Markov Processes Relat. Fields **25**, 329–358 (2019)

Markov Processes and Related Fields

Strong Stability and Uniform Ergodicity Estimates for the Waiting Process in Queuing Models with Impatient Customers

Zahir Mouhoubi[1] **and Djamil Aïssani**[2]

[1] Department of Mathematics of Édouard-Montpetit College, 945 Chemin de Chambly, Longueuil, QC J4H 3M6, Canada. E-mail: zahir.mouhoubi@cegepmontpetit.ca
[2] Laboratory of Modelization and Optimization of Systems, Faculty of Sciences and Engineer Sciences, University of Béjaïa, 06000, Algeria

Received January 23, 2018, revised September 7, 2018

Abstract. We provide sufficient conditions for the strong stability and the uniform ergodicity for the discrete-time waiting process describing the $GI/GI/1$ single-server queue with general patience time distribution. The rate of convergence to the stationarity and the potential of the chain are derived also. In addition, we obtain upper bounds of the deviation of the stationary and transition characteristics under perturbation of patience time distribution. The perturbation of the structure of the $GI/GI/1$ queueing system with ordinary customers which leads to the $GI/GI/\,1$ system with impatient customers is also considered.

Keywords: strong stability; uniform ergodicity; structural perturbation; renewal equation; potential of the chain; impatient queues

AMS Subject Classification: 60J05; 60J20; 60K25

1. Introduction

In modelling practical problems, the real system is often generally considered as a complex system which depends in complicated way on its parameters. Moreover, the parameters of the complex system are not often known exactly because they are obtained by statistical methods from empirical data. Therefore, the analysis of this type of complex systems does not allow us to obtain explicitly analytical formula. Also, if we are able to establish analytical results they are not generally useful in practice.

For this, we try usually to replace the complex system, by another which is more simpler in structure and/or component, and close to it in some particular sense. Thus a real complex system can be considered as a simpler one, with perturbed parameters, which can be considered as an ideal system. However, in order to justify these approximations and estimate the resultant error, it is essentially of interest and important to specify the kind and the type of this perturbation, and so the stability problem arises. In particular, the stability problem in queueing theory arises to establish the domain within which an ideal queueing system may be considered as a good approximation, in some sense, to the real complex queueing system under consideration.

The mentioned approximation must be precise in some sense. For this, numerous methods to investigate the stability problem are developed such as the metric method [48], test function method [23], method of the proximity points [22], renovating events or renewal method [7], weak convergence method applied to queueing system [42], stability method for continuous-time nonhomogeneous countable Markov chain [47], method of the uniform stability applied to a finite irreducible Markov chain [21] and the strong stability method for homogeneous Markov chain on a general state space [26]. In particular, we investigate what would be the effect on the stationary characteristics of Markov chain under the perturbation of its parameters, with respect to a given norm. These results are called perturbation bounds and are generally established in an explicit form. For this, various perturbation bounds of Markov chains are considered in literature. The first kind of results is devoted to finite Markov chains and obtained by using matrix analysis, see a nice review by Cho and Meyer [11] and some recent articles [19, 20, 27–30, 37, 40]. The latter results concern the estimate of the deviation of stationary distributions with respect to the total variation norm and the component-wise perturbation bounds. The second kind of results is devoted to general state Markov chains where probabilistic and operator-theoretic methods are used. More precisely, the results of the first group concern the estimates of the stationary and transition characteristics of homogeneous and inhomogeneous general state Markov chains with respect to the total variation norm (see Mitrophanov [33, 34] and Anisimov [3]). Another group is devoted to the same estimates with respect to a large class of norms (see Kartashov [24–26], Aïssani and Kartashov [2], Mouhoubi and Aïssani [36, 38]). These perturbation bounds are derived in terms of the ergodicity coefficients of the iterated transition kernel [3, 25, 33, 34], the rate of convergence to the stationarity [34] and the residual kernel [2, 24, 36]. Note that some numerical methods to compute the stationary distribution are considered by using the series expansion (see Heidergott and Hordijk [15] and Heidergott et al. [16, 17]).

The stability theory of queueing systems is devoted to establishing which model may be used as a good approximation of the real system under consideration. A class of queueing systems that is of particular interest are systems with impatient customers used usually as models which describe some computer systems and telecommunication networks. Some particular systems are investigated in [4, 5, 8, 9, 13, 31, 39, 43–46].

In this paper, we consider the embedded Markov chain, describing the waiting time in a $GI/GI/1$ queueing system with general impatient time distribution. To the best of our knowledge, there exist few papers which are devoted to investigation of qualitative properties of this process. Thus, Charlot and Pujolle in [10] consider the sufficient condition under which this process is recurrent in Harris sense. In the same way, Baccelli et al. in [4] have established a δ-irreducibility (see [32] for the definition of the irreducibility with respect to a measure), where δ represents the degenerate measure concentrated on zero. They also derived for the particular $M/G/1$ system with impatient customers an integral equation for the virtual waiting time. Moreover, they have established a relation between actual and virtual waiting-time distribution functions. However, transition and stationary characteristics of this process are not established if the impatient distribution function is not exponential and the distributions of the inter-arrival and service times are not of particular form.

The first purpose of this article is to clarify the sufficient conditions under which the embedded waiting time Markov chain in $GI/GI/1$ queueing system with general impatient time distribution is uniformly ergodic and strongly stable with respect to a judicious weighted norm. Moreover, we estimate the rate of convergence to the stationarity and derive the expression for the potential of the considered chain. Note that the uniform ergodicity is established using a special criterion, which will be given below, and not by considering the usual drift condition (see Meyn and Tweedie [32]). Also a straightforward proof is given to obtain the rate of convergence to the stationarity. The second goal is to show that under some assumptions, the stationary distribution of the waiting time in the perturbed $GI/GI/1$ queueing system with a general impatient time distribution can be approximated as such in the ideal system under consideration, which has the same structure but a different impatient time distribution. In some applications, the second model can be considered as the ideal one with exponential impatient time distribution since, in this case, some characteristics of the ideal system are well known. Furthermore, after clarifying the approximation conditions, we establish the uniform ergodicity and strong stability inequalities with exact computation of the constants. Similar estimates are derived for the approximation of the characteristics of the perturbed $GI/GI/1$ queueing system with general impatient time distribution by those of the ideal $GI/GI/1$ queueing system with ordinary customers. Note that the strong stability estimates for the Lindley waiting process in the $GI/GI/1$ queueing system with ordinary customers have been considered by Kartashov [26] and by Mouhoubi and Aïssani [35].

The paper is organized as follows. The next section presents a brief description of the model. Section 3 presents basic facts on strong stability method and some related results. The uniform ergodicity and the strong stability is discussed in Section 4. Uniform ergodicity estimates and some related properties are considered in Section 5. In Subsection 6.1, we perturb the patience time distribution in order to estimate the transient and stationary characteristics of the queue. The later estimates are also investigated under a structural perturbation in Subsection 6.2. Finally, some concluding remarks have been done.

2. Description of the model

We consider a general single server queue with impatient customers. That is, some customers leave the system if their waiting time exceeds a specified time interval. The description of the waiting time process in this queueing system is classically known (see Baccelli et al. [4]). Suppose that the customers arrive to the FIFO $GI/GI/1$ single-server queue system and the server does not stop freely the service. The customers are indexed by $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Let us denote by A_n the inter-arrival time between the $(n-1)$-th and the n-th customer. The 0-th customer arrives at time 0. The n-th customer, for $n \geq 1$, arrives at time $T_n = \sum_{i=1}^{n} A_i$. In this case, we can observe that $A_{n+1} = T_{n+1} - T_n$ for all $n \geq 0$, where $T_0 = 0$. Moreover, let us denote by B_n the service time of the $(n-1)$-th customer and by C_n the patient time of the $(n-1)$-th customer. Assume that the three sequences $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$ of the random variables are independent. Also we assume that each of those sequences is formed with the independent identically distributed random variables. It is well known [4] that the embedded Markov chain $W = \{W_n : n = 0, 1, \ldots\}$ of the waiting time process is given by the following non monotone recursive equation

$$W_{n+1} = \begin{cases} (W_n + B_{n+1} - A_{n+1})^+ & if \;\; W_n < C_{n+1}, \\ (W_n - A_{n+1})^+ & if \;\; C_{n+1} \leq W_n. \end{cases}$$

In the rest of this paper we denote by η and ϑ the probability distribution functions (defined on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$) of A_n and B_n respectively, and by γ the probability distribution function of C_n (defied on $(\overline{\mathbb{R}}_+, \mathcal{B}_{\overline{\mathbb{R}}_+})$). We consider

$$a = ess.\sup A_1 = \inf\{t \geq 0 : \eta([0, t]) = 1\},$$

$$b = ess.\inf B_1 = \sup\{t \geq 0 : \vartheta([t, +\infty]) = 1\}.$$

It is known that the process W is a homogeneous Markov process (see [10]). Then, in order to ensure the ergodicity of the process, it is sufficient to assume the following condition (see [4])

$$\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) < \mathbb{E}(A_1) \text{ and } b - a < 0. \tag{2.1}$$

In this case, we denote by π the unique invariant probability measure of the transition kernel P of the chain W. The transition kernel $\{P(x,\Gamma), x \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}_+)\}$ of the Markov chain W is given for all $(x,\Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$ as follows:

$$P(x,\Gamma) = (1-\gamma(x)) \int_{\mathbb{R}_+\times\mathbb{R}_+} \mathbb{I}_\Gamma([x+y-z]^+) d\vartheta(y) d\eta(z) + \gamma(x) \int_{\mathbb{R}_+} \mathbb{I}_\Gamma([x-z]^+) d\eta(z)$$

where $\mathbb{I}_\Gamma$ is the indicator function of the measurable set Γ. In the remainder of this paper, we assume the following Cramér condition:

$$m = \mathbb{E}[\xi_1] < 0, \, \exists \beta_0 > 0 : \mathbb{E}(\exp(\beta_0 \xi_1)) < \infty \tag{2.2}$$

where $\xi_k = B_k - A_k$ for all $k \geq 0$. Note that the condition $m = \mathbb{E}[\xi_1] < 0$ implies $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) < \mathbb{E}(A_1)$.

3. Preliminaries and notations

Note that all notations used in this paper are introduced in many references [26, 35, 38].

Let $W = (W_n, n \in \mathbb{N})$ be a homogenous Markov chain, describing the previous system, taking values in a measurable space $E = \mathbb{R}_+$ and defined on the phase space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, where $\mathcal{E} = \mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra. We introduce the trial function $v : E \longrightarrow [1, +\infty[$ such that $v(x) = \exp(\beta x)$ for all $x \in \mathbb{R}_+$ where β is a nonnegative parameter. Furthermore, we provide the space $m\mathcal{E}$ of finite measures on the σ-algebra $\mathcal{E}$ with a norm $\|\cdot\|_\beta$ which has the following form:

$$\|\mu\|_\beta = \int_0^{+\infty} \exp(\beta x) |\mu|(dx),$$

where $|\mu|$ is the full variation of the measure μ.

