How to obtain the continued fraction convergents of the number e by neglecting integrals

Bakir FARHI
bakir.farhi@gmail.com

Abstract

In this note, we show that any continued fraction convergent of the number $e = 2.71828\dots$ can be derived by approximating some integral $I_{n,m} := \int_0^1 x^n(1-x)^m e^x dx$ $(n, m \in \mathbb{N})$ by 0. In addition, we present a new way for finding again the well-known regular continued fraction expansion of e.

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1 Introduction, Notations and the Result

Throughout this paper the expression "*To neglect* a real number" will mean to approximate that number by 0.

A way of obtaining *good* rational approximations of the number e consists simply to neglect integrals of the form:

$$I_{n,m} := \int_0^1 x^n(1-x)^m e^x dx \qquad (n, m \in \mathbb{N},\ \text{sufficiently large}) \tag{1}$$

Actually, the neglect of such integrals is justified by the fact that $I_{n,m}$ tends to 0 as n and m tend to infinity. Indeed, by leaning on the Euler β-function, we have for all $n, m \in \mathbb{N}$:

$$I_{n,m} \leq e \int_0^1 x^n(1-x)^m dx = e \cdot \beta(n+1, m+1) = \frac{e}{(n+m+1)\binom{n+m}{n}},$$

which tends to 0 as n, m tend to infinity. Since $I_{n,m} \geq 0$, the claimed fact that $I_{n,m}$ tends to 0 as n, m tend to infinity follows.

For example, the calculation of $I_{2,2}$ gives: $I_{2,2} = 14e - 38$. So if we neglect $I_{2,2}$, we obtain the approximation $e \simeq \frac{19}{7}$ which is a *good* rational approximation of e since it is one of the *convergents* of its regular continued fraction expansion.

The purpose of this paper is essentially to show that any convergent of the regular continued fraction expansion of the number e can be obtained by neglecting some integral $I_{n,m}$. In addition, we present the paper in a way that the well-known regular continued fraction expansion of e, which is discovered for the first time by Euler (see e.g., [2]) and given by:

$$e = [2, 1, 2, 1, 1, 4, 1, \dots] = [2, \{1, 2n, 1\}_{n \geq 1}] \tag{2}$$

will be proved again. Let us define:

$$e' := [2, \{1, 2n, 1\}_{n \geq 1}]. \tag{3}$$

At the end of the paper, we show that $e' = e$ which provides a new proof of (2). Our main result is the following:

Theorem (the main theorem) *Let n be a natural number and m be a positive integer such that $|n - m| \leq 1$. Then the neglect of the integral $I_{n,m} := \int_0^1 x^n (1-x)^m e^x dx$ is equivalent to approximate the number e by some convergent of its continued fraction expansion.*
Reciprocally, any convergent of the regular continued fraction expansion of e can be obtained by neglecting some integral of the form $I_{n,m}$ with $n, m \in \mathbb{N}$, satisfying $|n - m| \leq 1$.

Now, we are going to give the result which details this theorem.
For all positive integer n, let $\frac{p_n}{q_n}$ (with $p_n, q_n \in \mathbb{N}$, $\gcd(p_n, q_n) = 1$) denotes the n^{th} convergent of the continued fraction expansion of e'. From (3), we have $e' := [a_1, a_2, \dots]$, where $a_1 = 2$ and

$$\begin{cases} a_{3k} = 2k \\ a_{3k+1} = a_{3k-1} = 1 \end{cases} \quad (\forall k \geq 1), \tag{4}$$

Next, according to the elementary properties of regular continued fraction expansions (see e.g., [2]), we have:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases} \quad (\forall n \geq 3). \tag{5}$$

The details of the main theorem are given by the following:

Theorem 1 (detailing the main theorem) *We have:*

$$I_{k,k} = (-1)^k k!\,(q_{3k-1}e - p_{3k-1}) \qquad (\forall k \geq 1) \tag{6}$$
$$I_{k,k+1} = (-1)^k k!\,(q_{3k+1}e - p_{3k+1}) \qquad (\forall k \in \mathbb{N}) \tag{7}$$
$$I_{k+1,k} = (-1)^{k+1} k!\,(q_{3k}e - p_{3k}) \qquad (\forall k \geq 1). \tag{8}$$

2 The Proofs

To prove Theorem 1, we need the following lemma:

Lemma 2 *For all positive integers n, m, we have:*

$$I_{n,m} = mI_{n,m-1} - nI_{n-1,m} \tag{9}$$
$$I_{n-1,m-1} = I_{n,m-1} + I_{n-1,m}. \tag{10}$$

