Non-Parametric Sensitivity Analysis of the Finite M/M/1 Queue

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Abstract: In this paper we establish a framework for robust sensitivity analysis of queues. Our leading example is the finite capacity M/M/1/N queue, and we analyze the sensitivity of this model with respect to the assumption that interarrival times are exponentially distributed.

Keywords: Robustness, Sensitivity analysis, Perturbation analysis for Markov chains, G/M/1/N system

1. INTRODUCTION

Sensitivity analysis studies the effect a perturbation of a design parameter of a model has on the performance of the model. This line of research dates back to Schweitzer's pioneering paper Schweitzer (1968), where the following question is addressed: Consider a Markov chain with discrete state space S, transition probability matrix P, and unique stationary distribution π_P; what is the effect of perturbing P by some matrix Δ on the stationary distribution, where Δ is such that $Q = P + \Delta$ is a well defined transition probability on S? Denote the unique stationary distribution of $P + \Delta$ by $\pi_{P+\Delta}$. Perturbation analysis of Markov chains (PAMC) studies bounds of the type

$$||\pi_{P+\Delta} - \pi_P|| \leq c||\Delta||,$$

where $|| \cdot ||$ denotes an appropriate norm (e.g., the total variation norm), and c is the so-called *condition number*. For a recent overview, see Cho and Meyer (2001). The above bound has the attractive feature that it provides a uniform bound on the ball $\{Q \in \mathbb{P}(S) : ||Q - P|| < \epsilon\}$, where $\mathbb{P}(S)$ denotes the set of all Markov transition probabilities over state space S. More specifically, it holds for the condition number that

$$\sup_{\{Q\in\mathbb{P}(S):||Q-P||<\epsilon\}} ||\pi_Q - \pi_P|| \leq \epsilon c. \qquad (1)$$

Put differently, c provides a robust perturbation bound since the norm distance of perturbing P by any Q, such that Q is no more than ϵ away from P, is bounded by ϵc. Inspired by (1), we define the *robust sensitivity of* π_P *with respect to* P by

$$\frac{d||\pi_P||}{d||P||} := \lim_{\epsilon\downarrow 0} \sup_{\{Q\in\mathbb{P}(S):||Q-P||<\epsilon\}} \frac{||\pi_Q - \pi_P||}{||Q - P||},$$

provided the limit exists.

In order to develop an approach to robust sensitivity analysis we will combine perturbation analysis of Markov chains with series expansions of Markov chains. The series expansion of Markov chains (SEMC) dates back to Schweitzer (1968), and for more recent work see Cao (1998), Heidergott and Hordijk (2003); Heidergott et al. (2006, 2007), Abbas et al. (2013). First, we introduce some basic concepts. We denote the ergodic projector of P by Π_P, where Π_P is a matrix such that $\mu\Pi_P = \pi_P$ for any probability distribution μ on S. For a stationary aperiodic ergodic chain, Π_P has rows identical to π_P. We write D for the deviation matrix associated with P, where

$$D = \sum_{m=0}^{\infty}(P^m - \Pi_P) = \sum_{m=0}^{\infty}(P - \Pi_P)^m - \Pi_P.$$

Element (i, j) of D can be interpreted as the expected total difference in number of visits to a state j from i compared to the number of visits to j starting in equilibrium. Many papers have been devoted to the investigation of the properties of D and the conditions for its existence (see for example Syski (2002); Coolen-Schrijner and Van Doorn (2002)). Note that D is finite for any aperiodic finite-state Markov chain.

Starting point for SEMC is the update formula

$$\pi_Q = \pi_P + \pi_Q(Q - P)D_P, \qquad (2)$$

which is easily checked. Inserting repeatedly the expression for π_Q on the right-hand side yields the following series expansion

$$\pi_Q = \pi_P + \pi_P \sum_{n=1}^{\infty}((Q - P)D_P)^n$$

$$= \pi_P + \pi_P((Q - P)D_P) \sum_{n=0}^{\infty}((Q - P)D_P)^n. \qquad (3)$$

Assume for the following that $||(Q - P)D|| < 1$, then applying norms to (3) it readily follows by SEMC that

$$\frac{||\pi_Q - \pi_P||}{||Q - P||} \leq ||\pi_P|| \frac{||D||}{1 - ||(Q - P)D||}. \qquad (4)$$

Hence, we obtain

$$\frac{d||\pi_P||}{d||P||}$$
$$\leq \lim_{\epsilon \downarrow 0} \sup_{\{Q:||Q-P||<\epsilon\}} ||\pi_P|| \frac{||D||}{1-||(Q-P)D||}.$$

It is worth noting that bounds of the type as displayed in Equation (4) are studied in the literature under the name of strong stability, see Kartashov (1996), and we will use techniques developed in strong stability theory in our technical analysis.