For the Banach subspace $\mathcal{M}_\beta = \{\mu \in m\mathcal{E} : \|\mu\|_\beta < \infty\}$, we introduce the dual space $\mathcal{J}_\beta$ of measurable functions on E. This norm induces a corresponding norm in the space $\mathcal{J}_\beta$ (with the finite norm) namely,

$$\|f\|_\beta = \sup_{x \geq 0} \exp(-\beta x) |f(x)| \text{ for all } f \in \mathcal{J}_\beta \tag{3.1}$$

as well as a norm in the space B_β of kernels Q which satisfy $\mathcal{M}_\beta Q \subset \mathcal{M}_\beta$ (and with finite norm), namely,

$$\|Q\|_\beta = \sup_{x \geq 0} \exp(-\beta x) \int_0^{+\infty} |Q|(x,dy) \exp(\beta y). \tag{3.2}$$

The action of each transition kernel Q on $\mu \in \mathcal{M}_\beta$ and $f \in \mathcal{J}_\beta$ is defined respectively for all $x \in E$ and $\Gamma \in \mathcal{E}$ as follows

$$\mu Q(\Gamma) = \int_0^{+\infty} Q(x,\Gamma)\mu(dx) \tag{3.3}$$

and

$$Qf(x) = \int_0^{+\infty} Q(x,dy)f(y). \tag{3.4}$$

Moreover, for all $\mu \in \mathcal{M}_\beta$ and $f \in \mathcal{J}_\beta$, the symbols μf and $f \otimes \mu$ denote respectively the integral and the kernel defined as follows:

$$\mu f = \int_0^{+\infty} f(y)\mu(dx), \tag{3.5}$$

$$(f \otimes \mu)(x,\Gamma) = f(x)\,\mu(\Gamma), \text{ for all } (x,\Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+). \tag{3.6}$$

For two kernels Q and K on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$, we define their product QK as the kernel

$$QK(x,\Gamma) = \int_0^{+\infty} Q(x,dy)K(y,\Gamma), \text{ for all } (x,\Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+). \tag{3.7}$$

When the Markov chain W is ergodic (admits a unique invariant probability), the stationary projector of the chain, Π, is given by the following identity: $\Pi = \mathbf{1} \otimes \pi$, where π is the invariant probability measure of the transition kernel P and $\mathbf{1}$ is the function identically equal to the unit [26].

Let us introduce the concept of the strong stability and the uniform ergodicity of the homogeneous Markov chain W.

Definition 3.1. The chain W is said to be strongly stable with respect to the norm $\|\cdot\|_\beta$ if

1) $\|P\|_\beta < \infty$.

2) Each transition kernel $Q \in \mathsf{B}_\beta$ in some neighborhood $\{Q : \|Q - P\|_\beta < \varepsilon\}$ has a unique invariant probability measure $\nu = \nu(Q) \in \mathcal{M}_\beta$ with $\|\pi - \nu\|_\beta \longrightarrow 0$, uniformly in this neighborhood, as $\|Q - P\|_\beta \longrightarrow 0$.

Definition 3.2. The aperiodic Markov chain W is said to be uniformly ergodic with respect to the weighted norm $\|\cdot\|_\beta$ if $P^t \longrightarrow \Pi$ in the induced operator weighted norm. More precisely, this is equivalent to the fact that there exist nonnegative constants ω and ε such that for all $t \geq 0$ we have

$$\|P^t - \Pi\|_\beta \leq \zeta\varepsilon^t.$$

The parameter ε is called an estimate of the rate of convergence and evaluates the speed of convergence to stationarity.

Recently, in [41] an asymptotic analysis by a nonlinear diffusion approximation was established. However, the rate of convergence (uniform ergodicity) and the sensitivity of the system parameters to small perturbations (strong stability) were not considered. Moreover, the approximations are obtained under some restrictions on the hazard rate function of the patience time distribution. For example, the distributions whose hazard rate tends to infinity on its support (see assumption 2 in [41]) were avoided. On the other hand, the diffusion approximation was established without Cramér condition (2.2). However, Cramér condition (2.2) is appropriate, because the kindness results in this paper (as compared to [41]) will require this. The same remarks may be done for the results obtained in [18].

Using our notations, we shall use in the sequel, the following results obtained for a large class of norms but expressed in this paper with respect to the weighted norm $\|\cdot\|_\beta$.

Theorem 3.1 (Theorem 2.3, [26]). *Assume that a Markov chain X with the regular transition operator P has a unique invariant probability measure π that satisfies the following conditions*

(A) $\|P\|_\beta < \infty$.

(B) *There exist a natural n, a nonnegative measure $\alpha \in \mathcal{M}_\beta$ and a nonnegative function $h \in \mathcal{J}_\beta$ such that $\pi h > 0$, $\alpha h > 0$, $\alpha \mathbf{1} = 1$ and the residual kernel $T = P^n - h \otimes \alpha$ is nonnegative.*

Then the Markov chain X is uniformly ergodic, strongly stable in the norm $\|\cdot\|_\beta$ and aperiodic if and only if for some n, α and h from (B), we have

(C) $\|T^m\|_\beta \leq \rho$ *for some $m \geq 1$ and $\rho < 1$.*

Furthermore, the uniform ergodicity and aperiodicity of the chain X under condition (A) imply that condition (C) is fulfilled for all n, α and h satisfying condition (B).

Remark 3.1. An other proof of this result is given by Mouhoubi and Aïssani in [36].

Remark 3.2. Observe that condition (B) is equivalent to the α-irreducibility of the Markov chain X.

Theorem 3.2 (Theorem 3.8, [26]). *Let conditions* (A), (B) *and* (C) *of Theorem* 3.1 *be satisfied for* $m = n = 1$ *with* $\sigma = \|\Pi\|_\beta < \infty$. *Then any stochastic transition kernel* $Q = P + \Delta$ *from a neighborhood*

$$\|Q - P\|_\beta = \|\Delta\|_\beta = \varepsilon < \frac{1-\rho}{1+\sigma\rho} = \varepsilon_0$$

has a unique invariant probability measure ν *and*

$$\|\nu - \pi\|_\beta = \frac{\|\pi\|_\beta}{\varepsilon_0 - \varepsilon}\,\varepsilon. \tag{3.8}$$

Theorem 3.3 (Theorem 6, [38]). *Let conditions* (A), (B) *and* (C) *of Theorem* 3.1 *be satisfied for* $m = n = 1$. *Then any transition kernel* Q *from a neighborhood*

$$\|Q - P\|_\beta = \|\Delta\|_\beta < 1 - \rho$$

has a unique invariant probability measure $\nu = \mu_\Delta/(\mu_\Delta \mathbf{1})$ *where the measure* μ_Δ *is equal to* $\sum_{i=0}^{+\infty} \alpha(\Delta + T)^i$. *Moreover, we have the following stability estimate*

$$\|\nu - \pi\|_\beta \le \frac{\|\alpha\|_\beta}{(1 - \rho - \|\Delta\|_\beta)^2}\Big(1 + \frac{\|\alpha\|_\beta\|\mathbf{1}\|_\beta}{1-\rho}\Big)\|\Delta\|_\beta, \tag{3.9}$$

where α and T are introduced in Theorem 3.1.

4. Ergodicity and stability analysis

We start our investigation by establishing sufficient conditions of the strong stability of the Markov process W. Before this, let us give a similar canonical decomposition of the transition kernel P as stated in Theorem 3.1. For all $(x, \Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$, let us denote $h(x) = (1 - \gamma(x))h_1(x) + \gamma(x)h_2(x)$ where $h_1(x) = \mathbb{P}(x + B_1 - A_1 \le 0)$, $h_2(x) = \mathbb{P}(x - A_1 \le 0)$ and $T(x, \Gamma) = (1 - \gamma(x))\,(\mathbb{P}(0 < x + \xi_1 \in \Gamma) + \gamma(x)\mathbb{P}(0 < x - A_1 \in \Gamma))$, where $\gamma(x) = \mathbb{P}(C_1 \le x)$. Moreover, W_n^x represents the waiting time W_n conditioning $W_0 = x$ for all $x \in \mathbb{R}_+$ and $n \ge 0$.

The following result may be obtained by a simple calculus and so the proof is omitted.

Lemma 4.1. *The transition kernel* P *admits the following canonical decomposition*

$$P(x, \Gamma) = T(x, \Gamma) + h(x)\delta(\Gamma), \tag{4.1}$$

where δ *is the Dirac measure concentrated on zero.*

Remark 4.1. It is easy to see that $T(x, \Gamma) = \mathbb{P}(0 < W_1^x \in \Gamma)$ and $h(x) = \mathbb{P}(W_1^x = 0)$ for all $(x, \Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$.

Let us denote by $\tau^x = \inf\{n \geq 1 : W_n^x = 0\}$ the first return time to state 0 where we set by convention $\inf \emptyset = +\infty$. We have for all $t \geq 0$

$$T^t(x, \Gamma) = \mathbb{P}(\tau^x > t, W_t^x \in \Gamma)$$

and

$$T^t h(x) = \mathbb{P}(\tau^x = t + 1).$$

Therefore, we obtain for all $t \geq 1$

$$p_t = \delta T^{t-1} h(x) = T^{t-1} h(0) = \mathbb{P}(\tau^0 = t).$$

We observe easily that for $t \geq 2$, we have

$$\begin{aligned} \lambda(t) &= \mathbb{P}(W_{t-1}^0 = 0) = \sum_{k=1}^{t-1} \mathbb{P}(\tau^0 = k, W_{t-1}^0 = 0) \\ &= \sum_{k=1}^{t-1} \mathbb{P}(\tau^0 = k)\mathbb{P}(W_{t-1-k}^0 = 0) = \sum_{k=1}^{t-1} \lambda(t-k) p_k. \end{aligned}$$

Moreover, it is obvious that $\lambda(1) = 1$ and $\lambda(t) = 0$ for all $t \leq 0$. Hence, the sequence $\lambda(t) = \mathbb{P}(W_{t-1}^0 = 0)$ satisfies the discrete renewal equation below

$$\begin{cases} \lambda(t) = \sum_{k=1}^{t-1} \lambda(k) p_{t-k} & \text{if } t \geq 2, \\ \lambda(1) = 1,\ \lambda(t) = 0 & \text{if } t \leq 0. \end{cases} \tag{4.2}$$

