An analog of the arithmetic triangle obtained by replacing the products by the least common multiples

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1 Introduction

The Al-Karaji arithmetic triangle is the triangle consisting of the binomial coefficients $\binom{n}{k}$ $(n, k \in \mathbb{N}, n \geq k)$. Precisely, for each $n \in \mathbb{N}$, the n^{th} row of that triangle is:

$$\binom{n}{0} \ \binom{n}{1} \ \cdots \ \binom{n}{n},$$

where

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{1 \times 2 \times \cdots \times k} \tag{1}$$

So the beginning of the arithmetic (or binomial) triangle is given by:

$$\begin{array}{lllllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}$$

Note that the construction of the triangle rests on the property that each number of a given row is the sum of the numbers which are situated just above. Explicitly, we have:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (\forall k, n \text{ such that } n \geq k \geq 1) \tag{2}$$

Historically, the first mathematician who discovered the binomial triangle was the pioneer arabic mathematician Al-Karaji (953 - 1029 AD). He drew this triangle until its 12^{th} row and noted the process of its recursive construction by pointing out (2). More interestingly, Al-Karaji discovered the binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (\forall n \in \mathbb{N}) \tag{3}$$

After Al-Karaji, several other mathematicians of the Islamic civilization reproduced that very important triangle (Al-Khayyam, Al-Samawal, Al-Tusi, Al-Farisi, Ibn Al-Banna, Ibn Munaim, Al-Kashi, ...). The same triangle have been discovered again in China (Yang Hui in the 13^{th} century). In Europ (16^{th} century), several mathematicians remarked the importance of Al-Karaji's triangle (Stifel, Tartaglia, Pascal, ...).

In this paper, we are going to obtain the analog of Al-Karaji's triangle by substituting in Formula (1) the products by the least common multiples. If we use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the lcm-analog of the binomial coefficient $\binom{n}{k}$ would be:

$$\frac{\operatorname{lcm}(1, 2, \ldots, n)}{\operatorname{lcm}(1, 2, \ldots, k) \times \operatorname{lcm}(1, 2, \ldots, n-k)}.$$

But this analogy is not quite interesting because those last numbers are not all integers. For example, for $n = 6, k = 3$, we have:

$$\frac{\operatorname{lcm}(1, 2, \ldots, 6)}{\operatorname{lcm}(1, 2, 3) \times \operatorname{lcm}(1, 2, 3)} = \frac{5}{3} \notin \mathbb{Z}.$$

In order to obtain an interesting analogy, we will use rather the formula $\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{1 \times 2 \times \cdots \times k}$. So, the lcm-analog of a binomial coefficient $\binom{n}{k}$ which we must consider is:

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{\operatorname{lcm}(n, n-1, \ldots, n-k+1)}{\operatorname{lcm}(1, 2, \ldots, k)} \tag{4}$$

(We naturally conventione that $\operatorname{lcm}(\emptyset) = 1$).

Notice that a table of the numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ was already given by A. Murthy (2004) and extended by E. Deutsch (2006) in the On-Line Encyclopedia of Integer Sequences (see the sequence A093430 of OEIS). However, to my knowledge, no property was already proved about those numbers in comparison with their analog binomial numbers.

2 Results

We begin with the easy result showing that the rational numbers $\left[{n \atop k}\right]$, defined by (4), are all integers. We have the following:

Proposition 1 *For all natural numbers n, k such that $n \geq k$, the positive rational number $\left[{n \atop k}\right]$ is an integer.*

Proof. Let n, k be natural numbers such that $n \geq k$. Among the k consecutive integers $n, n-1, \dots, n-k+1$, one at least is a multiple of 1, one at least is a multiple of 2, $\dots$, and one at least is a multiple of k. This implies that $\mathrm{lcm}(n, n-1, \dots, n-k+1)$ is a multiple of each of the positive integers $1, 2, \dots, k$. Consequently $\mathrm{lcm}(n, n-1, \dots, n-k+1)$ is a multiple of $\mathrm{lcm}(1, 2, \dots, k)$, which confirms that $\left[{n \atop k}\right]$ is an integer. The proposition is proved. ■

Definition. Throughout this paper, we call the numbers $\left[{n \atop k}\right]$: "the lcm-binomial numbers" and we call the triangle consisting of them: "the lcm-binomial triangle".

The beginning of the lcm-binomial triangle is given in the following:

$$
\begin{array}{llllllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 2 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 10 & 5 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}
$$

(Here the colored numbers in green are those that are different from their analog binomial numbers).

Now, we are going to establish less obvious results concerning the lcm-binomial numbers.