Consider the finite M/M/1 queue with capacity N, and assume that customers that arrive to a full queue are lost. In applications, the exponential assumption is made for ease of analytical tractability while the actual system typically violates this assumption. In this paper we develop a framework for robust sensitivity analysis of the M/M/1 queue with respect to the interarrival time distribution. In other words, we bound the sensitivity of the stationary distribution of the queue length process of the M/M/1/N queue with respect to the assumption that interarrival times are exponential. In order to do so, we will consider the queue length process embedded at service completions in the M/M/1/N loss queue.

Surprisingly enough, the theory developed for discrete state Markov chains in PAMC and SEMC can be made fruitful for the corresponding non-parametric perturbation of the interarrival time distribution (from "M" to "G," say). A key result for this will be that the norm distance of the transition kernel of the queue length process embedded at departures in the M/M/1/N queue, denoted by P, and the transition kernel of the queue length process embedded at departures in the G/M/1/N queue, denoted by Q, can be bounded in terms of the norm distance of the interarrival time distributions. Let E_λ denote the exponential distribution with rate λ, and denote by π_{E_λ} the stationary distribution of the queue length process embedded at departures in the M/M/1/N loss queue with service rate μ; let G denote a general service time distribution, and denote by π_G the stationary distribution of the queue length process embedded at departures in the G/M/1/N loss queue with service rate μ. Then we will establish a bound for

$$\frac{d||\pi_{E_\lambda}||}{d||G-E_\lambda||} := \lim_{\epsilon \downarrow 0} \sup_{\{G:||G-E_\lambda||<\epsilon\}} \frac{||\pi_G - \pi_{E_\lambda}||}{||G-E_\lambda||}.$$

The paper is organized as follows. Notations and preliminaries with the basic theorem on the series expansion method are provided in Section 2. In Section 3, we present the M/M/1/N and G/M/1/N models. In Section 4 we establish norm bounds for the G/M/1/N and M/M/1/N queue, and we compute an upper bound for the robust sensitivity of the M/M/1/N queue. Section 5 is devoted to sensitivity bounds for "directional" perturbations. We conclude with a discussion of further research.

2. PRELIMINARIES AND NOTATIONS

In this paper, we use the norm $\|.\|_v$, also called v-norm, where $v \in \mathbb{R}^S$ is such that $v(i) \geq 1$ for all $i \in S$. For a column vector $w \in \mathbb{R}^S$ the v-norm is given by

$$\|w\|_v = \sup_{i \in S} \frac{|w(i)|}{v(i)},$$

and for a row vector $u^\top \in \mathbb{R}^S$ the v-norm is given by

$$\|u\|_v = \sum_{i \in S} |u(i)| v(i).$$

As usual, we write distributions as row vectors and performance functions as column vectors.

For a matrix $A \in \mathbb{R}^{S \times S}$ the v-norm is given by

$$\|A\|_v = \sup_{i \in S} \frac{\sum_{j \in S} |A|(i,j)\ v(j)}{v(i)},$$

where $|A|(i,j)$ denotes the $(i,j)th$ element of the matrix of absolute values of A.

It is worth noting that letting $v(i) = 1$ for $i \in S$, recovers the total variation norm, which we denote by $\| \cdot \|_{tv}$. In the following we will work with the v-norm as it has the nice property that v-norm results on probability distributions carry over easily to performance mappings. More precisely, let $f \in \mathbb{R}^S$, then it holds

$$|\pi_G f - \pi_{E_\lambda} f| \leq ||\pi_G - \pi_{E_\lambda}||_v\, ||f||_v,$$

and

$$\lim_{\epsilon \downarrow 0} \sup_{\{G:||G-E_\lambda||<\epsilon\}} \frac{|\pi_G f - \pi_{E_\lambda} f|}{||G-E_\lambda||} \leq \frac{d||\pi_{E_\lambda}||}{d||G-E_\lambda||} ||f||, \quad (5)$$

provides a robust sensitivity bound on the effect of small perturbations of E_λ on the stationary cost measure f.