On the other hand, using the classical decomposition with respect to the first entrance, from the last exit decomposition [12] we obtain

$$\begin{aligned} P^t(x, \Gamma) &= \mathbb{P}(W_t^x \in \Gamma) = \mathbb{P}(W_t^x \in \Gamma, \tau^x > t) + \mathbb{P}(W_t^x \in \Gamma, \tau^x \leq t) \\ &= T^t(x, \Gamma) + \sum_{k=1}^{t} \mathbb{P}(\tau^x = k, W_t^x \in \Gamma) \\ &= T^t(x, \Gamma) + \sum_{k=1}^{t} \mathbb{P}(\tau^x = k, W_{t-k}^0 \in \Gamma). \end{aligned} \tag{4.3}$$

However, for all $s \geq 1$ we get

$$\begin{aligned} &\mathbb{P}(W_s^0 \in \Gamma) \\ &= \sum_{k=0}^{s} \mathbb{P}(W_k^0 = 0, W_{k+1}^0 > 0, W_{k+2}^0 > 0, \ldots, W_{s-1}^0 > 0, 0 < W_s^0 \in \Gamma) \\ &= \sum_{k=0}^{s} \mathbb{P}(W_k^0 = 0)\mathbb{P}(W_1^0 > 0, W_2^0 > 0, \ldots, W_{s-k-1}^0 > 0, 0 < W_{s-k}^0 \in \Gamma) \\ &= \sum_{k=0}^{s} \lambda(k) \delta T^{s-k}(\Gamma). \end{aligned}$$

This yields the following equality

$$P^t(x,\Gamma) = T^t(x,\Gamma) + \sum_{k=1}^{t} T^{k-1}h(x) \sum_{s=0}^{t-k} \lambda(s)\delta T^{t-k-s}(\Gamma).$$

Finally, we get for all $t \geq 1$ the decomposition below

$$P^t = T^t + \sum_{\substack{i,j\geq 0 \\ i+j\leq t-1}} \lambda(t-i-j)T^i h \otimes \delta T^j. \tag{4.4}$$

Remark 4.2. Using a different sketch of proof, the authors have established a similar decomposition as the identity (4.3) for general Markov chains which admit the decomposition given in condition (B) [36].

Observe that $\mathbb{P}(\xi_1 \leq 0) > 0$ and so $p_1 = \delta h = h(0) = \mathbb{P}(\xi_1 \leq 0) > 0$. Hence, we conclude that $d = gcd\{n : p_n > 0\} = 1$ and consequently the probability distribution is aperiodic. So from the local renewal Theorem [14] we get

$$\lim_{t\to+\infty} \lambda(t-i-j) = \lambda = \frac{1}{\sum_{k\geq 0} kp_k} = \frac{1}{\mathbb{E}[\tau^0]} = \pi(\{0\}).$$

Furthermore, according to Kalashnikov [23], there exist $\kappa \in [0,1[$ and $M > 0$ such that for all $t \geq 0$ we have

$$|\lambda(t) - \lambda| \leq M\kappa^t. \tag{4.5}$$

Finally, we need the following simple result.

Lemma 4.2. *Assume that condition* (2.1) *holds true. Then, we have* $\pi h = \pi(\{0\}) = \lambda$ *where* π *is the unique stationary probability distribution of the Markov chain* W.

Proof. The stationary probability measure π is the unique invariant probability for the transition operator P, i.e., π verifies the functional equation $\pi P = \pi$. Using the canonical decomposition (4.1), we deduce that

$$(\pi h)\delta = \pi(I-T)$$

where I is the unit kernel in B_β, that is for all $(x,\Gamma) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+)$, we have $IP(x,\Gamma) = PI(x,\Gamma) = P(x,\Gamma)$. In particular, we obtain

$$(\pi h)\delta(\{0\}) = \pi(I-T)(\{0\}).$$

Since $\pi I(\{0\}) = \pi(\{0\})$ and $T(x,\{0\}) = \mathbb{P}(0 < W_1^x \in \{0\}) = 0$, we obtain

$$\pi T(\{0\}) = \int_E \pi(dx)T(x,\{0\}) = 0.$$

Finally, we have

$$(\pi h)\delta(\{0\}) = \pi h = \pi(\{0\})$$

and the lemma is established. □

The following result establishes the uniform ergodicity and the strong stability of the embedded Markov chain W.

Theorem 4.1. *Assume that the Cramér condition* (2.2) *is satisfied. Then, there exits $0 < l \le +\infty$ such that for all $\beta \in]0, l[$, the process W is aperiodic, uniformly ergodic and strongly stable with respect to the norm $\|\cdot\|_\beta$.*

Proof. We must verify the assertions of Theorem 3.1.

1) From the expression of the kernel T, it is easy to see that $T \ge 0$, i.e., T is a non negative kernel.

2) We have

$$\begin{aligned} Tv(x) &= \mathbb{E}[e^{\beta W_1^x} \mathbb{1}_{\{W_1^x > 0\}}] \\ &= \mathbb{E}[e^{\beta(x+\xi_1)} \mathbb{1}_{\{C_1 \ge x, W_1^x > 0\}} + e^{\beta(x-A_1)} \mathbb{1}_{\{C_1 < x, W_1^x > 0\}}] \\ &\le e^{\beta x} \mathbb{E}[\exp(\beta \xi_1)] = \mathbb{E}[\exp(\beta\xi_1)] v(x). \end{aligned}$$

Setting $\rho(\beta) = \mathbb{E}[\exp(\beta\xi_1)]$, we obtain $Tv(x) \le \rho(\beta)v(x)$ for all $x \in \mathbb{R}^+$ and therefore $\|T\|_\beta \le \rho(\beta)$. Further, we have $\rho(0) = 1$ and $\rho'(0) = \mathbb{E}(\xi_1) = m < 0$. However, $\rho(\beta)$ is a convex function. Then it exists $l > 0$ such that $\rho(l) = 1$ and for all $\beta \in]0, l[$ we get $\rho(\beta) < 1$. Hence, the condition (C) is fulfilled for $m = 1$.

3) We verify easily that $h \in \mathcal{J}_\beta^+$, $\delta \in \mathcal{M}_\beta^+$, $\delta h = h(0) > 0$, $\delta \mathbf{1} = 1$ and according to Lemma 4.2, we get $\pi h = \pi(\{0\}) > 0$. The Cramér condition implies that $\|P\|_\beta < \infty$.

Then from Theorem 3.1 we get the wanted result. □

5. Uniform ergodicity estimates

We start our investigation by the intermediate following result.

Lemma 5.1. *Under the condition* (2.2)*, the stationary projector Π of the chain W has the following representation*

$$\Pi = \pi(\{0\}) \sum_{i \ge 0} \sum_{j \ge 0} T^i h \otimes \delta T^j$$

where π is the unique probability measure of the chain W, δ and h are introduced in (4.1).

Furthermore, the generalized potential R of the chain W coincides with the following σ-finite kernel

$$R = (I - \Pi) R_T (I - \Pi), \tag{5.1}$$

where $R_T = (I - T)^{-1}$.

Proof. Since π is an invariant measure for the transition operator P, we get $\Pi f(x) = \pi f$ for all $f \in \mathcal{J}^+$ where $\Pi = \mathbf{1} \otimes \pi$. Therefore, we have

$$\begin{aligned}
\pi f &= \frac{1}{\mathbb{E}(\tau^0)} \mathbb{E}\left[\sum_{k=0}^{\tau^0-1} f(W_k^0)\right] = \pi(\{0\}) \mathbb{E}\left[\sum_{k=0}^{\tau^0-1} f(W_k^0)\right] \\
&= \pi(\{0\}) \mathbb{E}\left[\sum_{k=0}^{+\infty} f(W_k^0) 1_{\tau^0 > k}\right] \\
&= \pi(\{0\}) \sum_{k=0}^{+\infty} T^k f(0).
\end{aligned}$$

However, since

$$\sum_{i\geq 0}\sum_{j\geq 0} T^i h(x) = \sum_{i\geq 0} \mathbb{P}(\tau^x = i+1) = 1,$$

we obtain

$$\begin{aligned}
\sum_{j\geq 0} T^j f(0) &= \sum_{j\geq 0} \delta T^j f \\
&= \Big(\sum_{j\geq 0} \delta T^j f\Big) \big(\sum_{i\geq 0} T^i h(x)\big) \\
&= \sum_{i\geq 0}\sum_{j\geq 0} T^i h(x) \delta T^j f \\
&= \Big(\sum_{i\geq 0}\sum_{j\geq 0} T^i h \otimes \delta T^j\Big)(f)(x).
\end{aligned}$$

From relations (5.24) and (5.29) in [26], for $n = 1$ and $C = \{0\}$, we obtain $R - (I - \Pi) R_T (I - \Pi) = (I - \Pi) Q R_q R_T (I - \Pi)$, where R_q is the potential of the chain on $(C, C \cap \mathcal{E}) = (\{0\}, \{0\})$ with the transition kernel $S(x, A) = Q(x, A \cap \{0\})$. Here $Q(x, A) = \mathbb{P}_x(W_{\tau^0} \in A)$ with the Markov moment $\tau^0 = \inf\{n \geq 1 : W_n = 0\}$. This chain has a unique invariant measure $\widetilde{\pi} = \delta$. Thus, R_q is equal to zero, since it coincides with the potential of the Markov chain W at the single-point set $C = \{0\}$. The lemma is therefore proved. □

The following result gives an estimate of $\|P^t - \Pi\|_\beta$ and the rate of convergence in the limit uniform ergodicity theorem.

Theorem 5.1. *Assume that conditions of Theorem* 4.1 *are fulfilled. For all* $\theta \in]\max(\kappa, \rho(\beta)), 1[$, $\beta \in]0, l[$ *and* $t \geq 0$, *we have*

$$\|P^t - \Pi\|_\beta \leq \rho(\beta)^t + \frac{\theta^{t+2}}{(\theta - \rho(\beta))^2} \max\{\pi(\{0\}), \Lambda(\theta)\}. \tag{5.2}$$

In particular, there exist $\zeta \geq 1$ *and* $\varepsilon < 1$ *such that for all* $t \geq 0$, *we get*

$$\|P^t - \Pi\|_\beta \leq \zeta \varepsilon^t, \tag{5.3}$$

where $\Lambda(\theta) = \sup_{t \geq 0} |\lambda(t) - \pi(\{0\})| \, \theta^{-t}$, l *and* κ *are introduced in Theorem* 4.1 *and the inequality* (4.5) *respectively, and* $\lambda(t)$ *is defined by the equation* (4.2).