Theorem 2 *For all natural numbers n, k such that $n \geq k$, the lcm-binomial number $\left[{n \atop k}\right]$ divides the binomial number $\binom{n}{k}$.*

Proof. Actually the theorem can be immediately showed by using a result of S. Hong and Y. Yang [3] which states that for all integers k, n (with $k \geq 0$, $n \geq 1$), the positive integer $g_k(1)$ divides the positive integer $g_k(n)$, where g_k denotes the Farhi arithmetical function[1] (see Lemma 2.4 of [3]). But in order to put the reader at their ease, we give in what follows an independent and complete proof. Let $n, k \in \mathbb{N}$ such that $n \geq 1$ and $n \geq k$. The statement of the theorem is clearly equivalent to the following inequalities:

$$v_p\left(\binom{n}{k}\right) \geq v_p\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) \quad \text{(for all prime number } p\text{)} \tag{5}$$

(where v_p denotes the usual p-adic valuation).
Let us show (5) for a given prime number p. On the one hand, we have:

$$\begin{aligned} v_p\left(\binom{n}{k}\right) &= v_p\left(\frac{n!}{k!(n-k)!}\right) \\ &= v_p(n!) - v_p(k!) - v_p((n-k)!) \\ &= \sum_{\alpha=1}^{\infty}\left\lfloor\frac{n}{p^\alpha}\right\rfloor - \sum_{\alpha=1}^{\infty}\left\lfloor\frac{k}{p^\alpha}\right\rfloor - \sum_{\alpha=1}^{\infty}\left\lfloor\frac{n-k}{p^\alpha}\right\rfloor \\ &= \sum_{\alpha=1}^{\infty}\left(\left\lfloor\frac{n}{p^\alpha}\right\rfloor - \left\lfloor\frac{k}{p^\alpha}\right\rfloor - \left\lfloor\frac{n-k}{p^\alpha}\right\rfloor\right) \end{aligned} \tag{6}$$

(where $\lfloor . \rfloor$ represents the integer part function).
It is important to stress that each of the terms $(\lfloor\frac{n}{p^\alpha}\rfloor - \lfloor\frac{k}{p^\alpha}\rfloor - \lfloor\frac{n-k}{p^\alpha}\rfloor)$ $(\alpha \geq 1)$, of the last sum, is nonnegative. indeed, for all positive integer α, we have:

$$\left\lfloor\frac{k}{p^\alpha}\right\rfloor + \left\lfloor\frac{n-k}{p^\alpha}\right\rfloor \leq \frac{k}{p^\alpha} + \frac{n-k}{p^\alpha} = \frac{n}{p^\alpha}.$$

But since $\lfloor\frac{k}{p^\alpha}\rfloor + \lfloor\frac{n-k}{p^\alpha}\rfloor$ is an integer, then we have even:

$$\left\lfloor\frac{k}{p^\alpha}\right\rfloor + \left\lfloor\frac{n-k}{p^\alpha}\right\rfloor \leq \left\lfloor\frac{n}{p^\alpha}\right\rfloor,$$

which confirms the stressed fact.
Now, on the other hand, we have:

$$\begin{aligned} v_p\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) &= v_p\left(\frac{\mathrm{lcm}(n, n-1, \dots, n-k+1)}{\mathrm{lcm}(1, 2, \dots, k)}\right) \\ &= a - b, \end{aligned}$$

where

$$\begin{aligned} a &:= v_p(\mathrm{lcm}(n, n-1, \dots, n-k+1)) \quad \text{and} \\ b &:= v_p(\mathrm{lcm}(1, 2, \dots, k)). \end{aligned}$$

[1]By definition: $g_k(n) := \frac{n(n+1)\cdots(n+k)}{\mathrm{lcm}(n,n+1,\dots,n+k)} \quad (\forall k, n)$.

Note that because $\left[{n \atop k}\right]$ is an integer (according to Proposition 1), we have $a \geq b$. By definition, a is the greatest exponent α of p for which p^α divides at least an integer of the range $(n-k, n]$. Since for all $\alpha \in \mathbb{N}$, the number of integers belonging to the range $(n-k, n]$, which are multiples of p^α, is exactly equal to $\lfloor \frac{n}{p^\alpha} \rfloor - \lfloor \frac{n-k}{p^\alpha} \rfloor$, then we have:

$$a = \max\left\{\alpha \in \mathbb{N} : \left\lfloor \frac{n}{p^\alpha} \right\rfloor - \left\lfloor \frac{n-k}{p^\alpha} \right\rfloor \geq 1\right\} \tag{7}$$

