2.3$, $N = 8$, $\gamma = 5.2$,

Table 3 Some performance measures for the Hypoexponential service

Measures	Model I	Model II	Measures	Model I	Model II
n_o	1.0060990	0.6493675	B_s	0.5212088	0.5522289
n_s	1.5273077	1.2015964	B_p	0.4614151	0.4893249
a_s	5.4726925	5.7984037	B_r	0.5984199	0.7331295
$\overline{\lambda}$	4.3781538	4.6387229	$\overline{w}$	0.2297998	0.1399884
$\overline{\gamma}$	5.0304947	3.2468374	ϖ	0.3488474	0.2590360

Table 4 Some performance measures for the Hyperexponential service

Measures	Model I	Model II	Measures	Model I	Model II
n_o	2.0397968	1.6502372	B_s	0.4751854	0.5062435
n_s	2.5149822	2.1564808	B_p	0.4205691	0.4509442
a_s	5.4850178	5.8435192	B_r	0.4997894	0.5505898
$\overline{\lambda}$	12.6155405	13.4400940	$\overline{w}$	0.1616892	0.1227847
$\overline{\gamma}$	10.6069441	8.5812340	ϖ	0.1993559	0.1604513

Table 5 Some performance measures for the Erlang service

Measures	Model I	Model II	Measures	Model I	Model II
n_o	4.2721696	2.3794110	B_s	0.6166735	0.8635551
n_s	4.8888431	3.2429662	B_p	0.5629232	0.8254101
a_s	4.1111569	5.7570338	B_r	0.6130662	0.9063280
$\overline{\lambda}$	6.1667352	8.6355505	$\overline{w}$	0.6927766	0.2755367
$\overline{\gamma}$	8.9715557	4.9967632	ϖ	0.7927765	0.3755367

$q = 0.35$, $\mu_1 = 30$, $\mu_2 = 25$, $p = 0.2$" and Erlang "E_2: $f^g(x) = 4\mu^2 x e^{-2\mu x}$" with the parameters "$\lambda = 1.5$, $N = 9$, $\gamma = 2.1$, $\mu = 10$, $p = 0.85''$) for the two models I and II.

In the following, we will present some numerical results in order to illustrate graphically the impact on the variation of different parameters of the studied models on the main characteristics of this models.

5.1 Comments and discussion of the results

In Fig. 10, it can be seen that the mean response time of the retrial queue *M/G/1/N/N* has a maximum. The location and the amplitude of this maximum depend on the retrial rate γ. We observe that, the arrival rate λ has a significant influence on the mean response time when the retrial rate γ is weak. Moreover, for high values of retrial rate, this influence is not significant. It can also be seen that all curves get close to each other for high value of the arrival rate λ.

In Fig. 11, we see that the increase in the arrival rate λ induces an increase on the server utilization.

In Fig. 12, we observe that the mean response time decreases when the retrial rate γ increases. This retrial rate has a significant effect on the mean response time when it is weak.