In case that $S \subset \mathbb{N}$ (like in our application), we let $v(i) = \beta^i$, for $i \in S$, for some $\beta \geq 1$. It is worth noting that β is a free parameter.

3. THE M/M/1/N AND G/M/1/N MODEL

Consider the G/M/1/N loss system where inter-arrival times are independently distributed with general distribution $G(t)$ and service times are distributed with $E_\mu(t)$ (exponential with parameter μ).

Let Y_n be the number of customers left behind in the system by the nth departure. It's easy to prove that Y_n forms a Markov chain with a transition matrix ($Q = (q_{i,j})_{i,j \in S}$), with entries

$$q_{ij} = \begin{cases} \alpha'_i & \text{if } j = 0, \\ b'_{i-j+1} & \text{if } 1 \leq j \leq i+1, \\ 0 & \text{if } i+1 \leq j \leq N, \end{cases}$$

for $0 \leq i \leq N-1$, and for $i = N$

$$q_{Nj} = \begin{cases} \alpha'_{N-1} & \text{if } j = 0, \\ b'_{N-j} & \text{if } 1 \leq j \leq N, \end{cases}$$

where

$$\alpha'_i = 1 - \sum_{k=0}^{i} b'_k,$$

and

$$b'_k = \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} dG(t).$$

The transition kernel ($P = (p_{i,j})_{i,j \in S}$) of the queue length process of the M/M/1/N loss system embedded at service completions is given by

$$p_{ij} = \begin{cases} \alpha_i & \text{if } j = 0, \\ b_{i-j+1} & \text{if } 1 \le j \le i+1, \\ 0 & \text{if } i+1 \le j \le N \end{cases},$$

for $0 \le i \le N-1$, and for $i = N$

$$p_{Nj} = \begin{cases} \alpha_{N-1} & \text{if } j = 0, \\ b_{N-j} & \text{if } 1 \le j \le N, \end{cases}$$

where

$$b_k = \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} dE_\lambda(t) = \frac{\lambda \mu^l}{(\lambda+\mu)^{l+1}},$$

and

$$\alpha_i = 1 - \sum_{k=0}^{i} b_k = \left(\frac{\mu}{\lambda+\mu} \right)^{i+1}.$$

We denote the taboo kernel of the M/M/1/N queue with taboo set $\{0\}$ by $(T = (T_{i,j})_{i,j \in S})$, i.e.,

$$T_{ij} = \begin{cases} p_{ij} & \text{if } i > 0 \text{ and } j \ge 0, \\ 0 & \text{if } i = 0 \text{ and } j \ge 0 \end{cases}.$$

In words, T is a deficient kernel representing an M/M/1/N system that avoids reaching the empty state. In the following section we will establish the main performance bounds for the M/M/1/N and the G/M/1/N queue.

4. ESTABLISHING NORM BOUNDS

Let

$$\hat{D} = \sum_{n \ge 0} T^n (I - \Pi_P),$$

where finiteness of $\hat{D}$ follows from the fact that the deviation exists for finite state-space models.

Theorem 1. Let

$$\eta := \frac{1 + \|\pi_P\|_v}{1 - \|T\|_v}.$$

If $||T||_v < 1$, then

$$||\hat{D}||_v \le \eta,$$

and, if in addition, $\eta ||Q - P|| < 1$, then

$$\|\pi_Q - \pi_P\|_v \le \|\pi_P\|_v \frac{\eta ||Q-P||_v}{1 - \eta ||Q-P||_v}.$$

Proof. We have

$$\pi_Q = \pi_P \sum_{n \ge 0} ((Q-P)D)^n \tag{6}$$

The following equality has been established in Kartashov (1996)

$$D = (I - \Pi_P) \sum_{n \ge 0} T^n (I - \Pi_P). \tag{7}$$

Note that $(Q-P)(I-\Pi_P) = (Q-P)$ and multiplying (7) by $(Q-P)$ yields

$$(Q-P)D = (Q-P)\hat{D} = (Q-P) \sum_{n \ge 0} T^n (I - \Pi_P).$$

From $||T||_v < 1$, we have

$$\begin{aligned} \|(Q-P)\hat{D}\|_v &\le \|(Q-P)\|_v \left\| \sum_{n \ge 0} T^n \right\|_v \|(I - \Pi_P)\|_v \\ &\le \|(Q-P)\|_v \frac{1}{1 - \|T\|_v} (1 + \|\pi_P\|_v) \\ &= \eta ||Q-P||. \end{aligned}$$