Similarly, b is (by definition) the greatest exponent α of p for which p^α divides at least an integer of the range $[1, k]$. But since for all $\alpha \in \mathbb{N}$, the number of integers belonging to the range $[1, k]$, which are multiples of p^α, is exactly equal to $\lfloor \frac{k}{p^\alpha} \rfloor$, then we have:

$$b = \max\left\{\alpha \in \mathbb{N} : \left\lfloor \frac{k}{p^\alpha} \right\rfloor \geq 1\right\} \tag{8}$$

Remarking that the sequence $\left(\lfloor \frac{n}{p^\alpha} \rfloor - \lfloor \frac{n-k}{p^\alpha} \rfloor\right)_{\alpha \in \mathbb{N}}$ is non-increasing (since each of the terms $\lfloor \frac{n}{p^\alpha} \rfloor - \lfloor \frac{n-k}{p^\alpha} \rfloor$ represents the number of integers lying in the range $(n-k, n]$, which are multiples of p^α), we have:

$$\forall \alpha \in \mathbb{N}, \alpha \leq a : \quad \left\lfloor \frac{n}{p^\alpha} \right\rfloor - \left\lfloor \frac{n-k}{p^\alpha} \right\rfloor \geq 1.$$

Further, from the definition of b, we have:

$$\forall \alpha \in \mathbb{N}, \alpha > b : \quad \left\lfloor \frac{k}{p^\alpha} \right\rfloor = 0.$$

Consequently, we have:

$$\forall \alpha \in \mathbb{N} \cap (b, a] : \quad \left\lfloor \frac{n}{p^\alpha} \right\rfloor - \left\lfloor \frac{n-k}{p^\alpha} \right\rfloor - \left\lfloor \frac{k}{p^\alpha} \right\rfloor \geq 1.$$

According to (6), it follows that:

$$\begin{aligned}
v_p\left(\binom{n}{k}\right) &= \sum_{\alpha=1}^{\infty}\left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor - \left\lfloor \frac{n-k}{p^\alpha} \right\rfloor - \left\lfloor \frac{k}{p^\alpha} \right\rfloor\right) \\
&\geq \sum_{b<\alpha\leq a}\left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor - \left\lfloor \frac{n-k}{p^\alpha} \right\rfloor - \left\lfloor \frac{k}{p^\alpha} \right\rfloor\right) \\
&\geq \sum_{b<\alpha\leq a} 1 \\
&= a - b \\
&= v_p\left(\left[\begin{matrix} n \\ k \end{matrix}\right]\right),
\end{aligned}$$

which confirms (5) and completes this proof. ■

Now, by Theorem 2, we see that the ratios $\binom{n}{k}/\left[{n \atop k}\right]$ are actually positive integers. But it certainly remains several other profound properties to discover about those numbers. We can ask for example about the couples (n, k) satisfying the equality $\binom{n}{k} = \left[{n \atop k}\right]$.
The following theorem shows a very important property for the ratios $\binom{n}{k}/\left[{n \atop k}\right]$. We derive from it for example that for a fixed column k, the numbers $\left(\binom{n}{k}/\left[{n \atop k}\right]\right)_{n \geq k}$ lie in a finite set of positive integers.

Theorem 3 *For all $k \in \mathbb{N}$, the sequence of positive integers $\left(\frac{\binom{n}{k}}{\left[{n \atop k}\right]}\right)_{n \geq k}$ is periodic and its smallest period T_k is given by:*

$$T_k = \prod_{p \text{ prime, } p < k} p^{\alpha_p},$$

where

$$\alpha_p = \begin{cases} 0 & \text{if } v_p(k) \geq \max_{1 \leq i < k} v_p(i) \\ \max_{1 \leq i < k} v_p(i) & \text{otherwise} \end{cases} \qquad (\forall p \text{ prime}, p < k).$$

As an important consequence, we derive the following:

Corollary 4 *For all $k \in \mathbb{N}$, the positive integer $\mathrm{lcm}(1, 2, \dots, k-1)$ is a period of the sequence $\left(\frac{\binom{n}{k}}{\left[{n \atop k}\right]}\right)_{n \geq k}$.*

Admitting Theorem 3, the proof of Corollary 4 becomes obvious: it suffices to remark that the exact period T_k, given by Theorem 3, of the sequence $\left(\binom{n}{k}/\left[{n \atop k}\right]\right)_{n \geq k}$ clearly divides $\mathrm{lcm}(1, 2, \dots, k-1)$.

