Funct. Approx. Comment. Math.
Volume **57**, Number 2 (2017), p. 277-278.

On the representation of an even perfect number as the sum of five cubes

Bakir Farhi
Laboratoire de Mathématiques appliquées
Faculté des Sciences Exactes
Université de Bejaia, 06000 Bejaia, Algeria
E-mail: bakir.farhi@gmail.com

Abstract

The aim of this note is to show that any even perfect number, other than 6, can be written as the sum of at most five positive integral cubes. We also conjecture that any such number can even be written as the sum of at most three positive integral cubes.

1 Introduction

A perfect number is a positive integer that is equal to the sum of its proper positive divisors. There are more than twenty five centuries that mathematicians discovered the perfect numbers and began to be interested in their study. Eucild (about 300 B.C) showed that if $(2^n - 1)$ is prime then the number $2^{n-1}(2^n - 1)$ is perfect. The converse of Euclid's theorem is not yet proved, but Euler proved in 1747 that any even perfect number N can be written as $N = 2^{p-1}(2^p - 1)$, where p and $(2^p - 1)$ are both primes, that is N is of Euclid's form (see [2]).

In this note, we are interested to the representation of an even perfect number as the sum of a limited number of cubes. In connection with Waring's problem for cubes (see [1, Chap 2]), we prove that the quantity of cubes which is necessary to represent a natural number as a sum of cubes, when this one is an even perfect number, can be reduced to five.

2010 *Mathematics Subject Classification*: Primary 11A25, 11B13.
Key words and phrases: Perfect numbers, Sum of cubes.

2 The result

Theorem 2.1. *Any even perfect number, other than 6, can be written as the sum of at most five positive integral cubes.*

Proof. The proof is based on the following identity:

$$2n^6 - 2 = \left(n^2 + n - 1\right)^3 + \left(n^2 - n - 1\right)^3 \tag{2.1}$$

which holds for any $n \in \mathbb{Z}$.

Now, let N be an even perfect number greater than 6. By Euler's theorem, N can be written as $N = 2^{p-1}(2^p - 1)$, where p and $(2^p - 1)$ are both prime numbers. Because $N > 6$, we have $p > 2$. For $p = 3$, we get $N = 28 = 1^3 + 3^3$, which is a sum of two positive integral cubes. For $p = 5$, we get $N = 496 = 4^3 + 6^3 + 6^3$, which is a sum of three positive integral cubes. For the following, assume that $p > 5$. So p has one of the two forms: $p = 6k + 1$ or $p = 6k + 5$ ($k \in \mathbb{N}$).

1st case: (if $p = 6k + 1$ for some positive integer k)

In this case, we have $N = 2^{p-1}(2^p - 1) = 2^{6k}(2^{6k+1} - 1)$. Taking $n = 2^k$ in (2.1), we get $2^{6k+1} - 2 = a^3 + b^3$, with $a = n^2 + n - 1$ and $b = n^2 - n - 1$. Hence:

$$N = 2^{6k}\left(2^{6k+1} - 1\right) = 2^{6k}\left(a^3 + b^3 + 1\right) = \left(2^{2k}a\right)^3 + \left(2^{2k}b\right)^3 + \left(2^{2k}\right)^3,$$

which is a sum of three positive integral cubes.

2nd case: (if $p = 6k + 5$ for some positive integer k)

In this case, we have:

$$\begin{aligned} N = 2^{p-1}\left(2^p - 1\right) = 2^{6k+4}\left(2^{6k+5} - 1\right) &= 2^{6k+3}\left(2^{6k+6} - 2\right) \\ &= 2^{6k+3}\left(64 \cdot 2^{6k} - 2\right). \end{aligned}$$

Since $64 = 3^3 + 3^3 + 2^3 + 2$, it follows that:

$$\begin{aligned} N &= 2^{6k+3}\Big((3^3 + 3^3 + 2^3 + 2)2^{6k} - 2\Big) \\ &= (2^{2k+1})^3\Big((3 \cdot 2^{2k})^3 + (3 \cdot 2^{2k})^3 + (2 \cdot 2^{2k})^3 + (2 \cdot 2^{6k} - 2)\Big) \end{aligned} \tag{2.2}$$

Next, taking $n = 2^k$ in (2.1), we get $2 \cdot 2^{6k} - 2 = a^3 + b^3$ (with $a = n^2 + n - 1$ and $b = n^2 - n - 1$), which when reported in (2.2) gives:

$$\begin{aligned} N &= (2^{2k+1})^3\Big((3 \cdot 2^{2k})^3 + (3 \cdot 2^{2k})^3 + (2 \cdot 2^{2k})^3 + a^3 + b^3\Big) \\ &= (3 \cdot 2^{4k+1})^3 + (3 \cdot 2^{4k+1})^3 + (2 \cdot 2^{4k+1})^3 + (2^{2k+1}a)^3 + (2^{2k+1}b)^3, \end{aligned}$$

which is a sum of five positive integral cubes. This achieves the proof. □

We end this note by a conjecture, based on numerical calculations.

Conjecture 2.2. *Any even perfect number, other than* 6*, can be written as the sum of at most three positive integral cubes.*

References

[1] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics, Vol. 164, Springer-Verlag, New York, 1996.

[2] W. Sierpiński, *Elementary theory of numbers*, Chap IV, Panstowowe Wydawnictwo Naukowe, Warsaw, 1964.