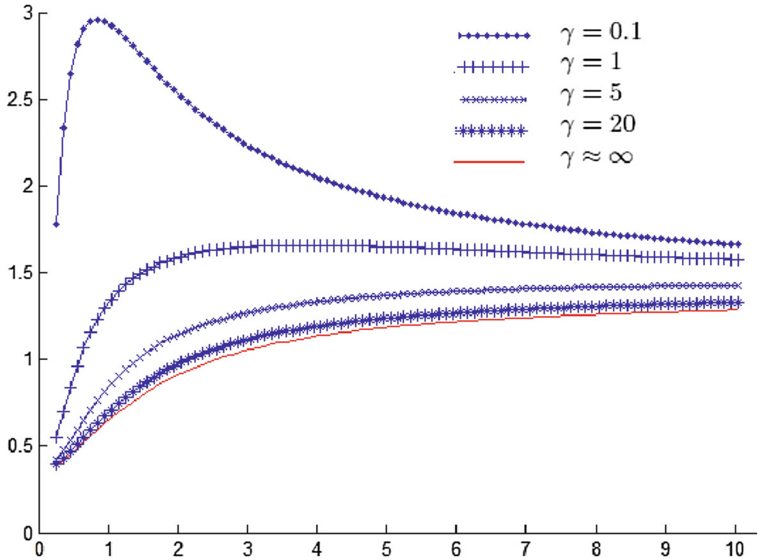


Fig. 10 Effect of arrival rate on model I mean response time; "$N = 4$, $\lambda = 0.15, \ldots, 10$, $Hypo_2(7; 5)$"

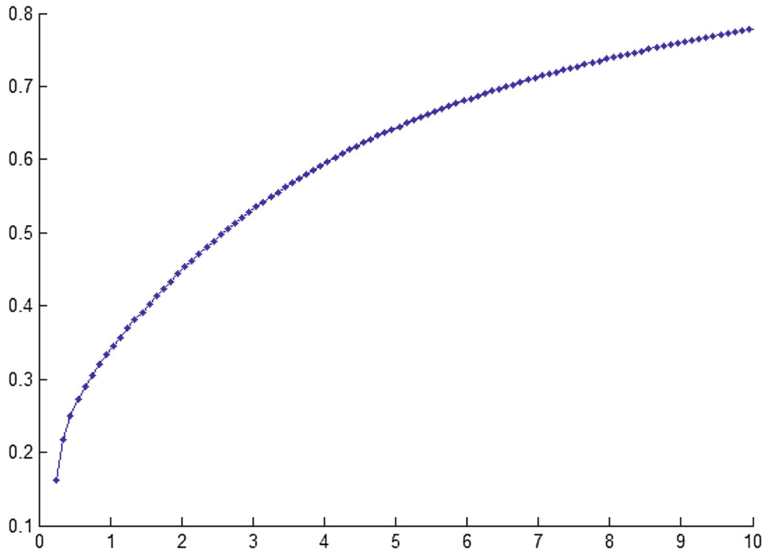


Fig. 11 Effect of arrival rate on the model I server utilization; "$N = 4$, $\lambda = 10^{-1}, \ldots, 20$, $Hypo_2(7; 5)$, $\gamma = 0.25$"

From Fig. 13, we see that the mean number of customers in the orbit decreases with the increase of the retrial rate γ. This decrease becomes slow with the intensifying of the repeated calls.

From Figs. 14 and 15, we observe that in contrast to Figs. 10 and 12 the variance of the service time has notable influence on mean response time. This influence decreases with rising retrial rate γ.

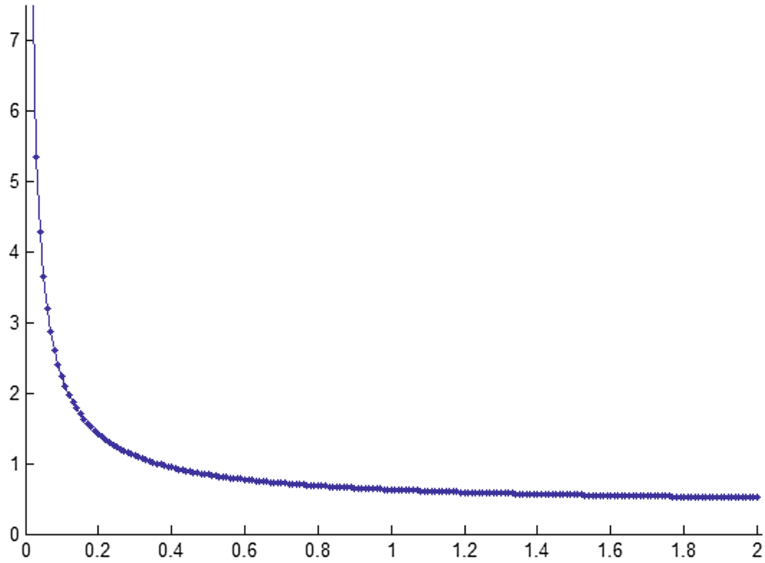


Fig. 12 Effect of retrial rate on model I mean response time; "$N = 4$, $\lambda = 0.2$, $Hypo_2(7; 5)$, $\gamma = 10^{-2}, \ldots, 2$"