To prove Theorem 3, we use the arithmetical functions g_k ($k \in \mathbb{N}$) introduced by the author in [1] and studied later by Hong and Yang [3] and by Farhi and Kane [2]. For a given $k \in \mathbb{N}$, the function g_k is defined by:

$$\begin{array}{rcl} g_k : \mathbb{N} \setminus \{0\} & \longrightarrow & \mathbb{N} \setminus \{0\} \\ n & \longmapsto & g_k(n) := \frac{n(n+1)\cdots(n+k)}{\mathrm{lcm}(n, n+1, \dots, n+k)} \end{array}.$$

In [1], it is just remarked that g_k is periodic and that $k!$ is a period of g_k. Then Hong and Yang [3] improved that period to $\mathrm{lcm}(1, 2, \dots, k)$ and recently, Farhi and Kane [2] have obtained the exact period of g_k which is given by:

$$P_k = \prod_{p \text{ prime, } p \leq k} p^{\begin{cases} 0 & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i) \\ \max_{1 \leq i \leq k} v_p(i) & \text{otherwise} \end{cases}}.$$

Knowing this result, the proof of Theorem 3 becomes easy:

Proof of Theorem 3. For a fixed $k \in \mathbb{N}$, a simple calculus shows that for any $n \in \mathbb{N}$, we have:

$$\frac{\binom{n}{k}}{\left[{n \atop k}\right]} = \frac{g_{k-1}(n-k+1)}{g_{k-1}(1)}.$$

This last identity clearly shows that for any given $k \in \mathbb{N}$, the sequence $\left(\binom{n}{k} / \left[{n \atop k}\right]\right)_{n \geq k}$ is periodic and that its exact period is equal to the exact period of g_{k-1}. So by the Farhi-Kane theorem, the exact period of $\left(\binom{n}{k} / \left[{n \atop k}\right]\right)_{n \geq k}$ is P_{k-1}, as claimed in Theorem 3. ■

We end this section by giving the lcm-binomial triangle until its 12th row.

1													
1	1												
1	2	1											
1	3	3	1										
1	4	6	2	1									
1	5	10	10	5	1								
1	6	15	10	5	1	1							
1	7	21	35	35	7	7	1						
1	8	28	28	70	14	14	2	1					
1	9	36	84	42	42	42	6	3	1				
1	10	45	60	210	42	42	6	3	1	1			
1	11	55	165	330	462	462	66	33	11	11	1		
1	12	66	110	165	66	462	66	33	11	11	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋱

The lcm-analog of Al-Karaji's triangle

Note that The lcm-binomial numbers colored in green are those that are different from their analog binomial numbers.

3 Some remarks and open problems about the lcm-binomial numbers

1) Can we prove Theorem 2 without use prime number arguments?

2) Describe the set of all the couples (n, k) $(n \geq k \geq 0)$ satisfying $\left[{n \atop k}\right] = \binom{n}{k}$.

3) Let $n \in \mathbb{N}$. Since for any $k \in \{0, 1, \dots, n\}$, we have $\left[{n \atop k}\right] \leq \binom{n}{k}$ (because $\left[{n \atop k}\right]$ divides $\binom{n}{k}$, according to Theorem 2) then for all nonnegative real number x, we have:

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] x^k \ \leq \ \sum_{k=0}^{n} \binom{n}{k} x^k \ = \ (1+x)^n,$$

that is:

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] x^k \ \leq \ (1+x)^n \qquad (\forall x \geq 0) \ . \tag{9}$$

Taking $x = 1$ in (9), we deduce in particular that for all $n \in \mathbb{N}$, we have $\left[{n \atop \lceil n/2 \rceil}\right] \leq 2^n$ (where $\lceil . \rceil$ denotes the ceiling function). But since $\left[{n \atop \lceil n/2 \rceil}\right] = \frac{\mathrm{lcm}(n, n-1, \dots, n - \lceil n/2 \rceil + 1)}{\mathrm{lcm}(1, 2, \dots, \lceil n/2 \rceil)}$ is an integer (according to Proposition 1), then $\mathrm{lcm}(n, n-1, \dots, n - \lceil n/2 \rceil + 1)$ is a multiple of $\mathrm{lcm}(1, 2, \dots, \lceil n/2 \rceil)$. Consequently we have $\mathrm{lcm}(n, n-1, \dots, n - \lceil n/2 \rceil + 1) = \mathrm{lcm}(n, n-1, \dots, n - \lceil n/2 \rceil + 1; 1, 2, \dots, \lceil n/2 \rceil) = \mathrm{lcm}(1, 2, \dots, n)$. So $\left[{n \atop \lceil n/2 \rceil}\right] \leq 2^n$ gives:

$$\mathrm{lcm}(1, 2, \dots, n) \leq 2^n \mathrm{lcm}(1, 2, \dots \lceil n/2 \rceil) \qquad (\forall n \in \mathbb{N}).$$

