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On the Representation of the Natural Numbers as the Sum of Three Terms of the Sequence $\lfloor \frac{n^2}{a} \rfloor$

Bakir Farhi
Department of Mathematics
University of Béjaia
Béjaia
Algeria
bakir.farhi@gmail.com

Abstract

In this note, we study the representation of a natural number as the sum of three natural numbers having the form $\lfloor \frac{n^2}{a} \rfloor$ ($n \in \mathbb{N}$), where a is a fixed positive integer and $\lfloor . \rfloor$ denotes the integer-part function. By applying Gauss's triangular number theorem, we show that every natural number is the sum of three numbers of the form $\lfloor \frac{n^2}{8} \rfloor$ ($n \in \mathbb{N}$). And by applying Legendre's theorem, we show that every natural number is the sum of three numbers of the form $\lfloor \frac{n^2}{4} \rfloor$ ($n \in \mathbb{N}$) and that every natural number $N \not\equiv 2 \pmod{24}$ is the sum of three numbers of the form $\lfloor \frac{n^2}{3} \rfloor$ ($n \in \mathbb{N}$). On the other hand, we show that every even natural number is the sum of three numbers of the form $\lfloor \frac{n^2}{2} \rfloor$ ($n \in \mathbb{N}$). We also propose two conjectures on the subject.

1 Introduction

Throughout this note, we let $\mathbb{N}$ denote the set of the non-negative integers and we let $\lfloor . \rfloor$ and $\langle . \rangle$ denote respectively the integer-part and the fractional-part functions.

Many results on the representation of a natural number as the sum of a fixed number of squares (or more generally quadratic progressions) are known. Lagrange [3] proved that every natural number is the sum of at most four squares. Gauss [2] proved that every natural number is the sum of at most three triangular numbers $\frac{k^2+k}{2}$ ($k \in \mathbb{N}$), or equivalently, that every natural number $N \equiv 3 \pmod{8}$ is the sum of three odd squares. Actually the Lagrange and Gauss theorems constitute particular cases of a general result asserted by Fermat and proved later by Cauchy. Cauchy's polygonal number theorem [1] states that for $m = 1, 2, 3, \ldots$ every natural number is the sum of $(m+2)$ polygonal numbers of the order $(m+2)$ (that is numbers of the form $\frac{m}{2}(k^2-k)+k$, with $k \in \mathbb{N}$). A short and easy proof of the theorem of Cauchy is given by Nathanson [5]. Legendre [4, p. 340–356] refined the theorem of Cauchy by proving that every natural number is the sum of five polygonal numbers of order $m+2$, one of which is either 0 or 1. On the other hand, Legendre [4, p. 331–339] refined the theorem of Lagrange and the theorem of Gauss by proving the following very interesting result:

> Every natural number not of the form $4^h(8k+7)$ $(h, k \in \mathbb{N})$ can be represented as the sum of three squares of natural numbers.

In this note, we study the representation of natural numbers as the sum of three numbers of the form $\lfloor \frac{n^2}{a} \rfloor$ $(n \in \mathbb{N})$, where a is a fixed positive integer. We first apply Gauss's triangular number theorem to prove that any natural number can be represented as the sum of three numbers of the form $\lfloor \frac{n^2}{8} \rfloor$ $(n \in \mathbb{N})$. Then we apply Legendre's theorem to prove that every natural number can be represented as the sum of three numbers of the form $\lfloor \frac{n^2}{4} \rfloor$ and that every natural number $N \not\equiv 2 \pmod{24}$ can be represented as the sum of three numbers of the form $\lfloor \frac{n^2}{3} \rfloor$. On the other hand, we prove (as application of Legendre's theorem) that every even natural number can be represented as the sum of three numbers of the form $\lfloor \frac{n^2}{2} \rfloor$ $(n \in \mathbb{N})$. Some natural conjectures on the subject are also proposed.

2 The Results

Theorem 1. *Every natural number can be written as the sum of three numbers of the form* $\lfloor \frac{n^2}{8} \rfloor$ $(n \in \mathbb{N})$.

Proof. By Gauss's triangular number theorem, every natural number can be written as the sum of three numbers of the form $\frac{k^2+k}{2}$ $(k \in \mathbb{N})$. To conclude, it suffices to remark that: $\frac{k^2+k}{2} = \lfloor \frac{n^2}{8} \rfloor$ for $n = 2k+1$. $\square$

Theorem 2. *Every natural number can be written as the sum of three numbers of the form* $\lfloor \frac{n^2}{4} \rfloor$ $(n \in \mathbb{N})$.