Inserting the above expression into (6)

$$\|\pi_Q - \pi_P\|_v \le \|\pi_P\|_v \sum_{n \ge 1} (\eta ||Q-P||)^n.$$

If $\eta ||Q-P|| < 1$, we obtain

$$\|\pi_Q - \pi_P\|_v \le \|\pi_P\|_v \frac{\eta ||Q-P||}{1 - \eta ||Q-P||},$$

which proves the claim. □

Theorem 2. Suppose that $\rho := \lambda/\mu < 1$. For all β such that

$$1 \le \beta < \frac{\mu}{\lambda},$$

it holds that

$$||T||_v \le \frac{\lambda \beta}{\lambda + \mu - \frac{\mu}{\beta}} \left(1 - \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^N \right) < 1$$

and

$$\|\pi_P\|_v = \frac{(1-\rho)(1-(\rho\beta)^{N+1})}{(1-\rho^{N+1})(1-\rho\beta)}.$$

Proof. We will show that there exists some constant $\psi < 1$ such that $Tv(l) \le \psi v(l)$ for all $l \in S$.
For $0 \le l \le N-1$, by computation

$$\begin{aligned} (Tv)(l) &= \sum_{j=0}^{N} \beta^j T_{lj} = \sum_{j=1}^{N} \beta^j p_{lj} = \sum_{j=1}^{l+1} \beta^j b_{l+1-j} \\ &= \sum_{j=0}^{l} \beta^{l+1-j} b_j = \frac{\lambda}{\lambda+\mu} \beta^{l+1} \sum_{j=0}^{l} \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^j \\ &= \frac{\lambda}{\lambda+\mu} \beta^{l+1} \frac{1 - \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^{l+1}}{1 - \frac{\mu}{\beta(\lambda+\mu)}} \\ &= \lambda \beta^{l+1} \frac{1 - \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^{l+1}}{\lambda + \mu - \frac{\mu}{\beta}}. \end{aligned}$$

For $l = N$,

$$\begin{aligned} (Tv)(N) &= \sum_{j=1}^{N} \beta^j p_{Nj} = \sum_{j=1}^{N} \beta^j d_{N-j} \\ &= \sum_{j=0}^{N-1} \beta^{N-j} b_j = \frac{\lambda}{\lambda+\mu} \beta^N \sum_{j=0}^{N-1} \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^j \\ &= \lambda \beta^N \frac{1 - \left(\frac{\mu}{\beta(\lambda+\mu)} \right)^N}{\lambda + \mu - \frac{\mu}{\beta}}, \end{aligned}$$

and for $0 \le l \le N$

$$(Tv)(l) \le \beta^l \frac{\lambda\beta}{\lambda+\mu-\frac{\mu}{\beta}} \left(1-\left(\frac{\mu}{\beta(\lambda+\mu)}\right)^N\right),$$

for all $\beta \ge 1$. We have

$$\left(1-\left(\frac{\mu}{\beta(\lambda+\mu)}\right)^N\right) < 1,$$

and, assuming that $\lambda/\mu < 1$ and $1 \le \beta < \mu/\lambda$, we obtain

$$\frac{\lambda\beta}{\lambda+\mu-\frac{\mu}{\beta}} < 1.$$

Hence,

$$\psi = \frac{\lambda\beta}{\lambda+\mu-\frac{\mu}{\beta}} \left(1-\left(\frac{\mu}{\beta(\lambda+\mu)}\right)^N\right) < 1.$$

Therefore, there exists $\psi < 1$ such that for all $l \in S$ $(Tv)(l) \le \beta^l \psi$, for all β such that $1 \le \beta < \mu/\lambda$, which proves the first part of the claim.

For the second part of the proof note that the stationary distribution of the queue length processes at departure moments in the M/M/1/N loss queue is equal to that in arrival moments, which by the PASTA property, is equal to the stationary distribution of the queue length process, and has a well known closed form solution. Computing the v-norm of this distribution is completes the proof. □

For the following, let

$$W := ||G - E_\lambda||_{tv} = \int_0^\infty |G - E_\lambda|(dt),$$

and observe that

$$W \le ||G - E_\lambda||_{tv}$$

for $\beta \ge 1$.