The iteration of the last inequality gives:

$$\mathrm{lcm}(1, 2, \dots, n) \leq 2^{n + \lceil n/2 \rceil + \lceil n/4 \rceil + \dots} \leq 2^{2n + \log_2(n)} = n 4^n \qquad (\forall n \geq 1).$$

Hence:

$$\mathrm{lcm}(1, 2, \dots, n) \leq n 4^n \qquad (\forall n \geq 1),$$

which is a nontrivial upper bound of $\mathrm{lcm}(1, 2, \dots, n)$.
The question which we pose is the following:

> Can we more judiciously use Relation (9) to prove a nontrivial upper bound for the least common multiple of consecutive integers that is significatively better than the previous one?

4) It is easy to see that unfortunately there is no an internal composition law $\star$ of $\mathbb{N}$ which satisfies for any positive integers n, k ($n \geq k$):

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \left[\begin{array}{c} n-1 \\ k-1 \end{array} \right] \star \left[\begin{array}{c} n-1 \\ k \end{array} \right]$$

(the analog of (2)).
Indeed, if we suppose that such a law $\star$ exists then we would have on the one hand $\left[{2 \atop 1}\right] \star \left[{2 \atop 2}\right] = \left[{3 \atop 2}\right]$, that is $2 \star 1 = 3$ and on the other hand $\left[{4 \atop 3}\right] \star \left[{4 \atop 4}\right] = \left[{5 \atop 4}\right]$, that is $2 \star 1 = 5$; which gives a contradiction.
The problem which we pose is the following:

Find an iterative construction (i.e., a construction row by row) for the lcm-binomial triangle.

5) For a given positive integer d, let $\Omega(d)$ denote the number of prime factors of d, counting with their multiplicities.
In this item, we look at the diagonals of the lcm-binomial triangle. We constat that the first diagonal (which we note by D_0) contains only the 1's; in other words, we have:

$$\forall d \in D_0 : \Omega(d) = 0 \leq 0.$$

The second diagonal (noted D_1) is consisted only on the 1's and the prime numbers; in other words, we have:

$$\forall d \in D_1 : \Omega(d) \leq 1.$$

Also, the third diagonal of the lcm-binomial triangle (noted D_2) is consisted of positive integers having at most two prime factors (counting with their multiplicities); in other words, we have:

$$\forall d \in D_2 : \Omega(d) \leq 2.$$

More generally, we have the following:

Proposition 5 *For $k \in \mathbb{N}$, let D_k denote the $(k+1)^{th}$ diagonal of the lcm-binomial triangle. Then, we have:*

$$\forall d \in D_k : \quad \Omega(d) \leq k.$$

The proof of this proposition is actually very easy and leans only on the following simple fact:

$$\forall n \in \mathbb{N} : \frac{\mathrm{lcm}(1,2,\dots,n,n+1)}{\mathrm{lcm}(1,2,\dots,n)} = \begin{cases} p & \text{if } n+1 \text{ is a power of a prime } p \\ 1 & \text{otherwise} \end{cases}.$$

Proof of Proposition 5. Let $k \in \mathbb{N}$ fixed and let $d \in D_k$. So, we can write d as: $d = \left[{n+k \atop n}\right] = \frac{\mathrm{lcm}(k+1,k+2,\dots,k+n)}{\mathrm{lcm}(1,2,\dots,n)}$ (for some $n \in \mathbb{N}$). It follows that d divides the positive integer $\frac{\mathrm{lcm}(1,2,\dots,n+k)}{\mathrm{lcm}(1,2,\dots,n)}$. But we constat that the last number is the product of the k positive integers $\frac{\mathrm{lcm}(1,2,\dots,n+i)}{\mathrm{lcm}(1,2,\dots,n+i-1)}$ $(1 \leq i \leq k)$ each of which is either a prime number or equal to 1 (according to the fact mentioned just before this proof). So, it follows that:

$$\Omega(d) \leq \Omega\left(\frac{\mathrm{lcm}(1,2,\dots,n+k)}{\mathrm{lcm}(1,2,\dots,n)}\right) \leq k.$$

The proposition is proved. ∎

Note that by using prime number theory, we can improve the obvious upper bound of Proposition 5 to:

$$\forall d \in D_k : \quad \Omega(d) \leq \mathrm{c}\frac{k}{\log k},$$

where c is an absolute positive constant (effectively calculable).

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