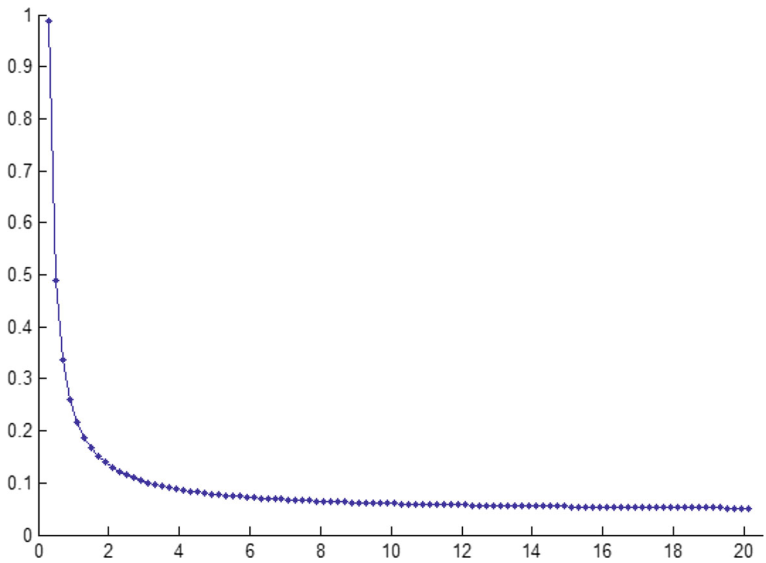


Fig. 13 Effect of retrial rate on model I mean number of customers in the orbit; "$N = 4$, $\lambda = 0.2$, $Hypo_2(7; 5)$, $\gamma = 10^{-1}, \ldots, 20$"

Finally, from the model I graphs, we see that when the retrial rate tends to ∞ all the considered characteristics (the mean response times, the mean number of customer in the orbit, and the server utilization) approach those of the classical *M/G/1/N/N* queue which has the optimistic performance bound.

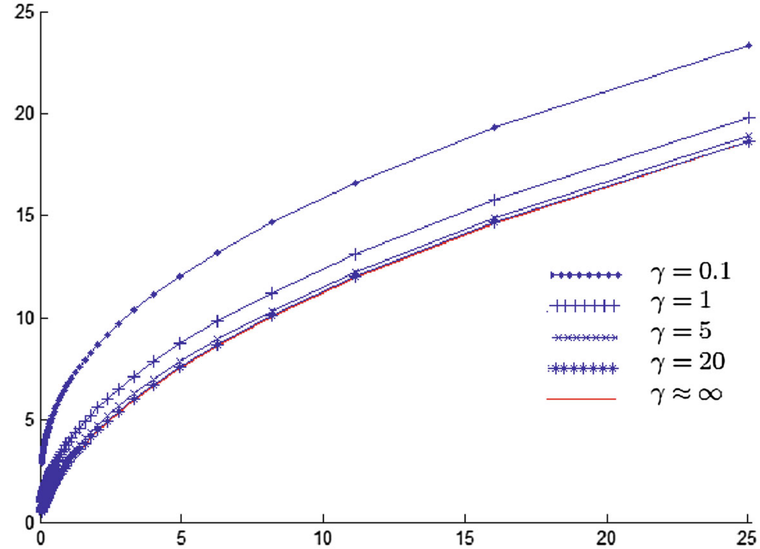


Fig. 14 Effect of the variance σ^2 of the service time on model I mean response time; "$N = 4$, $\lambda = 0.5$, $Hypo_2(\mu_1; \mu_2)$ with $\sigma^2 = 5.10^{-2}, \ldots, 25$"