Theorem 3. If $1 \le \beta < \mu/\lambda$, then it holds for $\beta \ge 1$ that

$$\|(P-Q)\|_v \le (1+\beta)W.$$

Proof. For all β such that $1 \le \beta < \mu/\lambda$, we have

$$\begin{aligned}\|P-Q\|_v &= \max_{0\le l\le N} \frac{1}{\beta^l}\sum_{j=0}^N \beta^j |p_{lj}-q_{lj}|,\\ &= \max_{0\le l\le N} \frac{1}{\beta^l}\left(|p_{l0}-q_{l0}| + \sum_{j=1}^N \beta^j|p_{lj}-q_{lj}|\right),\\ &\le \max_{0\le l\le N}\frac{1}{\beta^l}|p_{l0}-q_{l0}| + \max_{0\le l\le N}\frac{1}{\beta^l}\sum_{j=1}^N |p_{lj}-q_{lj}|.\end{aligned}$$

We set $A = \max_{0\le l\le N} \frac{1}{\beta^l}|p_{l0}-q_{l0}|$ and

$$B = \max_{0\le l\le N}\frac{1}{\beta^l}\sum_{j=1}^N |p_{lj}-q_{lj}|.$$

For $0 \le l \le N-1$

$$\begin{aligned}|p_{l0}-q_{l0}| &= |\sum_{j=0}^l b_j - \sum_{j=0}^l b'_j|\\ &\le \sum_{j=0}^l \int_0^\infty \frac{1}{j!}e^{-\mu t}(\mu t)^j |G-E_\lambda|(dt)\\ &= \int_0^\infty \sum_{j=0}^l \frac{(\mu t)^j}{j!}e^{-\mu t}|G-E_\lambda|(dt)\\ &\le \int_0^\infty |G-E_\lambda|(dt) = W.\end{aligned}$$

For l = N

$$\begin{aligned}|p_{N0}-q_{N0}| &= |\sum_{j=0}^{N-1} b_j - \sum_{j=0}^{N-1} b'_j|\\ &\le \int_0^\infty \sum_{j=0}^{N-1}\frac{(\mu t)^j}{j!}e^{-\mu t}|G-E_\lambda|(dt)\\ &\le \int_0^\infty |G-E_\lambda|(dt) = W.\end{aligned}$$

So, we have $A \le W$. For $0 \le l \le N-1$

$$\begin{aligned}&\frac{1}{\beta^l}\sum_{j=1}^N |p_{lj}-q_{lj}|\\ &\le \frac{1}{\beta^l}\sum_{j=1}^{l+1}\beta^j \int_0^\infty \frac{1}{(l+1-j)!}e^{-\mu t}(\mu t)^{l+1-j}|G-E_\lambda|(dt)\\ &\le \beta\int_0^\infty e^{-\mu t}\sum_{j=1}^{l+1}\frac{1}{(l+1-j)!}\left(\frac{\mu t}{\beta}\right)^{l+1-j}|G-E_\lambda|(dt)\\ &\le \beta\int_0^\infty |G-E_\lambda|(dt).\end{aligned}$$

For l = N

$$\begin{aligned}&\frac{1}{\beta^N}\sum_{j=1}^N |p_{Nj}-q_{Nj}|\\ &\le \frac{1}{\beta^N}\sum_{j=1}^N \beta^j\int_0^\infty \frac{1}{(N-j)!}e^{-\mu t}(\mu t)^{N-j}|G-E_\lambda|(dt)\\ &\le \int_0^\infty e^{-\mu t}\sum_{j=1}^N \frac{1}{(N-j)!}\left(\frac{\mu t}{\beta}\right)^{N-j}|G-E_\lambda|(dt)\\ &\le \int_0^\infty |G-E_\lambda|(dt).\end{aligned}$$

Therefore, $B \le \beta W$, and we obtain $\|P-Q\|_v \le (1+\beta)W$, which concludes the proof. □

5. ROBUST SENSITIVITY ANALYSIS

Elaborating on the norm bounds provided in the previous section, we will establish in the next theorem the robust sensitivity bound.