Proof. Using Lemma 5.1, we have

$$\begin{aligned} \|P^t - \Pi\|_\beta &= \Big\| T^t + \sum_{i \geq 0} \sum_{j \geq 0} (\lambda(t-i-j) - \lambda) \; T^i h \otimes \delta T^j \Big\|_\beta \\ &\leq \rho(\beta)^t + \theta^t \sup_{i \geq 0, j \geq 0} \big(\theta^{i+j-t} |\lambda(t-i-j) - \pi(\{0\})| \big) \, \Xi \end{aligned}$$

where

$$\Xi = \sum_{i \geq 0} \sum_{j \geq 0} \theta^{-i} \|T^i h\|_\beta \, \|\delta T^j\|_\beta \, \theta^{-j}.$$

Since $\|T\|_\beta < \rho(\beta) < 1$, $\|\delta\|_\beta = 1$ and $\|h\|_\beta \leq 1$, we obtain the obvious estimations

$$\sum_{i \geq 0} \theta^{-i} \|T^i h\|_\beta \leq \frac{1}{1 - \rho(\beta)\theta^{-1}}, \tag{5.4}$$

$$\sum_{j \geq 0} \theta^{-j} \|\delta T^j\|_\beta \leq \frac{1}{1 - \rho(\beta)\theta^{-1}}. \tag{5.5}$$

Let us denote $\Phi_t(\theta) = \theta^t \sup_{i \geq 0, j \geq 0} [\theta^{i+j-t} |\lambda(t-i-j) - \pi(\{0\})|]$, then we have

$$\begin{aligned} \Phi_t(\theta) &= \theta^t \max\big\{ \sup_{i+j<t} H(i,j,t), \sup_{i+j \geq t} H(i,j,t) \big\} \\ &\leq \theta^t \max\big\{ \pi(\{0\}), \sup_{s \geq 0} \theta^{-s} |\lambda(s) - \pi(\{0\})| \big\}, \end{aligned}$$

with $H(i,j,t) = \theta^{i+j-t} |\lambda(t-i-j) - \pi(\{0\})|$. Hence, we get

$$\Phi_t(\theta) \leq \theta^t \max\big\{ \pi(\{0\}), \sup_{s \geq 0} \theta^{-s} |\lambda(s) - \pi(\{0\})| \big\}. \tag{5.6}$$

Moreover, the quantity $\Lambda(\theta) = \sup_{s \geq 0} |\lambda(s) - \pi(\{0\})| \, \theta^{-s}$ is finite. Effectively, we have from the inequality (4.5)

$$|\lambda(s) - \pi(\{0\})| = |\lambda(s) - \lambda| = O(\kappa^s).$$

However, since $\varrho = \theta^{-1}\kappa < 1$, we obtain $\theta^{-s}|\lambda(s)-\lambda| = o(\varrho^s)$. From estimations (5.4)–(5.6), the estimate (5.2) is established. The inequality (5.3) is a simple consequence of (5.2), where

$$\zeta = 1 + \frac{\varepsilon^2}{(\varepsilon - \rho(\beta)^2} \max\{(\pi(\{0\})), \Lambda(\varepsilon)\} > 1 \tag{5.7}$$

and ε is any nonnegative real number such that $\max(\kappa, \rho(\beta)) < \varepsilon < 1$. □

Remark 5.1. Note that from the proof, the hypothesis $\kappa < \theta$ is a sufficient condition to ensure the finiteness of the quantity $\Phi_t(\theta)$. However, the inequality holds even if this condition is not satisfied provided that $\Phi_t(\theta)$ is finite.

Let us give an estimate for the norm of the invariant stationary measure π and the generalized potential R.

Theorem 5.2. *Under conditions and notations of Theorem* 5.1, *we have*

$$\|\pi\|_\beta = \int_0^{+\infty} e^{\beta x}\pi(dx) \le \frac{\pi(\{0\})}{1-\rho(\beta)} \tag{5.8}$$

and

$$\|R\|_\beta \le \min\Big(\frac{\zeta}{1-\varepsilon}, \frac{(1+\pi(\{0\})-\rho(\beta))^2}{(1-\rho(\beta))^3}\Big)$$

where $\varepsilon \in]\kappa, 1[$ *and* ζ *is given by relation* (5.7).

Proof. The identity (4.1) yields the relation $\pi = (\pi h)\delta R_T$ and consequently we obtain

$$\|\pi\|_\beta \le \pi(\{0\})\,\|R_T\|_\beta \le \frac{\pi(\{0\})}{1-\rho(\beta)}.$$

The estimate (5.8) is so proved.

From Theorem 1.4 in [26] and using Theorem 4.1, the generalized potential coincides analytically with the sum of an operator series as follows

$$R = \sum_{t\ge 0}(P^t - \Pi). \tag{5.9}$$

Hence, we obtain from the inequality (5.3) the following estimate

$$\|R\|_\beta \le \frac{\zeta}{1-\varepsilon}. \tag{5.10}$$

Moreover, from the identity (5.1), we get

$$\|R\|_\beta \le \frac{(1+\|\pi\|_\beta)^2}{1-\rho(\beta)}. \tag{5.11}$$

Therefore, from estimates (5.8) and (5.11), we conclude that

$$\|R\|_\beta \leq \frac{(1+\pi(\{0\})-\rho(\beta))^2}{(1-\rho(\beta))^3}. \tag{5.12}$$

Finally, combining the estimates (5.10) and (5.12), we complete the proof. □

The following result gives an estimate of a deviation between some mean of functionals of the Markov chain over t steps and of the stationary distribution π.

Theorem 5.3. *Assume that conditions of Theorem* 4.1 *are fulfilled. Then for all* $\beta \in]0, l[$*, as* $t \to +\infty$*, the asymptotic expansion*

$$E_x[f(W_t), W_t \in \Gamma] - \int_\Gamma f(y)\pi(dy) = O(e^{\beta x}\varepsilon^t) \tag{5.13}$$

holds true uniformly with respect to $x \geq 0$, $\Gamma \in \mathcal{B}_+$ *and for all functions* f *such that* $\|f\|_\beta < \infty$*. More precisely, we get*

$$\left| E_x[f(W_t), W_t \in \Gamma] - \int_\Gamma f(y)\pi(dy) \right| \leq \zeta\, \|f\|_\beta\, \varepsilon^t. \tag{5.14}$$

In particular, we have for all $t \geq 0$

$$\big| E_x[W_t] - E_x[W_\infty] \big| \leq \frac{\zeta}{\beta\, e} e^{\beta x}\, \varepsilon^t, \tag{5.15}$$

where $\varepsilon \in]\kappa, 1[$ *and* ζ *is given by relation* (5.7).

Proof. Observe that for all $t \geq 0$, we have

$$\|P^t g - \Pi\, g\|_\beta = \sup_{x \in E} \frac{|E_x[f(W_t), W_t \in \Gamma] - \int_\Gamma f(y)\pi(dy)|}{e^{\beta x}}, \tag{5.16}$$

where $g = f\mathbb{1}_\Gamma$. Hence, the inequalities (5.13) and (5.14) follow directly from inequality (5.3) of Theorem 5.1. Furthermore, if we consider $g = Id_{\mathbb{R}_+}$ in the left hand side of relation (5.16) then using the fact that $\|g\|_\beta = 1/(\beta\, e)$, from inequality (5.3) of Theorem 5.1, we obtain immediately the estimate (5.15). □

6. Stability estimates

In this section, we first perturb the patience time distribution and, second, the structure of the system. In each case, the upper stability bounds are derived for the stationary and the transient characteristics of the perturbed system.

6.1. Perturbation of the patience time distribution

The assumption (2.1) implies the existence of the stationary distribution of the embedded Markov chain W. In this case, there are fairly accurate asymptotic formulas for the stationary distribution of the waiting time in the $GI/GI/1$ system with impatient customers if the impatient time distribution is exponential. Unfortunately, no analytic or asymptotic expression is available for the stationary distribution of the waiting time in $GI/GI/1$ queues with general impatient time distribution. So, if we suppose that the impatient time distribution in the $GI/GI/1+G$ system is close to the exponential distribution, then we can approximate the $GI/GI/1+G$ system characteristics by those of the $GI/GI/1+M$ with prior estimation of the corresponding approximation error.

In this section we consider some stability estimates for the Markov chain W related to waiting times in a queueing system $GI/GI/1+G$ with impatient customers under the assumption that perturbations of the chain are of the same structure. More precisely, we obtain estimates with respect to perturbations of the patient time distribution. In practice, we assume that the patience time distribution is exponential and general, respectively in the unperturbed and perturbed systems described by the embedded chain W and $\overline{W}$, that is, that the patience time distribution in the $GI/GI/1+GI$ is close to the exponential one. For this, let us consider an other discrete waiting time process $\overline{W}^x = \{\overline{W}^x_n, n = 0,1,2,\dots\}$ in the $GI/GI/1$ queueing system with impatient customers with the same inter-arrival time A_n, the service time B_n and with a different impatient waiting time $\overline{C}_n$ with the probability distribution $\overline{\gamma}$. Then, the chain $\overline{W}$, which takes values in the same state space $E = \mathbb{R}^+$ as the chain W, satisfies the following recursive relation

$$\overline{W}_{n+1} = \begin{cases} (\overline{W}_n + B_{n+1} - A_{n+1})^+ & if \ \ \overline{W}_n < \overline{C}_{n+1}, \\ (\overline{W}_n - A_{n+1})^+ & if \ \ \overline{W}_n \geq \overline{C}_{n+1}. \end{cases}$$

We denote by $\overline{P}$ the transition kernel of the chain $\overline{W}$ with the invariant probability measure $\overline{\pi}$. Actually, this process describes the perturbation of the first one where the perturbation concerns the patience time distribution. Then, in the following proposition we obtain an estimate of the deviation between the two corresponding transition kernels, P and $\overline{P}$, of the chains W^x and $\overline{W}$ respectively. For this, we use the proximity measure $\Psi = \sup_{x\in\mathbb{R}^+} |\gamma(x) - \overline{\gamma}(x)| = \|\gamma - \overline{\gamma}\|_\infty$.

Proposition 6.1. *Suppose that conditions of Theorem* 4.1 *are satisfied. Then, for all* $\beta \in]0, l[$, *we have*

$$\varepsilon(\beta) = \|\overline{P} - P\|_\beta \leq \varpi(\beta)\, \Psi = \varpi(\beta)\, \|\gamma - \overline{\gamma}\|_\infty \tag{6.1}$$

where $\varpi(\beta) = \mathbb{E}[\exp(\beta\xi_1)] + \sup_{x\in\mathbb{R}_+} \exp(-\beta x)\mathbb{P}(x \leq A_1 < x + B_1)$ *and* l *is introduced in Theorem* 4.1.