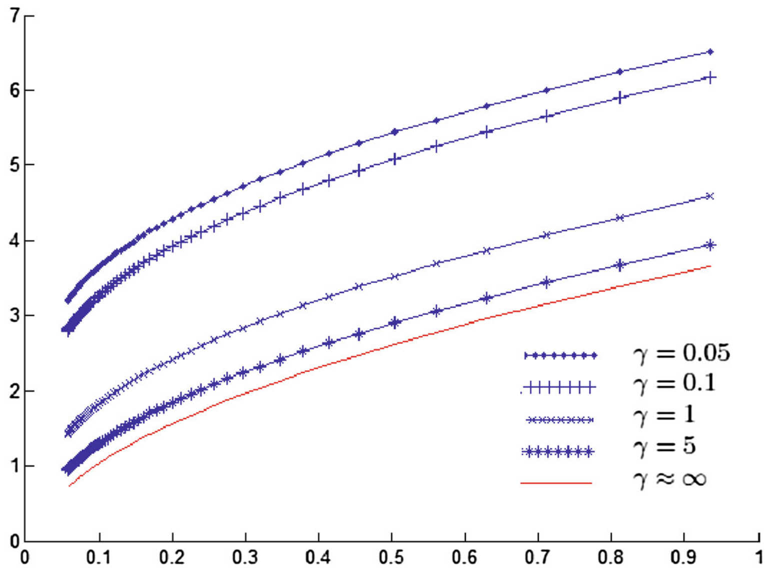


Fig. 15 Effect of the variance σ^2 of the service time on model I mean response time; "$N = 4$, $\lambda = 1.2$, $Hypo_2(\mu_1; \mu_2)$ with $\sigma^2 = 5.10^{-2}, \ldots, 1$"

In Figs. 16 and 17, Furthermore, we notice that the mean response time of the retrial queue *M/G/1/N/N* with orbital search has a maximum. The location and the amplitude of this maximum depend on the retrial rate γ and the search probability p. For higher values of the search probability p or lower values of retrial rate γ, the maximum becomes less dominant. The arrival rate λ has a significant influence on the mean response time when the retrial rate

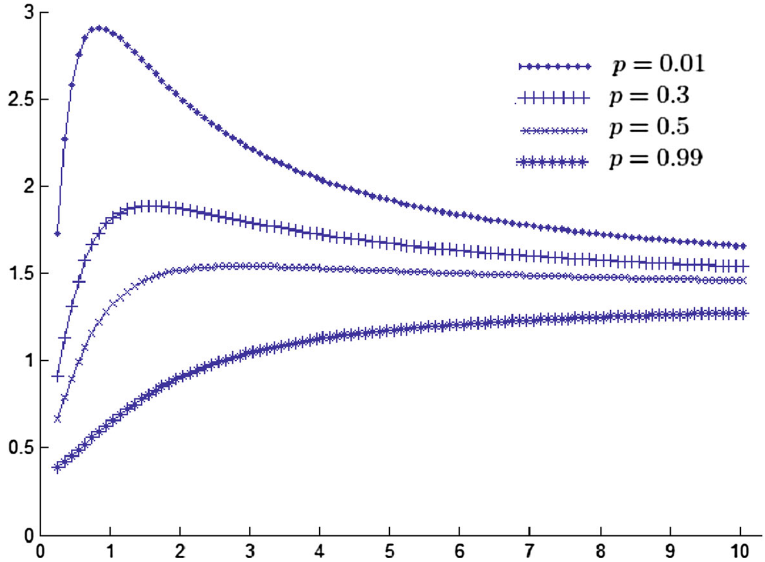


Fig. 16 Effect of arrival rate on model II mean response time; "$N = 4$, $\lambda = 0.15, \ldots, 10$, $Hypo_2(7; 5)$, $\gamma = 0.1$"