Theorem 4. If

$$(1+\beta)W\frac{1+\|\pi_{E_\lambda}\|_v}{1-\|T\|_v} < 1,$$

then it holds that

$\lambda = 1$		$\lambda = 5$	
β	$d\|\pi_{E_\lambda}\|/d\|E_\lambda\|$	β	$d\|\pi_{E_\lambda}\|/d\|E_\lambda\|$
0.01	7.0683	1.01	44.5544
1.04	7.0335	1.04	42.6722
1.06	7.0240	1.06	42.1231
1.08	7.0237	1.08	41.9956
1.2	7.1664	1.1	42.2163
1.5	8.1219	1.2	47.3913
2	10.7104	1.4	78.6925
5	67.1664	1.6	171.1264
8	812.4779	1.9	539.4901

Table 1. The robust perturbation bound for varying β.

$\lambda = 1$		$\lambda = 5$	
β	$(d\|\pi_{E_\lambda}\|/d\|E_\lambda\|)\|f\|$	β	$(d\|\pi_{E_\lambda}\|/d\|E_\lambda\|)\|f\|$
0.01	39.9521	1.01	251.8336
1.04	33.3522	1.04	202.3467
1.08	26.5569	1.08	158.7860
1.2	14.4000	1.1	142.9801
1.5	7.2195	1.2	95.2274
2	5.3552	1.4	86.0340
3	6.4319	1.5	99.6969
5	13.4333	1.7	196.8188
8	101.5597	1.9	754.8870

Table 2. The robust perturbation bound for the mean queue length for varying β.

$$\limsup_{\|G-E_\lambda\|\to 0} \frac{\|\pi_Q - \pi_P\|_v}{\|G - E_\lambda\|_v} \leq \inf_{\beta \geq 1} ||\pi_{E_\lambda}||_v (1+\beta)\, \frac{1 + \|\pi_{E_\lambda}\|_v}{1 - \|T\|_v}.$$

Proof. Noting that D in (3) can be replaced without loss of generality by $\hat{D}$, rewriting (4) yields

$$||\pi_G - \pi_{E_\lambda}||_v \leq ||\pi_{E_\lambda}||_v \frac{||Q-P||_v\, ||\hat{D}||_v}{1 - ||(Q-P)||_v\, ||\hat{D}||_v}. \qquad (8)$$

By Theorem 1 it holds

$$||\hat{D}||_v \leq \frac{1 + \|\pi_{E_\lambda}\|_v}{1 - \|T\|_v}.$$

and by Theorem 3 we have $||Q-P|| \leq (1+\beta)W$. Inserting the bounds into (8) yields

$$||\pi_G - \pi_{E_\lambda}||_v \leq ||\pi_{E_\lambda}||_v \frac{(1+\beta)W\, \frac{1+\|\pi_{E_\lambda}\|_v}{1-\|T\|_v}}{1 - (1+\beta)W \frac{1+\|\pi_{E_\lambda}\|_v}{1-\|T\|_v}}. \qquad (9)$$

Dividing by W and letting W tend to zero gives

$$\lim_{\epsilon \downarrow 0} \sup_{G: ||G-E_\lambda|| < \epsilon} \frac{||\pi_G - \pi_{E_\lambda}||_v}{||G - E_\lambda||_v} \leq ||\pi_{E_\lambda}||_v (1+\beta)\, \frac{1 + \|\pi_{E_\lambda}\|_v}{1 - \|T\|_v},$$

which proves the claim. □

Using the fact that the total variational norm is smaller or equal to the v-norm for $\beta > 1$, we arrive at the following corollary.

Corollary 5. In the special case of the total variational norm, the statement put forward in Theorem 4 simplifies to

$$\limsup_{||G-E_\lambda||_{tv}\to 0} \frac{\|\pi_Q - \pi_P\|_{tv}}{\|F - E_\lambda\|_{tv}} \leq \frac{4(\lambda+\mu)^N}{\mu^N}.$$

In the following, we will present numerical results. We consider an M/M/1/N system, with Poisson arrival rate λ, service rate $\mu = 10$, and $N = 6$. In Table 1 we present numerical values for the bound put forward in Theorem 4.

As the table illustrates, the value for β that minimizes the robust bound depends on λ. For $\lambda = 1$, $\beta = 1.08$ is the best choice yielding 7.0237 for the robust sensitivity, and for $\lambda = 5$, $\beta = 1.08$ is the best choice yielding 41.995 for the robust sensitivity.

Following (5), we turn as next to our robust bound for the mean queue length. We consider again an M/M/1/N system, with Poisson arrival rate λ, service rate $\mu = 10$, and $N = 6$. In Table 2 we present numerical values for the bound obtained from combining the bound in Theorem 4 with (5).