Proof. First, let us denote $\bar{h}(x) = h_2(x) - h_1(x) = \mathbb{P}(x \le A_1 < x + B_1)$. We have

$$\begin{aligned}
\int_{\mathbb{R}_+} \mathbb{P}(0 < x + \xi_1 \in dy) \exp(\beta y) &= \mathbb{E}[\exp(\beta(\xi_1 + x)), x + \xi_1 > 0] \\
&\le \mathbb{E}[\exp(\beta(\xi_1 + x))], \\
\int_{\mathbb{R}_+} \mathbb{P}(0 < x - A_1 \in dy) \exp(\beta y) &= \mathbb{E}[\exp(\beta(x - A_1)), x - A_1 > 0] \\
&\le \mathbb{E}[\exp(\beta(x - A_1))], \\
\int_{\mathbb{R}_+} \mathbb{P}(x + \xi_1 \le 0) \exp(\beta y) \delta(dy) &= \mathbb{P}(x + \xi_1 \le 0)
\end{aligned}$$

and

$$\int_{\mathbb{R}_+} \mathbb{P}(x - A_1 \le 0) \exp(\beta y) \delta(dy) = \mathbb{P}(x - A_1 \le 0).$$

Consequently, we get

$$\begin{aligned}
\varepsilon(\beta) &\le \sup_{x \ge 0} e^{-\beta x} \int_{\mathbb{R}_+} |\gamma(x) - \overline{\gamma}(x)| \\
&\quad \times \big|\mathbb{P}(0 < x + \xi_1 \in dy) - \mathbb{P}(0 < x - A_1 \in dy) - \bar{h}(x)\delta(dy)\big| e^{\beta y} \\
&\le \Psi \sup_{x \ge 0} e^{-\beta x} \left(\bar{h}(x) + \int_{\mathbb{R}_+} (\mathbb{P}(0 < x + \xi_1 \in dy) - \mathbb{P}(0 < x - A_1 \in dy)\, e^{\beta y}) \right) \\
&\le \Psi \sup_{x \ge 0} e^{-\beta x} \big(\mathbb{E}[\exp(\beta \xi_1 + \beta x), x + \xi_1 > 0] + \bar{h}(x)\big) \\
&\le \Psi \sup_{x \ge 0} \big(\mathbb{E}[\exp(\beta \xi_1)] + e^{-\beta x} \bar{h}(x)\big) \\
&= \Psi\big(\mathbb{E}[\exp(\beta \xi_1)] + \sup_{x \ge 0} e^{-\beta x} \bar{h}(x)\big).
\end{aligned}$$

Hence the result is proved. □

Remark 6.1. We may use an other proximity measure. Indeed, according to the coupled argument, we have

$$\mathbb{P}(W_1^x \in \Gamma, C_1 \wedge \overline{C}_1 \ge x) = \mathbb{P}(\overline{W}_1^x \in \Gamma, C_1 \wedge \overline{C}_1 \ge x)$$

and

$$\mathbb{P}(W_1^x \in \Gamma, C_1 \vee \overline{C}_1 < x) = \mathbb{P}(\overline{W}_1^x \in \Gamma, C_1 \vee \overline{C}_1 < x).$$

Hence, we get

$$\begin{aligned}
&|P(x,\Gamma) - \overline{P}(x,\Gamma)| \\
&\quad = |\mathbb{P}((x+\xi_1)^+ \in \Gamma, C_1 \le x \le \overline{C}_1) - \mathbb{P}((x-\Gamma_1)^+ \in \Gamma, C_1 \le x \le \overline{C}_1)| \\
&\qquad + |\mathbb{P}((x+\xi_1)^+ \in \Gamma, \overline{C}_1 \le x \le C_1) - \mathbb{P}((x-\Gamma_1)^+ \in \Gamma, \overline{C}_1 \le x \le C_1)|.
\end{aligned}$$

Since $v : t \to e^{\beta t}$ is an increasing function and $(x - A_1)^+ \le (x+\xi_1)^+$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+} e^{\beta t}|P(x,dt) - \overline{P}(x,dt)| \\
&\quad = \mathbb{E}[(e^{\beta(x+\xi_1)^+} - e^{\beta(x-A_1)^+})(1_{C_1\le x\le \overline{C}_1} + 1_{\overline{C}_1\le x\le C_1})] \\
&\quad = \mathbb{E}[e^{\beta(x+\xi_1)^+} - e^{\beta(x-A_1)^+}](\gamma(x) + \overline{\gamma}(x) - \gamma(x)\overline{\gamma}(x)) \\
&\quad \le \mathbb{E}[(e^{\beta(x+\xi_1)} - e^{\beta(x-A_1)})1_{x+\xi_1\ge 0}](\gamma(x) + \overline{\gamma}(x) - 2\gamma(x)\overline{\gamma}(x)) \\
&\quad = \{\mathbb{E}[e^{\beta(x+\xi_1)} - e^{\beta(x-A_1)}] - \mathbb{E}[(e^{\beta(x+\xi_1)} - e^{\beta(x-A_1)})1_{x+\xi_1<0}]\}L(x) \\
&\quad \le \{\mathbb{E}[e^{\beta(x+\xi_1)} - e^{\beta(x-A_1)}] + h_1(x)\}L(x)
\end{aligned}$$

where $L(x) = (\gamma(x)+\overline{\gamma}(x)-2\gamma(x)\overline{\gamma}(x))$ for all $x \in \mathbb{R}_+$. This yields the following inequality :

$$\|P - \overline{P}\|_\beta \le \sup_{x\in\mathbb{R}_+} \{(\gamma(x) + \overline{\gamma}(x) - 2\gamma(x)\overline{\gamma}(x))(\mathbb{E}[e^{\beta\xi_1} - e^{-\beta A_1}] + e^{-\beta x}h_1(x))\}.$$

In order to obtain the deviation between the stationary distributions π and $\overline{\pi}$ of the embedded Markov chains W and $\overline{W}$ respectively, we need first the following lemma.

Lemma 6.1. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all $\beta \in]0, l[$ such that the following condition*

$$\varepsilon(\beta) = \|P - \overline{P}\|_\beta < 1 - \rho(\beta) \tag{6.2}$$

holds true, we have the inequality

$$\sup_{i\ge 0} \|\overline{P}^i\|_\beta \le \frac{1}{(1-\rho(\beta)-\varepsilon(\beta))^2} \tag{6.3}$$

where $\rho(\beta)$ and l are defined in Theorem 4.1.

Proof. It is obvious that for all $i \ge 0$, we have $\delta\overline{P}^i h \le 1$ since $h(x) = \mathbb{P}(W_1^x = 0) \le 1$. Let us denote $\Delta = \overline{P} - P$, then we can remark that

$$\delta\overline{P}^i = \delta\overline{P}^{i-1}(\Delta + T) + (\delta\overline{P}^{i-1}h)\delta \ \text{ for all } i \ge 1.$$

Hence, $\|\delta\overline{P}^i\|_\beta \leq \|\delta\overline{P}^{i-1}\|_\beta\,(\varepsilon(\beta)+\rho(\beta)) + \|\delta\|_\beta$ for all $i \geq 1$ and by induction we obtain the estimate

$$\sup_{i\geq 0}\|\delta\overline{P}^i\|_\beta \leq \frac{1}{1-\rho(\beta)-\varepsilon(\beta)}. \tag{6.4}$$

Moreover, we observe that

$$\begin{aligned}\overline{P}^i &= \overline{P}^i - P\overline{P}^{i-1} + (T+h\otimes\delta)\overline{P}^{i-1} = \overline{P}^i - P\overline{P}^{i-1} + T\overline{P}^{i-1} + h\otimes\delta\overline{P}^{i-1}\\ &= (\Delta+T)\overline{P}^{i-1} + h\otimes\delta\overline{P}^{i-1}\\ &= (\Delta+T)\overline{P}^{i-1} + h\otimes\delta\overline{P}^{i-1}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|\overline{P}^i\|_\beta &\leq \|\overline{P}^{i-1}\|_\beta\|\Delta+T\|_\beta\|\overline{P}^{i-1}\|_\beta + \|h\|_\beta\|\delta\overline{P}^{i-1}\|_\beta\\ &\leq \|\overline{P}^{i-1}\|_\beta\|\Delta+T\|_\beta\|\overline{P}^{i-1}\|_\beta + \|\delta\overline{P}^{i-1}\|_\beta\end{aligned}$$

and by induction we get

$$\begin{aligned}\|\overline{P}^i\|_\beta &\leq \sum_{t=1}^{i}\|\delta\overline{P}^t\|_\beta\,\|\Delta+T\|_v^{t-1} + \|\Delta+T\|_v^i\\ &\leq \sup_t\|\delta\overline{P}^t\|_\beta\sum_{t=1}^{i}\|\Delta+T\|_\beta^{t-1} + \|\Delta+T\|_v^i.\end{aligned}$$

Consequently, we have

$$\sup_t\|\overline{P}^t\|_\beta \leq \sup_t\|\delta\overline{P}^t\|_\beta\sum_{t=1}^{+\infty}\|\Delta+T\|_\beta^{t-1}.$$

However, the condition (6.2) implies that $\|\Delta\|_\beta < 1-\rho(\beta)$. Therefore, $\|\Delta + T\|_\beta \leq \|\Delta\|_\beta + \|T\|_\beta < 1-\rho(\beta)+\rho(\beta) = 1$, which implies that the geometric series $\sum_t\|\Delta+T\|_\beta^{t-1}$ converges and so,

$$\sum_t\|\Delta+T\|_\beta^{t-1} = \frac{1}{1-\|\Delta+T\|_\beta} \leq \frac{1}{1-\varepsilon(\beta)-\rho(\beta)}.$$

This yields the following estimate

$$\sup_t\|\overline{P}^t\|_\beta \leq \frac{\sup_t\|\delta\overline{P}^t\|_\beta}{1-\|\Delta+T\|_\beta}$$

and therefore

$$\sup_i\|\overline{P}^i\|_\beta \leq \frac{1}{1-\varepsilon(\beta)-\rho(\beta)}\,\sup_t\|\delta\overline{P}^t\|_\beta. \tag{6.5}$$

Finally, using inequalities (6.4) and (6.5), the result is immediately established. □

The following theorem concerns the estimate of the deviation of the transition probabilities over t steps for the two Markov processes W and $\overline{W}$ for all $t \in \mathbb{Z}_+$.

