République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université A.MIRA-BEJAIA

جامعة بجاية
Tasdawit n Bgayet
Université de Béjaïa

Faculté des Sciences Exactes
Département de Physique
Laboratoire de Physique Théorique

THÈSE

EN VUE DE L'OBTENTION DU DIPLOME DE DOCTORAT

Domaine : Science de la matière **Filière :** Physique

Spécialité : Physique théorique

Présentée par
Mr. DJEHICHE Abdelmadjid

Thème

Spacetime and the electromagnetic interaction

Soutenue le : 08/04/2021 **Devant le Jury composé de :**

Nom et Prénom	**Grade**		
Mr. GHARBI Abdelhakim	**Prof**	**Univ. de Béjaia**	**Président**
Mr. BOUDA Ahmed	**Prof**	**Univ. de Béjaia**	**Rapporteur**
Mr. BELABBAS Abdelmoumene	**MCA**	**Univ. de Béjaia**	**Co-rapporteur**
Mr. BELALOUI Nadir	**Prof**	**Univ. de Constantine**	**Examinateur**
Mr. FOUGHALI Taoufik	**MCA**	**Univ. de Béjaia**	**Examinateur**
Mr. HAOUAT Salah	**Prof**	**Univ. de Jijel**	**Examinateur**

Année Universitaire : 2020/2021

“Get rid of the thinking constraints to start thinking” *Abdelmadjid*

DEDICATION

In the name of Allah, the most gracious, the most merciful. I would love to dedicate this thesis to the soul of my mother. The women who glorify knowledge.

ACKNOWLEDGEMENT

I am particularly grateful to my supervisor **Pr. Ahmed Bouda** for the valuable support he provided me along the research period.

I also owe special thanks to my Co-supervisor **Mr. Abdelmoumene Belabbas** for his permanent assistance and his patience while discussing some points. Especially when he resisted the improvised ideas that I came with.

My thanks goes to every members in theoretical physics laboratory of Bejaia University.

I would give many thanks as well to my family especially my parents, to my friends (**Nabil**, **Hichem**, **Mouad**…. especially my soulmate **Abdelhak Asli**).

Without forgetting to thank **Pr. Salah Haouat** for his near and far contributions.

I also do not forget to thank the professors **Mr. Abdelhakim Gharbi**, **Mr. Nadir Belaloui** and **Mr. Taoufik Foughali**

I will forever be thankful to my lover fiancée.

Finally, I would like to thank whoever deserves to be thanked.

Contents

Chapter 1

General Introduction

Our universe has been created from one point or from only one event as the Big Bang theory says. This is what prompted most of physicists to think that there exist only one system, which describes the universe or more appropriately one single law that governs this universe. As well known today the physical universe is described by four interactions:

Gravity this effect is weak and very clear; appears at large scale, resulting from the interaction between two masses, it is always attractive (there is one kind of masses).

Electromagnetism resulting from the interaction between two charges in different ways, attractive or repulsive according to the charges sign. In particle physics the electromagnetic force is the result of the fact that charged particles exchange photons with one another.

Weak interaction (weak nuclear force) this interaction occurs between subatomic particles, and it is the responsible for the atoms radioactive decay. In particle physics, fermions can exchange a three kinds of massive bosons $W^+, W^-$and Z. In beta decay a neutron transforms to a proton with emission of W^-particle, which then decays into electron and neutrino. The same process appears in the fusion of hydrogen into helium at the core of our sun. Weak interaction is weak because the massive bosons $W^+, W^-$and Z have a rest mass much bigger than those of the leptons and the quarks (except the Top quark).

Strong interaction Neutrons and protons combined in one nucleus by strong interaction. In particle physics the proton consists of three quarks, each of quarks have three states or three colors (green, red, blue). The strong nuclear force, according to the standard model, is the result of the fact that particles with colors charges exchange gluons one another and this exchange holds quarks together inside protons. The gluon state defined by two colors (red and anti red, blue and anti green, green and anti green....) then the gluon can split up into quark and its associated anti-quark also a quark annihilates its associated anti-quark to produce a gluon. Therefore, the proton does not just have a three quarks in addition there are a virtual quarks and anti-quarks created and annihilated inside this proton. The gluons attract themselves because gluons can absorb and emit other gluons unlike photons. Moreover, if we apply amount of energy to separate two quarks this energy is converted to create a quark and anti quark and these quarks stay bounded so an isolated quark could never be observed. Unlike electromagnetic force the strong nuclear force between quarks does not decrease as we increase the distance between them. The strong nuclear force inside a nucleus is very strong compared to the electric repulsive force between protons inside this nucleus.