Proof. Let N be a natural number. Since $(4N+1)$ has not the form $4^h(8k+7)$ $(h, k \in \mathbb{N})$ then by Legendre's theorem $(4N+1)$ can be written as the sum of three squares of natural numbers. Let

$$4N+1 = a^2+b^2+c^2 \qquad (a, b, c \in \mathbb{N}).$$

By dividing on 4, we have:

$$N + \frac{1}{4} = \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4},$$

that is:

$$N + \frac{1}{4} = \left\lfloor \frac{a^2}{4} \right\rfloor + \left\lfloor \frac{b^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor + \left(\left\langle \frac{a^2}{4} \right\rangle + \left\langle \frac{b^2}{4} \right\rangle + \left\langle \frac{c^2}{4} \right\rangle \right).$$

Now, since the quadratic residues modulo 4 are 0 and 1 then $\langle \frac{a^2}{4} \rangle + \langle \frac{b^2}{4} \rangle + \langle \frac{c^2}{4} \rangle \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. So by taking the integer part in the two hand-sides of the last equality, we get:

$$N = \left\lfloor \frac{a^2}{4} \right\rfloor + \left\lfloor \frac{b^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor,$$

as required. The theorem is proved. □

Theorem 3. *Every natural number* $N \not\equiv 2 \pmod{24}$ *can be written as the sum of three numbers of the form* $\lfloor \frac{n^2}{3} \rfloor$ $(n \in \mathbb{N})$.

Proof. Let N be a natural number satisfying $N \not\equiv 2 \pmod{24}$. We distinguish the following two cases:

1st case: if $N \not\equiv 2 \pmod{8}$.

In this case, we can find $r \in \{1, 2\}$ such that $3N + r \not\equiv 0, 4, 7 \pmod{8}$, so $(3N + r)$ is not of the form $4^h(8k + 7)$ $(h, k \in \mathbb{N})$. It follows by Legendre's theorem that $(3N + r)$ can be written as:

$$3N + r = a^2 + b^2 + c^2 \qquad \text{(with } a, b, c \in \mathbb{N}).$$

By dividing by 3 and by separating the integer and the fractional parts, we get:

$$N + \frac{r}{3} = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left(\left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle \right) \tag{1}$$

Now, since the quadratic residues modulo 3 are 0 and 1 then $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. But on the other hand, we have (according to (1)): $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle \equiv \frac{r}{3} \pmod{1}$. Hence $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle = \frac{r}{3}$ and by reporting this in (1), we get (after simplifying):

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor,$$

as required.

2nd case: if $N \equiv 2 \pmod{8}$.

In this case, we have $3N + 3 \equiv 1 \pmod{8}$. It follows by Legendre's theorem that $(3N + 3)$ can be written as:

$$3N + 3 = a^2 + b^2 + c^2 \tag{2}$$

(with $a, b, c \in \mathbb{N}$). By dividing by 3 and by separating the integer and the fractional parts, we get:

$$N + 1 = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left(\left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle \right) \tag{3}$$

Now, since $a^2 + b^2 + c^2 \equiv 0 \pmod{3}$ (according to (2)) then we have either $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod{3}$ or $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod{3}$. Let us prove that the alternative $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod{3}$ cannot hold. Suppose that $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod{3}$, then $a \equiv b \equiv c \equiv 0 \pmod{3}$. So we can write $a = 3a'$, $b = 3b'$, $c = 3c'$ $(a', b', c' \in \mathbb{N})$. By reporting this in (2), we obtain (after simplifying):

$$N + 1 = 3a'^2 + 3b'^2 + 3c'^2.$$

This implies that $N+1 \equiv 0 \pmod{3}$, so that $N \equiv 2 \pmod{3}$. But ($N \equiv 2 \pmod{8}$ and $N \equiv 2 \pmod{3}$) is equivalent to $N \equiv 2 \pmod{24}$ which contradicts the hypothesis $N \not\equiv 2 \pmod{24}$. So, the alternative $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod{3}$ is impossible. Therefore, we have $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod{3}$, which implies that $\langle \frac{a^2}{3} \rangle + \langle \frac{b^2}{3} \rangle + \langle \frac{c^2}{3} \rangle = 1$. By reporting this in (3) and by simplifying, we get:

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor,$$

as required. The theorem is proved. □

Corollary 4. *Every natural number is the sum of four numbers of the form* $\lfloor \frac{n^2}{3} \rfloor$ $(n \in \mathbb{N})$, *one of which is either* 0 *or* 1.