Theorem 6.1. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality* (6.2) *holds true, we have the inequality*

$$\sup_{t \geq 0} \|\overline{P}^t - P^t\|_\beta \leq \frac{\mathcal{H}\,\varepsilon(\beta)}{(1-\rho(\beta)-\varepsilon(\beta))^2}. \tag{6.6}$$

In particular, for all β *such that* $\Psi < \dfrac{1-\rho(\beta)}{\varpi(\beta)}$, *we have:*

$$\sup_{t \geq 0} \|\overline{P}^t - P^t\|_\beta \leq \frac{\mathcal{H}\,\varpi(\beta)}{(1-\rho(\beta)-\varpi(\beta)\,\Psi)^2}\,\Psi \tag{6.7}$$

where Ψ, $\varpi(\beta)$ *are defined in Proposition* 6.1 *and*

$$\mathcal{H} = \min\Big(\frac{\zeta}{1-\varepsilon}, \frac{(1+\pi(\{0\})-\rho(\beta))^2}{(1-\rho(\beta))^3}\Big).$$

Proof. Let us denote $\Delta_t = \overline{P}^t - P^t$ for all $t \geq 0$. Then, we have

$$\Delta_t = \overline{P}(\overline{P}^{t-1} - P^{t-1}) + (\overline{P} - P)P^{t-1} = \overline{P}\Delta_{t-1} + \Delta_1 P^{t-1}.$$

Moreover, since $\Delta\Pi = 0$, we get by induction

$$\Delta_t = \Delta(P^{t-1} - \Pi) + \ldots + \overline{P}^{t-2}\Delta(P - \Pi) + \overline{P}^{t-1}\Delta.$$

This implies that

$$\|\Delta_t\|_\beta \leq \|\Delta\|_\beta \sup_{i \geq 1} \|\overline{P}^i\|_\beta \sum_{i=0}^{\infty} \|P^i - \Pi\|_\beta.$$

The inequality (6.6) derives straightly from Lemma 6.1 and Theorem 5.2. The inequality (6.7) follows from combining together the estimate (6.1) of Proposition 6.1 and the inequality (6.6). This completes the proof. □

Remark 6.2. Note that the inequality obtained by N.V. Kartashov for general state Markov chains in [26, Chap. 3, p. 46], holds only in some small neighborhood

$$\Big\{\varepsilon(\beta) = \|\overline{P} - P\|_\beta < \frac{1-\rho(\beta)}{c}\Big\}$$

where c is a family of constants strictly greater than 1 which depend on different conditions on the perturbation $\Delta = P - \overline{P}$. In contrast, the inequality (6.6) obtained for our particular case is valid in a wide neighborhood of the transition kernel P defined by $\{\varepsilon(\beta) = \|\overline{P} - P\|_\beta < 1 - \rho(\beta)\}$.

The following result gives an estimate of the deviation of the stationary distributions π and $\overline{\pi}$.

Theorem 6.2. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality* (6.2) *is fulfilled, the transition kernel* $\overline{P}$ *of the perturbed Markov chain* $\overline{W}$ *admits a unique invariant probability measure given by the identity* $\mu_\Delta/(\mu_\Delta \mathbf{1}) = \overline{\pi}$ *where* $\Delta = \overline{P} - P$ *and* $\mu_\Delta = \sum_{i=0}^{+\infty} \delta(\Delta + T)^i$. *Moreover, we have the following stability estimate*

$$\|\overline{\pi} - \pi\|_\beta \leq \frac{\varepsilon(\beta)}{(1-\rho(\beta)-\varepsilon(\beta))^2}\Big(\frac{2-\rho(\beta)}{1-\rho(\beta)}\Big). \tag{6.8}$$

More specifically, for all $\beta \in]0, l[$ *such that* $\Psi < (1-\rho(\beta))/\varpi(\beta)$ *is fulfilled, we have:*

$$\|\overline{\pi} - \pi\|_\beta \leq \frac{\varpi(\beta)}{(1-\rho(\beta)-\varpi(\beta)\,\Psi)^2}\Big(\frac{2-\rho(\beta)}{1-\rho(\beta)}\Big)\,\Psi \tag{6.9}$$

where Ψ *and* $\varpi(\beta)$ *are defined in Proposition* 6.1 *and* l *is introduced in Theorem* 4.1.

Proof. The existence and the unicity of the measure $\overline{\pi}$ follows from Theorem 3.3. Furthermore, according to the idendities $\|\alpha\|_\beta = \|\mathbf{1}\|_\beta = 1$, the inequality (3.9) of Theorem 3.3 yields the inequality (6.8). In addition, the inequality (6.9) follows from combining together the estimate (6.1) of Proposition 6.1 and the inequality (6.8). The theorem is finally established. □

Now let us consider a smaller perturbation, with respect to the weighted norm $\|\cdot\|_\beta$, than those considered in Theorem 6.2. More exactly, we have the following theorem.

Theorem 6.3. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality*

$$\varepsilon(\beta) < \frac{1-\rho(\beta)}{1+\Theta\,\rho(\beta)} \tag{6.10}$$

holds true, we have the following strong stability estimate

$$\|\overline{\pi} - \pi\|_\beta \leq \min\Big\{\chi(\beta), \frac{\Theta(1+\Theta\,\rho(\beta))}{1-\rho(\beta)-(1+\Theta\,\rho(\beta))\,\varepsilon(\beta)}\Big\}\,\varepsilon(\beta). \tag{6.11}$$

More specifically, for all $\beta \in]0, l[$ *such that*

$$\Psi < \frac{1-\rho(\beta)}{\varpi(\beta)\ (1+\Theta\,\rho(\beta))} \tag{6.12}$$

we have

$$\|\overline{\pi}-\pi\|_\beta \le \min\Big\{\overline{\chi}(\beta), \frac{\Theta(1+\Theta\,\rho(\beta))}{1-\rho(\beta)(\beta)-(1+\Theta\,\rho(\beta))\varpi(\beta)\,\Psi}\Big\}\,\varpi(\beta)\,\Psi \tag{6.13}$$

where

$$\Theta = \frac{\pi(\{0\})}{1-\rho(\beta)}, \quad \chi(\beta) = \frac{1}{(1-\rho(\beta)-\varepsilon(\beta))^2}\Big(\frac{2-\rho(\beta)}{1-\rho(\beta)}\Big)$$

and

$$\overline{\chi}(\beta) = \frac{1}{(1-\rho(\beta)-\varpi(\beta)\,\Psi)^2}\Big(\frac{2-\rho(\beta)}{1-\rho(\beta)}\Big).$$

Proof. According to Theorem 5.2, we have $\|\pi\|_\beta \le \Theta$. Hence, from Theorem 3.1 we get

$$\|\overline{\pi}-\pi\|_\beta \le \frac{\Theta(1+\Theta\,\rho(\beta))\,\varepsilon(\beta)}{1-\rho(\beta)-\,(1+\Theta\,\rho(\beta))\,\varepsilon(\beta)}. \tag{6.14}$$

The estimates (6.8), (6.14) and (6.1) imply the inequality (6.11). In the same manner, from the estimate (6.1), the inequalities (6.12) and (6.14), we derive

$$\|\overline{\pi}-\pi\|_\beta \le \frac{\Theta(1+\Theta\,\rho(\beta))\,\varpi(\beta)\,\Psi}{1-\rho(\beta)-\,(1+\Theta\,\rho(\beta))\,\varpi(\beta)\,\Psi}. \tag{6.15}$$

Observe that since the inequality (6.12) is satisfied it implies that the following inequality $\Psi < (1-\rho(\beta))/\varpi(\beta)$ is also fulfilled. It follows that the inequality (6.9) holds true. Hence, by combining (6.9) and (6.15), we obtain the inequality (6.13). The proof is completed. □

Now let us try to estimate some stationary and non stationary characteristics of the perturbed Markov chain $\overline{W}$ with respect to the weighted norm $\|\cdot\|_\beta$ in terms of the parameter of the unperturbed Markov chain W. For this, we start with the stationary case. We denote by W_∞ and $\overline{W}_\infty$ the generic random variables which have the probability distribution π and $\overline{\pi}$ respectively.

Theorem 6.4. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0,l[$ *such that the inequality* (6.2) *is fulfilled and for all* $\Gamma \in \mathcal{E} = \mathcal{B}(\mathbb{R}_+)$, *we have*

$$|\mathbb{P}(W_\infty \in \Gamma) - \mathbb{P}_x(\overline{W}_\infty \in \Gamma)| = |\overline{\pi}-\pi|(\Gamma) \le \frac{\chi(\beta)}{\upsilon_\Gamma}\,\varepsilon(\beta). \tag{6.16}$$

However, if the hypothesis (6.10) *holds, then we get*

$$|\overline{\pi}-\pi|(\Gamma) \le \frac{1}{\upsilon_\Gamma}\,\min\Big\{\chi(\beta), \frac{\Theta(1+\Theta\,\rho(\beta))}{1-\rho(\beta)(\beta)-(1+\Theta\,\rho(\beta))\varepsilon(\beta)}\Big\}\,\varepsilon(\beta) \tag{6.17}$$

where $\upsilon_\Gamma = \max(\Bbbk_\Gamma, 2)$ *with* $\Bbbk_\Gamma = \inf_{x\in\Gamma} e^x$ *for all* $\Gamma \in \mathcal{B}(\mathbb{R}_+)$.

Proof. Observe that for all $a \in \mathbb{R}$, we have

$$|(\pi-\overline{\pi})(\Gamma)| = |(\pi-\overline{\pi})\mathbb{1}_\Gamma| = |(\pi-\overline{\pi})(\mathbb{1}_\Gamma - a\mathbf{1}| \le \|\pi-\overline{\pi}\|_\beta \sup_{x\ge 0} |\mathbb{1}_\Gamma(x) - a|. \quad (6.18)$$

But it should be noted that the right hand side of the last inequality must not depend on the parameter a. Moreover, from the monotonicity property of the absolute value, the supremum is necessary either $|1-a|$ or $|a|$, for all $a \in \mathbb{R}$. However, for $a = 1/2$, we get $|1-a| = |a| = 1/2$. Hence, we obtain

$$|(\pi-\overline{\pi})(\Gamma)| \le \frac{1}{2}\,\|\pi-\overline{\pi}\|_\beta. \quad (6.19)$$

Moreover, we have obviously

$$\begin{aligned} \Bbbk_\Gamma |\overline{\pi}-\pi|(\Gamma) &= \Bbbk_\Gamma \int_{x\in\Gamma} |\overline{\pi}-\pi|(dx) \\ &\le \int_{x\in\Gamma} e^x\,|\overline{\pi}-\pi|(dx) \le \int_{x\in E} e^x\,|\overline{\pi}-\pi|(dx) \\ &= \|\overline{\pi}-\pi\|_\beta. \end{aligned} \quad (6.20)$$

Hence, from estimates (6.19)–(6.20), we obtain

$$|\mathbb{P}(W_\infty \in \Gamma) - \mathbb{P}_x(\overline{W}_\infty \in \Gamma)| = |\overline{\pi}-\pi|(\Gamma) \le \frac{\|\pi-\overline{\pi}\|_\beta}{\upsilon_\Gamma}. \quad (6.21)$$

Finally, according to estimates (6.8) and (6.21), the inequality (6.16) is established. In the same way, the combination of estimates (6.8) and (6.21) leads to estimate (6.17). The result is established. □

Now let us extend the measures $\mu \in \mathcal{E} = \mathcal{B}(\mathbb{R}_+)$ to $\mathcal{B}(\mathbb{R}_-)$ by putting $\mu(\Gamma) = 0$ for all $\Gamma \in \mathcal{B}(\mathbb{R}_-)$.