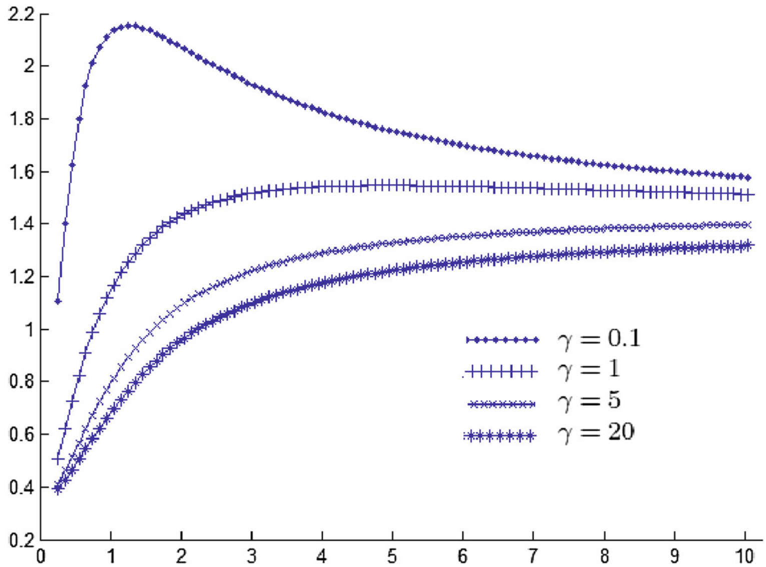


Fig. 17 Effect of arrival rate on model II mean response time; "$N = 4$, $\lambda = 0.15, \ldots, 10$, $Hypo_2(7; 5)$, $p = 0.2$"

γ is weak and/or ($p \in \{0.01, 0.3, 0.5\}$). Furthermore, we constat that all curves get close to each other for high value of the arrival rate λ.

In Fig. 18, we observe that the increase of the arrival rate λ induces an increase on the server utilization. For high values of the probability search p the server utilization approaches to 1.

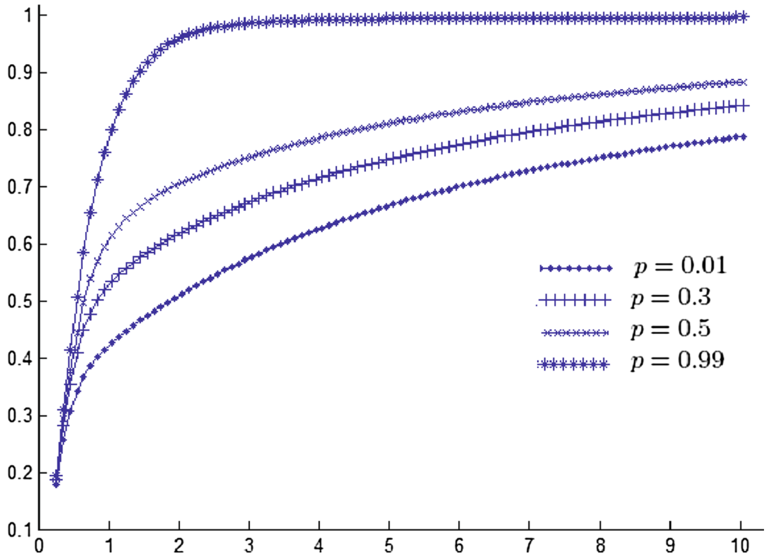


Fig. 18 Effect of arrival rate on the model II server utilization; "$N = 4$, $\lambda = 0.15, \ldots, 10$, $Hypo_2(7; 5)$, $\gamma = 0.25$"