Proof. Let us prove (9). By integring by parts, we have for all positive integers n, m:

$$\begin{aligned}
I_{n,m} &:= \int_0^1 x^n(1-x)^m e^x dx \\
&= \int_0^1 x^n(1-x)^m (e^x)' dx \\
&= \left[x^n(1-x)^m e^x\right]_0^1 - \int_0^1 \left(x^n(1-x)^m\right)' e^x dx \\
&= 0 - \int_0^1 \left\{nx^{n-1}(1-x)^m - m(1-x)^{m-1}x^n\right\} e^x dx \\
&= m\int_0^1 x^n(1-x)^{m-1}e^x dx - n\int_0^1 x^{n-1}(1-x)^m e^x dx \\
&= mI_{n,m-1} - nI_{n-1,m},
\end{aligned}$$

as required. Now, let us prove (10). For all positive integers n, m, we have:

$$\begin{aligned}
I_{n,m-1} + I_{n-1,m} &= \int_0^1 x^n(1-x)^{m-1}e^x dx + \int_0^1 x^{n-1}(1-x)^m e^x dx \\
&= \int_0^1 x^{n-1}(1-x)^{m-1}\left\{x + (1-x)\right\} e^x dx \\
&= \int_0^1 x^{n-1}(1-x)^{m-1} e^x dx \\
&= I_{n-1,m-1},
\end{aligned}$$

as required. The lemma is proved. ■

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. We proceed by induction on $k \in \mathbb{N}$. For $k = 0, 1$, we easily check the validity of the relations of Theorem 1. Now, suppose that the relations (6), (7) and (8) of Theorem 1 hold for some $k \geq 1$ and let us show that they also hold for the integer $(k+1)$. Using Formula (9) of Lemma 2 together with the induction hypothesis, we have:

$$\begin{aligned} I_{k+1,k+1} &= (k+1)\left(I_{k+1,k} - I_{k,k+1}\right) \\ &= (k+1)\left\{(-1)^{k+1}k!(q_{3k}e - p_{3k}) - (-1)^k k!(q_{3k+1}e - p_{3k+1})\right\} \\ &= (-1)^{k+1}(k+1)!\left\{(q_{3k} + q_{3k+1})e - (p_{3k} + p_{3k+1})\right\}. \end{aligned}$$

But according to (4) and (5), we have:

$$p_{3k} + p_{3k+1} = p_{3k+2} \text{ and } q_{3k} + q_{3k+1} = q_{3k+2}.$$

Hence:

$$I_{k+1,k+1} = (-1)^{k+1}(k+1)!\left(q_{3k+2}e - p_{3k+2}\right),$$

which confirms the validity of Relation (6) of Theorem 1 for $(k+1)$.

Next, by using Formulas (9) and (10) of Lemma 2 together with the induction hypothesis, we have:

$$\begin{aligned} I_{k+1,k+2} &= (k+2)I_{k+1,k+1} - (k+1)I_{k,k+2} \\ &= (k+2)I_{k+1,k+1} - (k+1)\left(I_{k,k+1} - I_{k+1,k+1}\right) \\ &= (2k+3)I_{k+1,k+1} - (k+1)I_{k,k+1} \\ &= (2k+3)(k+1)(I_{k+1,k} - I_{k,k+1}) - (k+1)I_{k,k+1} \\ &= (k+1)\Big\{(2k+3)I_{k+1,k} - (2k+4)I_{k,k+1}\Big\} \\ &= (k+1)\Big\{(2k+3)(-1)^{k+1}k!(q_{3k}e - p_{3k}) \\ &\qquad -(2k+4)(-1)^k k!(q_{3k+1}e - p_{3k+1})\Big\} \\ &= (-1)^{k+1}(k+1)!\Big\{(2k+3)(q_{3k}e - p_{3k}) + (2k+4)(q_{3k+1}e - p_{3k+1})\Big\} \\ &= (-1)^{k+1}(k+1)!\Big\{\left((2k+3)q_{3k} + (2k+4)q_{3k+1}\right)e \\ &\qquad - \left((2k+3)p_{3k} + (2k+4)p_{3k+1}\right)\Big\} \end{aligned} \tag{11}$$