Proof. Let N be a natural number. If $N \not\equiv 2 \pmod{24}$ then according to Theorem 3, N can be written as:

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left\lfloor \frac{0^2}{3} \right\rfloor,$$

as required.
Now, if $N \equiv 2 \pmod{24}$, then $N - 1 \equiv 1 \pmod{24} \not\equiv 2 \pmod{24}$ and according to Theorem 3, $(N-1)$ can be written as: $N - 1 = \lfloor \frac{a^2}{3} \rfloor + \lfloor \frac{b^2}{3} \rfloor + \lfloor \frac{c^2}{3} \rfloor$ $(a, b, c \in \mathbb{N})$. Hence:

$$N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor,$$

as required. The corollary is proved. □

We believe that the excluded case ($N \equiv 2 \pmod{24}$) of Theorem 3 is not significant. This leads us to make the following conjecture:

Conjecture 5. Every natural number can be written as the sum of three numbers of the form $\lfloor \frac{n^2}{3} \rfloor$ $(n \in \mathbb{N})$.

More generally, we propose the following conjecture:

Conjecture 6. Let $k \geq 2$ be an integer. Then there exists a positive integer $a(k)$ satisfying the following property:

> Every natural number can be written as the sum of $(k+1)$ numbers of the form $\lfloor \frac{n^k}{a(k)} \rfloor$ $(n \in \mathbb{N})$.

Theorems 1 and 2 show that the last conjecture is true for $k = 2$ and we can take $a(2) = 8$ or 4. Furthermore, if we believe in the truth of Conjecture 5, the smallest valid value of $a(2)$ is $a(2) = 3$ (see below).

Now, because the numbers $\lfloor \frac{n^2}{2} \rfloor$ $(n \in \mathbb{N})$ are all even, we cannot write any natural number as the sum of a fixed number of that numbers but the question we can ask is that: is it true that any even natural number is the sum of a fixed number of the numbers having the form $\lfloor \frac{n^2}{2} \rfloor$?. The following theorem answers this question affirmatively:

Theorem 7. *Every even natural number can be written as the sum of three numbers of the form* $\lfloor \frac{n^2}{2} \rfloor$ $(n \in \mathbb{N})$.

Proof. Let N be an even natural number. Then $2N+1 \equiv 1 \pmod 4$. It follows by Legendre's theorem that $(2N+1)$ is the sum of three squares of natural numbers. Write

$$2N + 1 = a^2 + b^2 + c^2 \qquad (a, b, c \in \mathbb{N}).$$

Hence:

$$N = \left\lfloor \frac{a^2}{2} \right\rfloor + \left\lfloor \frac{b^2}{2} \right\rfloor + \left\lfloor \frac{c^2}{2} \right\rfloor + \left(\left\langle \frac{a^2}{2} \right\rangle + \left\langle \frac{b^2}{2} \right\rangle + \left\langle \frac{c^2}{2} \right\rangle - \frac{1}{2} \right) \tag{4}$$

Now, since each of $\langle \frac{a^2}{2} \rangle$, $\langle \frac{b^2}{2} \rangle$, $\langle \frac{c^2}{2} \rangle$ lies in $\{0, \frac{1}{2}\}$ then $\langle \frac{a^2}{2} \rangle + \langle \frac{b^2}{2} \rangle + \langle \frac{c^2}{2} \rangle - \frac{1}{2}$ lies in $\{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$. But since (according to (4)) $\langle \frac{a^2}{2} \rangle + \langle \frac{b^2}{2} \rangle + \langle \frac{c^2}{2} \rangle - \frac{1}{2}$ is an even integer (because N, $\lfloor \frac{a^2}{2} \rfloor$, $\lfloor \frac{b^2}{2} \rfloor$, $\lfloor \frac{c^2}{2} \rfloor$ are even integers) then $\langle \frac{a^2}{2} \rangle + \langle \frac{b^2}{2} \rangle + \langle \frac{c^2}{2} \rangle - \frac{1}{2} = 0$. By reporting this in (4), we obtain:

$$N = \left\lfloor \frac{a^2}{2} \right\rfloor + \left\lfloor \frac{b^2}{2} \right\rfloor + \left\lfloor \frac{c^2}{2} \right\rfloor,$$

as required. The theorem is proved. □

Corollary 8. *Every natural number can be written as the sum of three numbers, each of which has one of the two forms* k^2 *or* $(k^2 + k)$ $(k \in \mathbb{N})$.

Proof. It suffices to remark that for $n \in \mathbb{N}$:

$$\left\lfloor \frac{n^2}{2} \right\rfloor = \begin{cases} 2k^2, & \text{if } n = 2k \ (k \in \mathbb{N}); \\ 2(k^2 + k), & \text{if } n = 2k + 1 \ (k \in \mathbb{N}). \end{cases}$$

The corollary immediately follows from Theorem 7. □

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