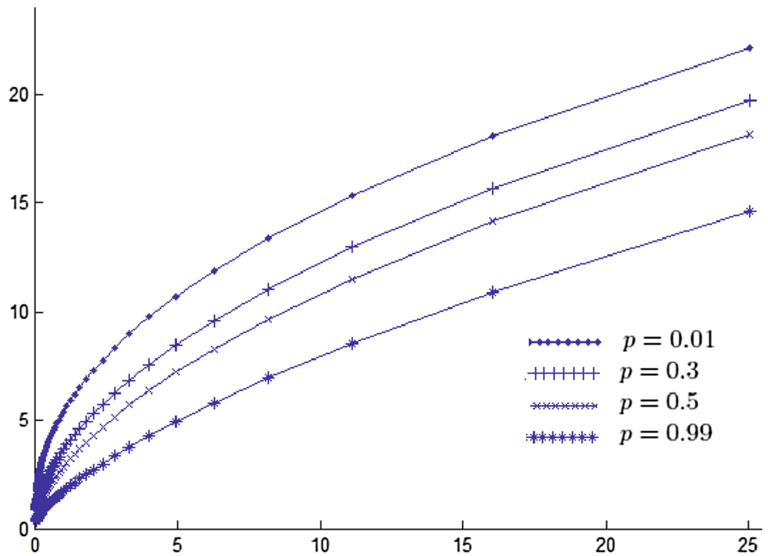


Fig. 19 Effect of variance σ^2 of the service time on model II mean response time; "$N = 4$, $\lambda = 0.5$, $Hypo_2(\mu_1; \mu_2)$ with $\sigma^2 = 5.10^{-2}, \ldots, 25$, $\gamma = 0.1$"

From Figs. 19 and 20, we observe that, the variance of the service time has notable influence on mean response time for all values of p. This influence decreases with rising the search probability p. The shortest mean response time is obtained when p tends to 1. Furthermore, from Figs. 21 and 22 we constat that the variance of the service time has notable influence on mean response time for all values of γ. This influence decreases with rising retrial rate γ.

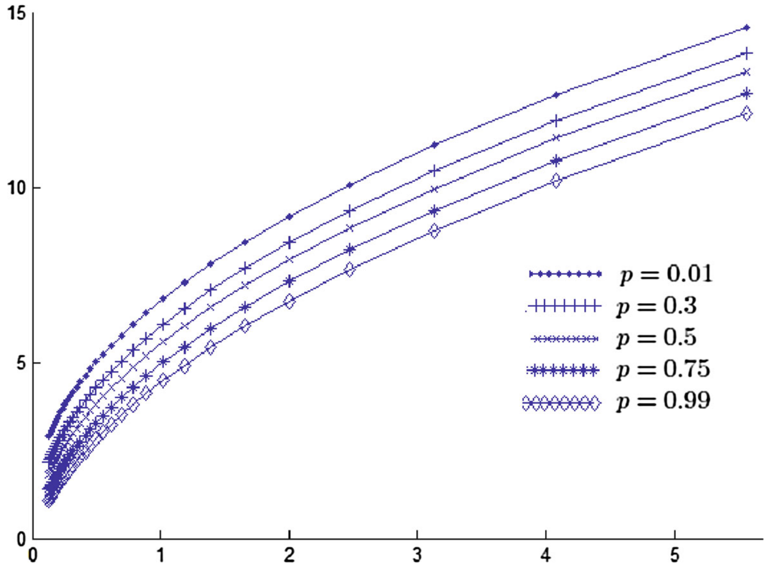


Fig. 20 Effect of variance σ^2 of the service time on model II mean response time; "$N = 4$, $\lambda = 0.8$, $E_2(\mu)$ with $\sigma^2 = 0.1, \ldots, 6$, $\gamma = 0.25$"