Corollary 6.1. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality* (6.2) *is fulfilled and for all* $x \ge 0$, *we have*

$$|\mathbb{P}(W_\infty \le x) - \mathbb{P}_x(\overline{W}_\infty \le x)| = |\overline{\pi}(]-\infty, x]) - \pi(]-\infty, x])| \le \frac{\chi(\beta)}{2}\,\varepsilon(\beta). \quad (6.22)$$

Moreover, if the hypothesis (6.10) *is satisfied, then we have*

$$\begin{aligned} &|\overline{\pi}(]-\infty, x]) - \pi(]-\infty, x])| \\ &\quad \le \frac{1}{2}\,\min\Big\{\chi(\beta), \frac{\Theta(1+\Theta\,\rho(\beta))}{1-\rho(\beta)(\beta)-(1+\Theta\,\rho(\beta))\varepsilon(\beta)}\Big\}\,\varepsilon(\beta). \end{aligned} \quad (6.23)$$

Theorem 6.5. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality* (6.2) *is fulfilled, the following estimate holds true for all* $t \geq 0$ *and* $x \in \mathbb{R}_+$*:*

$$\sup_{\Gamma \in \mathcal{B}(\mathbb{R}_+)} |\mathbb{P}_x[W_t \in \Gamma] - \mathbb{P}_x[\overline{W}_t \in \Gamma]| \leq \frac{e^{\beta x}}{2} \frac{\mathcal{H}}{(1 - \rho(\beta) - \varepsilon(\beta))^2} \varepsilon(\beta) \tag{6.24}$$

where $\mathcal{H}$ *is explicitly defined in Theorem* 6.1.

Proof. Let us put $f = 1\!\!1_\Gamma$ for all $\Gamma \in \mathcal{B}(\mathbb{R}_+)$. Then, for all $x \in E$ and $t \geq 0$, we have $|\mathbb{P}_x[W_t \in \Gamma] - \mathbb{E}_x[\overline{W}_t \in \Gamma]| = |P^t f - \overline{P}^t f|(x)$. However, we can remark that $(P^t - \overline{P}^t)\mathbf{1} = \mathbf{1} - \mathbf{1} = 0$ for all $t \geq 0$. Therefore, for all $a \in \mathbb{R}$ we get

$$\|\Delta_t f\|_\beta = \|\overline{P}(f - a\mathbf{1})\|_\beta \leq \|\Delta_t\|_\beta \|f - a\mathbf{1}\|_\beta.$$

Following the same sketch to prove (6.19), we can establish finally the estimate

$$|\mathbb{P}_x[W_t \in \Gamma] - \mathbb{P}_x[\overline{W}_t \in \Gamma]| \leq \frac{e^{\beta x}}{2} \|\overline{P}^t - P^t\|_\beta.$$

The result follows according to (6.6). □

Theorem 6.6. *Assume that the conditions put forward in Theorem* 4.1 *are satisfied. For all* $\beta \in]0, l[$ *such that the inequality* (6.2) *is satisfied, the following estimate holds true for all* $t \geq 0$ *and* $x \in \mathbb{R}_+$*:*

$$\sup_{t \geq 0} |\mathbb{E}_x[W_t] - \mathbb{E}_x[\overline{W}_t]| \leq \frac{e^{\beta x}}{\beta\, e} \frac{\mathcal{H}}{(1 - \rho(\beta) - \varepsilon(\beta))^2} \varepsilon(\beta). \tag{6.25}$$

Proof. Note that the function $f = Id_{\mathbb{R}_+} \in \mathcal{J}_\beta$ and $\|f\|_\beta = 1/\beta\, e$. Substituting f in the following evident estimate

$$\forall x \in \mathbb{R}_+ : |\mathbb{E}_x[f(W_t)] - \mathbb{E}_x[f(\overline{W}_t)]| \leq e^{\beta x} \|f\|_\beta \|P^t - \overline{P}^t\|_\beta.$$

and using the inequality (6.6), the wanted result is established. □

Remark 6.3. Observe that if the following inequality

$$\Psi < \frac{1 - \rho(\beta)}{\varpi(\beta)}$$

is satisfied, then the estimate (6.2) holds true. So the previous perturbation bounds established under condition (6.2) are still valid. In this case, we can obtain explicit estimates by substituting $\varpi(\beta)\Psi$ instead of $\varepsilon(\beta)$ in the bounds (6.16), (6.22), (6.24) and (6.25). In the same way, if

$$\Psi < \frac{1 - \rho(\beta)}{\varpi(\beta)\ (1 + \Theta\, \rho(\beta))}$$

holds true, then the estimate (6.10) is also satisfied. In this case, explicit estimates may be established using the same substitution in (6.17) and (6.23).

Remark 6.4. The stability bounds obtained in Theorems 6.4, 6.5 and Corollary 6.1 are established using a contraction inequality. In another work, we have established a generalization of this type of inequality and its dual version for general state Markov chains defined on a general phase space $(E, \mathcal{E})$ with respect to the weighted variation norms and $L^p(\varphi)$ norms where φ is a probability measure on $\mathcal{E}$. Since these results are not yet published, we have given here a direct proof for this type of inequalities.

6.2. Structural perturbation

Let us consider the random walk process (classical Lindley process) described by a chain $\tilde{W}$ satisfying the following recursive relation

$$\tilde{W}_{n+1} = (\tilde{W}_n + \xi_{n+1})^+, \;\; n \geq 0 \tag{6.26}$$

which takes values in the same state space $E = \mathbb{R}^+$ as the chain W, where for all $n \geq 1$, $\xi_n = B_n - A_n$. Actually, this process describes the waiting process of the classical $GI/GI/1$ queue system (with ordinary customers), see [26,35,38] with the same probability distribution of the inter-arrival time and service time as those of the stochastic process W. In fact this process describes the waiting time in the limit of the impatient model since we can consider $C_n = +\infty$ and therefore we conclude that $\mathbb{P}(C_1 = +\infty) = 1$. Then, the condition $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) < \mathbb{E}(A_1)$ becomes $m = \mathbb{E}[\xi_1] < 0$. It is known from Borovkov [6, Chapter 4] that the chain $\tilde{W}$ is ergodic and has a unique invariant probability measure $\tilde{\pi}$ provided that $m = \mathbb{E}[\xi_1] < 0$. Let us denote the transition kernel of the chain $\tilde{W}$ by $\tilde{P}$ which is defined as follows

$$\begin{aligned} \tilde{P}(x, \Gamma) &= \mathbb{P}(0 < x + \xi_1 \in \Gamma) + \mathbb{P}(x + \xi_1 \leq 0)\delta(\Gamma) \\ &= \mathbb{P}(0 < x + \xi_1 \in \Gamma) + h_1(x)\delta(\Gamma). \end{aligned}$$

We may consider the process W as the perturbed Lindley process where the perturbation concerns the structure of the system,i.e., it is the perturbation of the impatient infinite time in the $GI/GI/1$ queueing system.

Then, we obtain in the following proposition an estimate of the deviation between the two corresponding transition kernels, P and $\tilde{P}$, of the chains W and $\tilde{W}$ respectively. The proof can be easily established and so is omitted.

Proposition 6.2. *Under conditions of Theorem* 4.1 *for the Markov chain* $\tilde{W}$*, we have*

$$\tilde{\varepsilon}(\beta) = \|\tilde{P} - P\|_\beta \leq \varpi(\beta) \sup_{x \in \mathbb{R}^+} |\gamma(x)| = \varpi(\beta)\|\gamma\|_\infty \tag{6.27}$$

where $\varpi(\beta)$ *is defined in Proposition* 6.1.

The following theorem evaluates the deviation, with respect to the norm $\|\cdot\|_\beta$, between the stationary probability measures π and $\tilde{\pi}$ corresponding respectively to the Markov processes W and $\tilde{W}$. For this, it is of interest to specify that the approximation must be done in terms of the parameter of the Lindley process $\tilde{W}$ which characterizes the ideal (non perturbed) model. Note that the stationary distribution of the Markov process $\tilde{W}$ may be computed by numerical methods investigated by many authors, see [1]. Therefore, the bound of inequalities obtained in this section must depend on the parameters of the chain $\tilde{W}$. Thus we have to investigate in the first step the strong stability of the Lindley waiting process. However, the processes which were defined by the same recursive equation (6.26) as $\tilde{W}$ are strongly stable with respect to the same norm $\|\cdot\|_\beta$, under the same Cramér condition, as it is shown in [26,35,36]. Moreover, we have the same estimate of the residual operator norm $\|T\|_\beta \leq \rho(\beta) = \mathbb{R}[\exp(\beta\xi_1)]$, where here $\xi_1 = B_1 - A_1$.

So the sketch of proofs for the different strong stability estimates are the same as those given in the previous section. It suffices to substitute, in all previous inequalities, $\varepsilon(\beta)$ by $\tilde{\varepsilon}(\beta)$ or $\varpi(\beta)\,\Psi$ by $\varpi(\beta)\|\gamma\|_\infty$. For example, the result analogous to Theorem 6.2 is expressed as follows.

Theorem 6.7. *Under conditions and notations of Theorem* 4.1, *and for all* $\beta \in]0, l[$ *such that* $\tilde{\varepsilon}(\beta) = \|P - \tilde{P}\|_\beta < 1 - \rho(\beta)$, *we have the following inequality*

$$\|\tilde{\pi} - \pi\|_\beta \leq \frac{1}{(1-\rho(\beta)-\tilde{\varepsilon}(\beta))^2}\Big(\frac{2-\rho(\beta)}{2-\rho(\beta)}\Big)\,\varpi(\beta)\|\gamma\|_\infty.$$

Concluding remarks

In this paper, we have clarified sufficient uniform ergodicity and strong stability conditions for the waiting process describing the $GI/GI/1$ queues with general patience time distribution with respect to the norm $\|\cdot\|_\beta$. This result allows us to approximate the stationary and non stationary characteristics of the perturbed system under a conservative perturbation structure. Thus, it can be used in practice to estimate the characteristics of the $GI/GI/1$ queues with general patience time distribution by those of the $GI/GI/1$ queues with the specific impatient time distribution (eg. exponential) where its characteristics are well known. This may be done, for example, using the Little formulae and the estimates established in this paper. An estimate for the potential of the chain and the rate of the uniform convergence of the t-fold power of the transition kernel to the stationary projector with respect to the norm $\|\cdot\|_\beta$ have been also established. Moreover, since the waiting process in the $GI/GI/1$ queue described by the Lindley recursion is strongly stable with respect to the same weighted norm, the stationary and non stationary characteristics of the $GI/GI/1$ queues with general impatient time distribution can be approximated by the corresponding characteristics of the classical $GI/GI/1$ queue with ordinary customers.