As the table illustrates, the value for β that minimizes the robust bound depends on λ. For $\lambda = 1$, $\beta = 2$ is the best choice yielding 5.355 for the robust sensitivity, and for $\lambda = 5$, $\beta = 1.4$ is the best choice yielding 86.034 for the robust sensitivity.

6. DIRECTIONAL SENSITIVITIES

From a geometrical point of view, the robust sensitivity estimate is based on a worst case analysis. Indeed, the supremum in the definition requires that the interarrival time distribution that constitutes the worst case perturbation has to be found for given ϵ, which in itself is an optimization problem. Obviously, such a robust sensitivity indicator is an upper bound for the effect of any "directional" sensitivity that is obtained from perturbing E_λ with a concrete and fixed distribution.

There is a wealth of results in the literature provided condition numbers for particular queuing systems. In this line of research P and Q are fixed and the perturbation is used for bounding the effect on the stationary distribution of switching from P to Q. Elaborating on (9) we have the following perturbation bound

$$||\pi_G - \pi_{E_\lambda}||_v \leq ||\pi_{E_\lambda}||_v \frac{(1+\beta)W\, \frac{1+\|\pi_{E_\lambda}\|_v}{1-\|T\|_v}}{1 - (1+\beta)W \frac{1+\|\pi_{E_\lambda}\|_v}{1-\|T\|_v}}.$$

Unfortunately, this bound only applies under the condition that

$$W \leq \frac{1}{1+\beta} \frac{1 - \|T\|_v}{1 + \|\pi_{E_\lambda}}.$$

Letting $\beta = 1$, then $||\cdot||_v$ yields the total variational norm, for which it holds that $\|\pi_{E_\lambda}\|_{tv} = 1$. Inserting this bound together with the bound for $\|T\|_{tv}$ into the restriction for W, we arrive at

$$W \leq \frac{1}{4} \left(\frac{\mu}{\lambda + \mu} \right)^N.$$

Since $\mu < \lambda + \mu$ the bound on the above RHS converges geometrically fast to zero in N. Provided that W is bounded by the expression on the above RHS, we arrive at the overall bound

$$||\pi_G - \pi_{E_\lambda}||_{tv} \leq \frac{4W(\lambda+\mu)^N}{\mu^N - 4W(\lambda+\mu)^N}.$$

Even without considering the actual quality of the bound, the simple fact that it is only defined for a rather limited range of total variational distances suggests that this is not a good way to proceed for obtaining meaningful bounds.

However, starting from (2) and applying the total variational bound, i.e., $\beta = 1$, we obtain

$$||\pi_G - \pi_{E_\lambda}||_{tv} \leq \|P - Q\|_{tv} \|\hat{D}\|_{tv}$$

and inserting the previously obtained bounds in Theorem 3 and Theorem 1 we arrive at

$$||\pi_G - \pi_{E_\lambda}||_{tv} \leq 4W \| \frac{1}{1 - ||T||_{tv}}$$

Is is worth noting that this bound holds with the only restriction that $\lambda < \mu$. To see this, note for $\lambda < \mu$, $\beta = 1$ it holds that

$$1 - \frac{\mu}{(\lambda + \mu)} < 1,$$

which implies that the bound on $||T||_v$ displayed in Theorem 2 is applicable. Hence, the overall bound becomes

$$||\pi_G - \pi_{E_\lambda}||_{tv} \leq 4W \left(\frac{\lambda + \mu}{\mu} \right)^N .$$

Again, this bound will be only meaningful for small values of W. To see this, note that the total variational norm of $\pi_G - \pi_{E_\lambda}$ is bounded by 2, assuming, for example, $4 = \mu = 2\lambda$, it follows that the bound only drops below the trivial bound of 2 for

$$W \leq \frac{1}{2} \left(\frac{2}{3} \right)^N .$$

Consequently, letting $N = 6$ like in the previous example we obtain $W \leq 0.0438$. To summarize, the norm bound approach for establishing bounds on directional sensitivities fails to provide numerically convincing results.

7. CONCLUSION

In this paper, we have developed a framework for robust sensitivity estimates for the finite M/M/1 queue with respect to a perturbation of the interarrival time distribution. The extension of our framework to more complex queues and networks of queues is topic of further research.

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