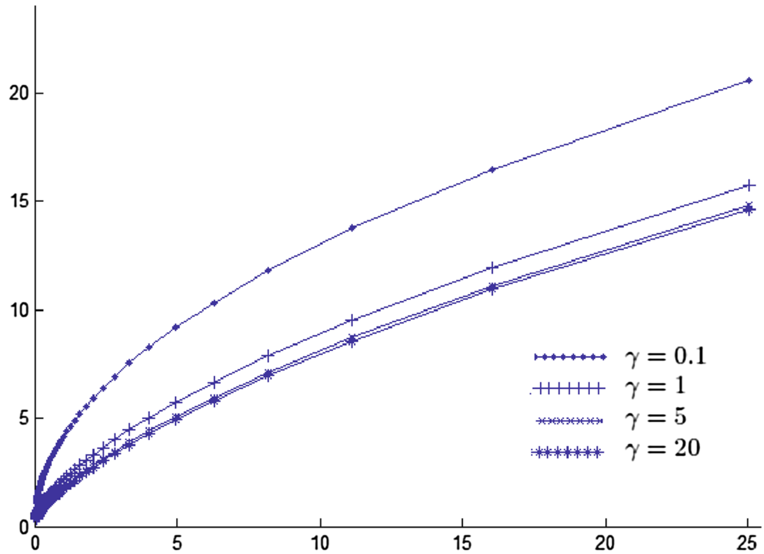


Fig. 21 Effect of variance σ^2 of the service time on model II mean response time; "$N = 4$, $\lambda = 0.5$, $Hypo_2(\mu_1; \mu_2)$ with $\sigma^2 = 5.10^{-2}, \ldots, 25$, $p = 0.2$"

In Fig. 23, we observe that the response time decreases with the intensity of the retrial rate γ, particularly when p tends to 1, γ has a negligible effect on mean response time. On the other hand, the influence of the retrial rate is more significant for smaller value of p.

From Fig. 24, we see that the mean number of customers in the orbit decreases with the increase of the retrial rate γ. This decrease is not considerable for high retrial rate values.

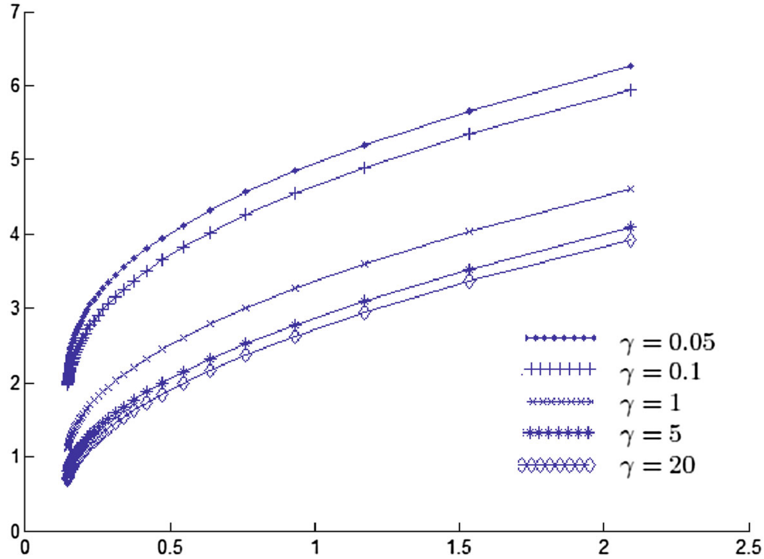


Fig. 22 Effect of variance σ^2 of the service time on model II mean response time; "$N = 4$, $\lambda = 1$, $H_2(q, \mu_1, \mu_2)$ with $\sigma^2 = 0.1, \dots, 2.5$, $p = 0.2$, $q = 0.6$"