But, according to (4) and (5), we have:

$$\begin{aligned} p_{3k+4} &= p_{3k+3} + p_{3k+2} \\ &= ((2k+2)p_{3k+2} + p_{3k+1}) + (p_{3k+1} + p_{3k}) \\ &= (2k+2)p_{3k+2} + 2p_{3k+1} + p_{3k} \\ &= (2k+2)(p_{3k+1} + p_{3k}) + 2p_{3k+1} + p_{3k} \\ &= (2k+3)p_{3k} + (2k+4)p_{3k+1} \end{aligned}$$

and similarly, we get:

$$q_{3k+4} = (2k+3)q_{3k} + (2k+4)q_{3k+1}.$$

It follows from (11) that:

$$I_{k+1,k+2} = (-1)^{k+1}(k+1)!(q_{3k+4}e - p_{3k+4}),$$

which confirms the validity of Relation (7) of Teorem 1 for the integer $(k+1)$.

Finally, by still using the formulas of Lemma 2 together with the formulas for $I_{k+1,k+1}$ and $I_{k+1,k+2}$ which we just proved above, we have:

$$\begin{aligned} I_{k+2,k+1} &= I_{k+1,k+1} - I_{k+1,k+2} \\ &= (-1)^{k+1}(k+1)!(q_{3k+2}e - p_{3k+2}) - (-1)^{k+1}(k+1)!(q_{3k+4}e - p_{3k+4}) \\ &= (-1)^{k+1}(k+1)!\left\{(q_{3k+2} - q_{3k+4})e - (p_{3k+2} - p_{3k+4})\right\}. \end{aligned}$$

But since (according to (4) and (5)): $q_{3k+4} = q_{3k+3} + q_{3k+2}$ and $p_{3k+4} = p_{3k+3} + p_{3k+2}$ then it follows that:

$$I_{k+2,k+1} = (-1)^{k+2}(k+1)!(q_{3k+3}e - p_{3k+3}),$$

which confirms the validity of Relation (8) of Theorem 1 for the integer $(k+1)$. The three relations of Theorem 1 thus hold for $(k+1)$. The proof of Theorem 1 is complete. ■

Theorem 1 permits us to establish a new proof for the fact that the regular continued fraction expansion for the number e is given by (2).

A new Proof of (2). Relation (6) of Theorem 1 shows that for all positive integer k, we have:

$$\begin{aligned} \left| e - \frac{p_{3k-1}}{q_{3k-1}} \right| &= \frac{|I_{k,k}|}{k! q_{3k-1}} \\ &\leq \frac{I_{k,k}}{k!} \quad \text{(since } I_{k,k} \geq 0 \text{ and } q_{3k-1} \in \mathbb{Z}_+^*) \qquad (12) \end{aligned}$$

Next, by using the simple inequalities $x(1-x) \leq \frac{1}{4}$ and $e^x \leq e$ ($\forall x \in [0,1]$), we have for all positive integer k:

$$I_{k,k} := \int_0^1 (x(1-x))^k e^x dx \leq \int_0^1 \frac{e}{4^k} dx = \frac{e}{4^k}.$$

It follows by inserting this in (12) that:

$$\left| e - \frac{p_{3k-1}}{q_{3k-1}} \right| \leq \frac{e}{4^k k!} \qquad (\forall k \geq 1),$$

which shows that $\frac{p_{3k-1}}{q_{3k-1}}$ tends to e as k tends to infinity. But since $\frac{p_{3k-1}}{q_{3k-1}}$ represents the $(3k-1)^{\text{th}}$ convergent of the regular continued fraction expansion of e', we have on the other hand $\lim_{k \to +\infty} \frac{p_{3k-1}}{q_{3k-1}} = e'$. Hence $e = e'$, which confirms (2). ■

3 Remarks about the analog of the main theorem concerning the number π:

The analogs of the integrals $I_{n,m}$ whose the neglect leads to approximate the number π by the convergents of its regular continued fraction expansion are not known in their general form. However, for the particular famous approximation $\pi \simeq \frac{22}{7}$, Dalzell [1] noticed that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

So the neglect of the later integral leads to the Archimedes approximation $\pi \simeq \frac{22}{7}$. For some other continued fraction convergents of π (like $\frac{333}{106}, \frac{355}{113}, \dots$), Lucas [3] experimentally obtained some integrals having the form:

$$\int_0^1 \frac{x^n(1-x)^m(a+bx+cx^2)}{1+x^2} dx \qquad (n, m, a, b, c \in \mathbb{N})$$

whose the neglect leads to approximate π by those convergents.

References

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[3] S. K. Lucas, Approximations to π Derived from Integrals with Nonnegative Integrands, *Amer. Math. Monthly* **116** (2009) 166-172.