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An Elementary Proof that any Natural Number can be Written as the Sum of Three Terms of the Sequence $\lfloor \frac{n^2}{3} \rfloor$

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Abstract

The aim of this note consists to give an elementary proof that any natural number can be written as the sum of three terms of the sequence $(\lfloor \frac{n^2}{3} \rfloor)_{n\in\mathbb{N}}$. This is a recent conjecture of the author which is very recently confirmed by S. Mezroui et al. who used a result due to P. T. Bateman and derived from the theory of modular forms. We also make some conjectures related to the subject.

1 Introduction

Throughout this paper, we let $\mathbb{N}^*$ denote the set of the positive integers and we let $\lfloor x \rfloor$ and $\langle x \rangle$ denote, respectively, the integer-part and the fractional-part of a real number x.

The representation of natural numbers as the sum of a fixed number of squares is one of the old and fascinate problems in Number Theory. Let us just cite the most important classical results which are due to Euler, Lagrange and Legendre. Euler proved that a natural number is the sum of two squares if and only if in its decomposition as product of primes, the powers of the primes having the form $(4k + 3)$ $(k \in \mathbb{N})$ are all even. Lagrange [3] proved that any natural number is the sum of four squares of integers and Legendre [4] proved the powerful result stating that: any natural number not of the form $4^h(8k + 7)$ $(h, k \in \mathbb{N})$ can be written as the sum of three squares.

In [2], by leaning on Legendre's theorem cited above, the author obtained some results related to the representation of the natural numbers as the sum of three terms of the sequence $(\lfloor \frac{n^2}{a} \rfloor)_{n\in\mathbb{N}}$ (where $a \in \mathbb{N}^*$ is a parameter). In particular, he proved that any natural number

$N \not\equiv 2 \pmod{24}$ can be written as the sum of three terms of the sequence $\left(\lfloor \frac{n^2}{3} \rfloor\right)_{n \in \mathbb{N}}$ and he conjectured that this result becomes true even if $N \equiv 2 \pmod{24}$. This conjecture is very recently confirmed by S. Mezroui et al. (cf. [5]). So, we have the following:

Theorem 1 (cf. [2, 5]). *Every natural number can be written as the sum of three numbers of the form $\lfloor \frac{n^2}{3} \rfloor$ ($n \in \mathbb{N}$).*

However, the proof of S. Mezroui et al. [5] is not elementary because it depends on a result of P. T. Bateman [1] which is related to the theory of modular forms. The aim of this note is to give an elementary proof of Theorem 1. The advantage of our proof is double: on the one hand, it is elementary and on the other hand, it gives us a method of finding explicitly the representation in question; that is the representation of a given natural number as $\lfloor \frac{a^2}{3} \rfloor + \lfloor \frac{b^2}{3} \rfloor + \lfloor \frac{c^2}{3} \rfloor$ ($a, b, c \in \mathbb{N}$).

2 An elementary proof of Theorem 1

The fundamental idea of our proof of Theorem 1 consists to use the identity:

$$(2x + 2y + z)^2 + (2x - y - 2z)^2 + (x - 2y + 2z)^2 \;=\; 9\left(x^2 + y^2 + z^2\right)$$

($\forall x, y, z \in \mathbb{Z}$). Once known, the verification of this identity is immediate.
We now give the details of our proof. We begin with two lemmas:

Lemma 2. *Let $a, b, c \in \mathbb{Z}$. If one at least of the three integers a, b and c is not a multiple of 3 then also one at least of the three integers $a + b - c$, $a + c - b$ and $b + c - a$ is not a multiple of 3.*

Proof. The system of congruences:

$$\begin{cases} a + b - c \equiv 0 \pmod 3 \\ a + c - b \equiv 0 \pmod 3 \\ b + c - a \equiv 0 \pmod 3 \end{cases}$$

has for determinant:

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4 \not\equiv 0 \pmod 3.$$

Since this determinant is invertible modulo 3, then we have

$$\begin{cases} a + b - c \equiv 0 \pmod 3 \\ a + c - b \equiv 0 \pmod 3 \\ b + c - a \equiv 0 \pmod 3 \end{cases} \Longleftrightarrow \begin{cases} a \equiv 0 \pmod 3 \\ b \equiv 0 \pmod 3 \\ c \equiv 0 \pmod 3 \end{cases},$$

which concludes this proof. □

Lemma 3 (The fundamental lemma). *For any natural number k, there exist natural numbers a, b, c, which are not all multiples of 3, such that:*

$$a^2 + b^2 + c^2 \;=\; 8k + 1.$$

Proof. We argue by induction on k.
• For $k = 0$, it suffices to take $(a, b, c) = (1, 0, 0)$.
• Let k be a positive integer. Suppose that the property of the lemma is true for all natural number $k' < k$ and let us show that it becomes true for k. We distinguish the following two cases:
1^st case: If $k \not\equiv 1 \pmod 9$
By Legendre's theorem, there exist $a, b, c \in \mathbb{N}$ such that:

$$a^2 + b^2 + c^2 = 8k + 1.$$

If a, b, c are all multiples of 3, we would have $a^2 + b^2 + c^2 \equiv 0 \pmod 9$; that is $8k + 1 \equiv 0 \pmod 9$. This gives $k \equiv 1 \pmod 9$, which contradicts the assumption of this first case. So a, b, c cannot all be multiples of 3, as required.
2^nd case: If $k \equiv 1 \pmod 9$
In this case, there exists $k' \in \mathbb{N}$ such that $k = 9k' + 1$. So, we have

$$8k + 1 = 9(8k' + 1).$$

Since $k' = \frac{k-1}{9} < k$, then by the induction hypothesis there exist $a', b', c' \in \mathbb{N}$, which are not all multiples of 3, such that:

$$a'^2 + b'^2 + c'^2 = 8k' + 1.$$

Next, by Lemma 2, one at least of the integers $a' + b' - c'$, $a' + c' - b'$, $b' + c' - a'$ is not a multiple of 3. By permuting a', b', c' if necessary, we can suppose that:

$$a' + b' - c' \not\equiv 0 \pmod 3 \tag{1}$$

Now, let

$$\begin{aligned} a &:= |2a' + 2b' + c'| \\ b &:= |2a' - b' - 2c'| \\ c &:= |a' - 2b' + 2c'|. \end{aligned}$$

A simple calculation shows that:

$$a^2 + b^2 + c^2 = 9(a'^2 + b'^2 + c'^2).$$

Hence:

$$a^2 + b^2 + c^2 = 9(8k' + 1) = 8k + 1.$$

To conclude, it remains to show that the natural numbers a, b, c are not all multiples of 3. We have

$$\begin{cases} a \equiv 0 \pmod 3 \\ b \equiv 0 \pmod 3 \\ c \equiv 0 \pmod 3 \end{cases} \iff \begin{cases} 2a' + 2b' + c' \equiv 0 \pmod 3 \\ 2a' - b' - 2c' \equiv 0 \pmod 3 \\ a' - 2b' + 2c' \equiv 0 \pmod 3 \end{cases} \iff a' + b' - c' \equiv 0 \pmod 3.$$

But since $a' + b' - c' \not\equiv 0 \pmod 3$ (according to (1)), we conclude that effectively a, b, c are not all multiples of 3. The proof is complete. □

Using Lemma 3, we are now able to prove Theorem 1 by the method we have introduced in [2].

Proof of Theorem 1

Let N be a fixed natural number. We shall prove that N can be represented as the sum of three terms of the sequence $(\lfloor \frac{n^2}{3} \rfloor)_{n\in\mathbb{N}}$. To do this, we distinguish the following two cases:
1$^{\text{st}}$ case: if $N \not\equiv 2 \pmod 8$.
In this case, we can find $r \in \{1, 2\}$ such that $3N + r \not\equiv 0, 4, 7 \pmod 8$, so $(3N + r)$ is not of the form $4^h(8k + 7)$ $(h, k \in \mathbb{N})$. It follows by Legendre's theorem that $(3N + r)$ can be written as:

$$3N + r = a^2 + b^2 + c^2 \qquad \text{(with } a, b, c \in \mathbb{N}\text{)}.$$

By dividing by 3 and by separating the integer and the fractional parts, we get

$$N + \frac{r}{3} = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left(\left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle \right) \tag{2}$$

Now, since the quadratic residues modulo 3 are 0 and 1 then $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. But on the other hand, we have (according to (2)): $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle \equiv \frac{r}{3} \pmod 1$. Hence $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle = \frac{r}{3}$ and by reporting this in (2), we get (after simplifying):

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor,$$

as required.