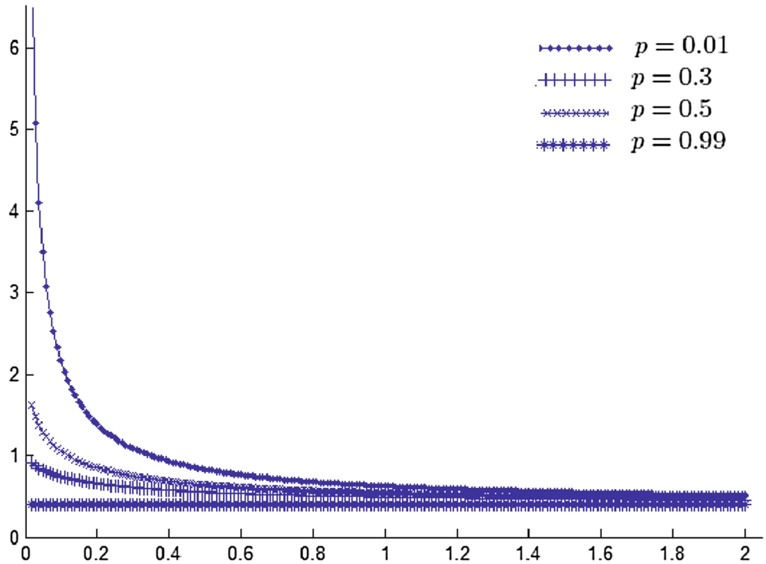


Fig. 23 Effect of retrial rate on model II mean response time; "$N = 4$, $\lambda = 0.2$, $Hypo_2(7; 5)$, $\gamma = 10^{-2}, \dots, 2$"

For high values of γ the behavior of finite source retrial queues with orbital search tends to the behavior of the classical *M/G/1/N/N* queueing system.

Finally, our study confirms that the server utilization increases with the increasing of the orbital research probability. This means that the server's idle time is reduced using the orbital search policy.

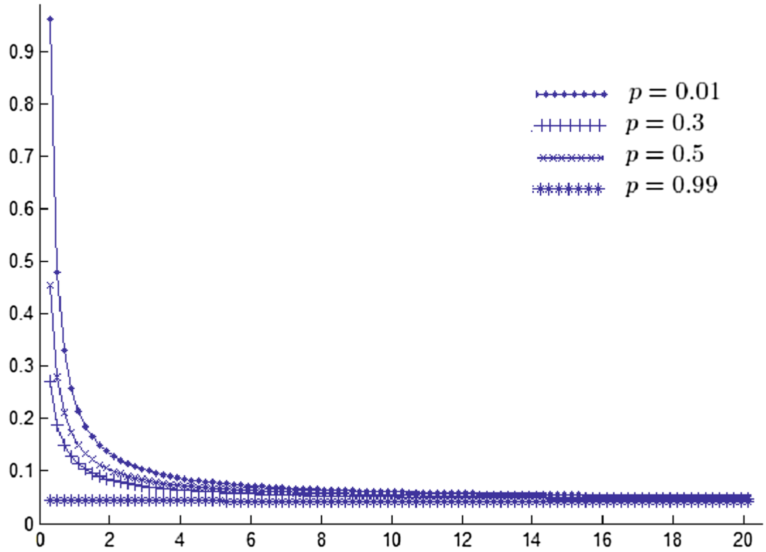


Fig. 24 Effect of retrial rate on model II mean number of customers in the orbit; "$N = 4$, $\lambda = 0.2$, $Hypo_2(7; 5)$, $\gamma = 10^{-1}, \ldots, 20$"

6 Conclusion

In this work, an alternative approach for modeling and analyzing Semi-Markovian retrial queues is presented. A qualitative and quantitative analysis of *M/G/1/N/N* with retrials is obtained by the approach based on the theory of *MRP*. As an example, the analysis of the retrial queue $M/Hypo_2/1/2/2$ is detailed. Furthermore, the *MRSPN* model of *M/G/1/N/N* with retrials is extended to orbital search mechanism. The characteristics of the above systems are computed and some graphical results are obtained by an algorithm elaborated in Matlab environment. It may be interesting to provide a detailed study on the transient analysis and/or to include, to the same models, the following phenomenons: vacation, breakdown, etc. Moreover, the $MRSPN - M/Hypo_2/1/N/N$ can be applied as a good approximation for *MRSPN* models associated to complex queues.

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