Note that the diffusion approximation established in [18,41] describes the fluid limit (convergence mode) and does not give us an estimate of the error of approximation to the stationarity such as the bounds obtained in Theorems 5.2 and 5.3 in Section 5. Moreover, in Theorems 5.2 and 5.3 there is no requirement for the patience time distribution, contrary to the results obtained in [41] where the approximation is given only under some restrictions on the hazard rate function of the patience time distribution (eg. distributions whose hazard rate tends to infinity on its support).

It should be noted that in practice, the results obtained in Section 6 can be used to approximate the stationary and transition characteristics of some complex systems such as $M/M/1$, $M/G/1$, $G/M/1$ and $GI/GI/1$ equipped with specific policies (vacation service, breakdowns of the server,...) by those of the systems $M/M/1$, $M/G/1$, $G/M/1$ and $GI/GI/1$ respectively with impatient customers provided that the processes have the same state space.

Furthermore, because the arrival rate in the n^{th} system is of order n, a sample path version of Little's law known as the snapshot principle suggests that we can connect the waiting process and the queue-length process (eg., see formula (3.5) in [41]). This allows us to use the estimates obtained in this paper to obtain the corresponding estimates for the stationary and transient characteristics for the queue-length process. However, this is out of scope of this paper.

Finally, it is worth noting that we are not able to extend the results of Section 6 to the case $m > 0$ (without Cramér condition). We conjecture that for $m > 0$ the process is not v-strongly stable. That is, the convergence to the stationarity is not uniform with respect to the weighted variation norm $\| \cdot \|_{\beta}$. That means that the characteristics of the process are very sensitive to a small perturbation of some parameters of the system. Unfortunately, we are not able to prove this until now.

Acknowledgement

We are deeply grateful to the anonymous referees for very constructive comments and suggestions, which allowed us to improve the presentation of this paper significantly. Furthermore, we would like to thank Francois Charlot for interesting remarks after reading the first version of this article.

References

[1] J. Abate, G.L. Choudhury and W. Whitt (2000) An introduction to numerical transform inversion and its application to probability models. In: *Computational Probability*, W.K. Grassmann (ed.). Kluwer Academic Publishers, Boston, MA, 257–323.

[2] D. Aïssani and N. Kartashov (1983) Ergodicity and stability of Markov chains with respect to operator topology in the space of transition kernels. *Doklady Akad. Nauk Ukr. SSR. ser. A11*, 3–5.

[3] V.V. Anisimov (1988) Estimates for the deviation of the transition characteristics of nonhomogeneous Markov processes. *Ukranian Math. J.* **40**, 588–592.

[4] F. Baccelli, P. Boyer and G. Hebuterne (1984) Single-server queues with impatient customers. *Adv. Appl. Prob.* **16**, 887–905.

[5] P.P. Bah Hacharya and A. Ephrernides (1991) Stochastic monotonicity properties of multiserver queues with impatient customers. *Adv. Appl. Prob.* **28**, 673–682.

[6] A.A. Borovkov (1972) *The Probabilistic Processes in Theory of Queueing Systems.* Nauka, Moscow.

[7] A.A. Borovkov (1998) *Ergodicity and Stability of Stochastic Processes.* Wiley and Sons, New York.

[8] O.J. Boxma and P.R. Waal (1994) Multiserver queues with impatient customers. In: *Proceedings of ITC-14*, North Holland, Amsterdam, 743–756.

[9] A. Brandt and M. Brandt (1999) On the $M(n)/M(n)/s$ queue with impatient calls. *Perform. Evaluat.* **35**, 1–8.

[10] F. Charlot and J. Pujolle (1978) Recurrence in single-server queues with impatient customers. *Ann. Inst. Henri Poincaré* **14** (4), 399–410.

[11] G.E. Cho and C.D. Meyer (2001) Comparison of perturbation bounds for the stationnary probabilities of a finite Markov chain. *Linear Algebra Appl.* **335**, 137–150.

[12] K.L. Chung (1960) *Markov Chains with Stationary Transition Probabilities.* Springer-Verlag, New York.

[13] A. De Kok and H.C. Tijms (1985) A queueing system with impatient customers. *J. Appl. Prob.* **22**, 688–696.

[14] W. Feller (1950) *An Introduction to Probability Theory and its Applications.* John Wiley, New York.

[15] B. Heidergott and A. Hordjik (2003) Taylor expansions for stationary Markov chains. *Adv. Appl. Prob.* **35**, 1046–1070.

[16] B. Heidergott, A. Hordjik and M. Van Uitert (2007) Series expansions for finite-state Markov chains. *Probability in Engineering and Informational Sciences* **21**, 381–400.

[17] B. Heidergott, A. Hordjik and H. Weisshaupt (2006) Measure-valued differentiation for stationary Markov chains. *Math. Operat. Res.* **31** (1), 154–172.

[18] Z.H. Huang, J. and J. Zhang (2016) A unified approach to diffusion analysis of queues with general patience-time distributions. *Math. Operat. Res.* **41** (3), 1135–1160.

[19] J.J. Hunter (2005) Stationary distributions and mean first passage times of perturbed Markov chains. *Linear Algebra Appl.* **410**, 217–243.

[20] J.J. Hunter (2006) Mixing times with applications to perturbed Markov chains. *Linear Algebra Appl.* **417**, 108–123.

[21] I.C.F. Ipsen and C.D. Meyer (1994) Uniform stability of Markov chains. *SIAM J. Matrix Anal. Appl.* **15** (4), 1061–1074.

[22] V.V. Kalashnikov (1974) Uniform closeness of components of Markov process under variation of its parameters. *Eng. Cyber.* **12**, 32–41.

[23] V.V. Kalashnikov (1977) Stability analysis in queueing problems by the method of trial functions. *Theory Prob. Appl.* **22** (1), 86–103.

[24] N.V. Kartashov (1985) Inequalities in theorems of ergodicity and stability for Markov chains with common phase space. I. *Theory Prob. Appl.* **30** (2), 247–259.

[25] N.V. Kartashov (1985) Inequalities in theorems of ergodicity and stability for Markov chains with common phase space. II. *Theory Prob. Appl.* **30** (3), 507–515.

[26] N.V. Kartashov (1996) *Strong Stable Markov Chains.* VSP, Utrecht TBIMC Scientific Publishers, Kiev.

[27] S.J. Kirkland (2002) On question concerning condition numbers for Markov chains. *SIAM J. Matrix Anal. Appl.* **23**, 1109–1119.

[28] S.J. Kirkland (2003) Conditioning properties of the stationary distribution for a Markov chain. *Electron. J. Linear Algebra* **10**, 1–15.

[29] S.J. Kirkland (2004) Diagraph based conditioning for Markov chains. *Linear Algebra App.* **336**, 45–65.

[30] S.J. Kirkland, M. Neumann and B.L. Shader (1998) Application of paz's inequality to perturbations for Markov chains. *Linear Algebra Appl.* **268**, 183–196.

[31] M. Martin and J.R. Artalejo (1995) Analysis of an $M/G/1$ queue with two type of impatient units. *Adv. Appl. Prob.* **27** (3), 840–861.

[32] S.P. Meyn and R.L. Tweedie (1993) *Markov Chains and Stochastic Stability.* Springer-verlag.

[33] A.Y. Mitrophanov (2003) Stability and exponential convergence of continous-time Markov chains. *J. Appl. Prob.* **40**, 970–979.

[34] A.Y. Mitrophanov (2005) Sensitivity and convergence of uniformly ergodic Markov chains. *J. Appl. Prob.* **42**, 1003–1014.

[35] Z. Mouhoubi and D. Aïssani (2003) On the uniform ergodicity and strong stability estimates of waiting process. In: *Bull. Inter. Stat. Inst.* **LX(2)**, 97–98.

[36] Z. Mouhoubi and D. Aïssani (2005) Some inequalities of the uniform ergodicity and strong stability of homogeneous Markov chains. *Pliska Stud. Math. Bulgar.* **17**, 171–189.

[37] Z. Mouhoubi and D. Aïssani (2010) New perturbation bounds for denumerable Markov chains. *Linear Algebra Appl.* **432**, 1627–1649.

[38] Z. Mouhoubi and D. Aïssani (2014) Uniform ergodicity and strong stability estimates of homogeneous Markov chains. *J. Math. Sci.* **20**, 452–461.

[39] P. Moyal (2010) The queue with impatience: construction of the stationary workload under FIFO. *J. App. Prob.* **47** (4), 498–512.

[40] M. Neumann and J. Xu (2004) Improved bounds for a condition number of Markov chains. *Linear Algebra Appl.* **386**, 225–241.

[41] J.E. Reed and A.R. Ward (2008) Approximating the $GI/GI/1+GI$ queue with a nonlinear drift diffusion: Hazard rate scaling in heavy traffic. *Math. Oper. Res.* **33** (3), 205–216.

[42] D. Stoyan (1984) *Comparaison Methods for Queues and Other Stochastic Models.* John Wiley, Sons, New York.

[43] S. Subba Rao (1965) Queueing models with balking, reneging and interruptions. *Operat. Res.* **13**, 596–608.

[44] S. Subba Rao (1967) A queueing model with balking and reneging. *Ann. Inst. Stat. Math. Jpn.* **19**, 55–71.

[45] L. Tadj, L. Benkherouf and L. Aggoun (1998) On a delayed service queueing system with random server capacity and impatient customers. *Internat. J. Appl. Stat. Sci.* **3**, 309–328.

[46] B. Van Houdt, R.B. Lenin and C. Blondia (2003) Delay distribution of impatient customers in a discrete time. *Queue. Syst. Theo. Appl.* **45** (1), 59–73.

[47] A.I. Zeifman (1985) Stability for continous-time nonhomogeneous Markov chains. *Lecture Notes Math.* **1115**, 401–414.

[48] V.M. Zolotarev (1975) Qualitatitve estimates for continuity property of queueing systems. *Theory Prob. Appl.* **20** (1), 211–213.