2$^{\text{nd}}$ case: if $N \equiv 2 \pmod 8$.
In this case, we have $3N + 3 \equiv 1 \pmod 8$. It follows by Lemma 3 that there exist $a, b, c \in \mathbb{N}$, which are not all multiples of 3, such that:

$$3N + 3 = a^2 + b^2 + c^2 \tag{3}$$

By dividing by 3 and by separating the integer and the fractional parts, we get

$$N + 1 = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left(\left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle \right) \tag{4}$$

Now, since a, b, c are not all multiples of 3 and $a^2 + b^2 + c^2 \equiv 0 \pmod 3$ (according to (3)), then we have inevitably:

$$a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod 3.$$

This implies that $\langle \frac{a^2}{3} \rangle = \langle \frac{b^2}{3} \rangle = \langle \frac{c^2}{3} \rangle = \frac{1}{3}$. By reporting this in (4), we finally obtain (after simplifying):

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor,$$

which is the required representation of N. The proof is complete. □

3 Some related conjectures

Up to now, we just know (according to [2], [5] and the present paper) that for $a \in \{3, 4, 8\}$, any natural number can be represented as the sum of three terms of the sequence $(\lfloor \frac{n^2}{a} \rfloor)_{n \in \mathbb{N}}$. However, a simple Visual Basic program leads us to believe that this property holds for any integer $a \geq 3$. We make the following:

Conjecture 4. Let $a \geq 3$ be an integer. Then every natural number can be represented as a sum of three terms of the sequence $(\lfloor \frac{n^2}{a} \rfloor)_{n \in \mathbb{N}}$.

Next, it is conjectured in [2] the following:

Conjecture 5. (cf. [2]) For any integer $k \geq 2$, there exists a positive integer $a(k)$ satisfying the following property:

> Every natural number can be represented as the sum of $(k+1)$ terms of the sequence $(\lfloor \frac{n^k}{a(k)} \rfloor)_{n \in \mathbb{N}}$.

So, for the moment, the only value of k for which Conjecture 5 is known to be true is $k = 2$ and we can take $a(2) = 8$, 4 or 3.
Now, we suppose Conjecture 5 true and we interest to the smallest possible value of $a(k)$ $(k \geq 2)$ in that conjecture. We make the following:

Conjecture 6. Suppose that Conjecture 5 is true and denote by $\alpha(k)$ ($k \geq 2$ an integer) the smallest possible value of $a(k)$ in Conjecture 5. Then, we have

$$k! \ < \ \alpha(k) \ < \ k^k.$$

By taking $k = 2$ in Conjecture 6, we simply obtain $\alpha(2) = 3$, which is true (according to [2], [5] and the present paper). For $k = 3$, a Visual Basic program leads us to believe that $\alpha(3) = 17$, which satisfies the double inequality of Conjecture 6.
Further, we believe that we can take in Conjecture 5: $a(k) = k^k$ (for any $k \geq 2$). So, we make the following conjecture:

Conjecture 7. Let $k \geq 2$ be an integer. Then, every natural number can be represented as the sum of $(k+1)$ terms of the sequence $(\lfloor (\frac{n}{k})^k \rfloor)_{n \in \mathbb{N}}$.

Up to now, Conjecture 7 is only confirmed for $k = 2$ (cf. [2]).

We end this note by making a last conjecture which is stronger than Conjecture 5 and generalizes at the same time Conjecture 4.

Conjecture 8. For any integer $k \geq 2$, there exists a positive integer $b(k)$ such that for any integer $b \geq b(k)$, we have the following property:

> Every natural number can be represented as the sum of $(k+1)$ terms of the sequence $(\lfloor \frac{n^k}{b} \rfloor)_{n \in \mathbb{N}}$.

If we believe the truths of Conjecture 4 and Conjecture 8 and we denote by $\beta(k)$ the smallest possible value of $b(k)$ in Conjecture 8 then we have $\beta(2) = 3 = \alpha(2)$. Note that for other values of k ($k \geq 3$), we just have (obviously) $\beta(k) \geq \alpha(k)$.

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