The last three interactions were combined in standard model [1] as a gauge theory. The model is based on quantum mechanics which governs the particles world. The rest is the gravity which is connected to spacetime geometry. The latter is neither a gauge theory nor a quantum mechanical theory. Therefore, string theory seeks a quantum mechanical theory of gravity, and thence gives a unified description to the four interactions.

The general relativity [2, 3, 4, 5] is one of the most important and fundamental theories. This is confirmed with the observations data such as the perihelion advance, the gravitational lensing, the gravitational waves (detected recently by LIGO experiment [6]) and more recently (April 2019) the detection of black hole image created by the Event Horizon Telescope. For these reasons, and based on general relativity concepts, we can look for another way of unification. Some physicists led by Einstein were speculating about a unified theory in a geometrical way. Basically, the unified geometrical theory is frequently a modification of general relativity geometry (non-Riemannian geometry) such as the use of anti- or non- symmetric metric, conformal metric (Weyl theory [7]) and affine connection [8] , furthermore, by the need of extra dimensions, as in Kaluza-Klein theory [9, 10].

On the other hand, in classical physics, we can determine the distance traveled by a body just when we have or know its amount of energy. Thus, if we know the amount of energy of a physical system, whatever the type of the interaction occurs in it, we can figure out the dynamics of particles or bodies in this system. Nature often shows us that it is founded on similarity, harmony and uniformity. In this regard, it is worth mentioning here the "Principle of Solidarity"[1][11] which entails a mutual interaction between phenomena (the four interactions) and the spacetime structure. As we can notice in [12], this principle may seem somewhat fundamental to a geometric view of interactions. This leads to a reasonable thought so that every interaction has its own spacetime with a generalized Minkowski metric, which is occupied by a dynamic character. Accordingly, we find ourselves on the face of the suggested principles of DSR (Deformed Special Relativity) theory [11, 13, 14]. The theory is based on the deformation of the Minkowski spacetime. In the same way as the Galilean transformations are invalid in the relativistic speed regime, and they are replaced by those of Lorentz. Likewise, the Lorentz transformations

[1] This principle was stated by Italian mathematician Bruno Finzi, who says that "It is necessary to consider space-time TO BE SOLIDLY CONNECTED with the physical phenomena occurring in it, so that its features and its very nature do change with the features and the nature of those. In this way not only space-time properties affect phenomena, but reciprocally phenomena do affect space-time properties. One thus recognizes in such an appealing "Principle of Solidarity" between phenomena and space-time that characteristic of mutual dependence between entities, which is peculiar to modern science."

become invalid at the Planck scale energies and must be modified. Therefore, the metric becomes dependent on energy in a dynamic role [15].

In general relativity theory, Einsteins' equations have succeeded in giving the relation between the geometry of spacetime and the energy-momentum. So, whatever the kind of energy (of all interactions) in Einsteins' equations right hand side, it becomes the source of gravitational field. Well, let's take the example of the Einstein-Maxwell equations where the electromagnetism energy-impulsion contribute to affect the structure of spacetime. Besides, in classical physics where the acceleration has a physical meaning, we can consider acceleration as an aspect of nature harmony. This appears very clear in Newton's second law of dynamics. He assumes that all forces with its different kinds can provide an acceleration to the particles (despite its properties) multiplied by an inertial term (till this day science does not give a clear picture to the true nature of the inertial mass). Although neither force nor acceleration has physical meaning in quantum mechanics. However, having a look at the relationship between acceleration in the classical case, and its equivalent in quantum mechanics, is not impossible (in particle accelerators the energy of particles increase with acceleration) but it can be resurrected (see [16, 17] for knowledge). In addition, we must not forget the perfect equivalence between gravity and accelerated frame, which lets acceleration get another meaning in quantum gravity.

Moreover, based on general relativity theory achievements, another thinking about geometrization has been shown in [18, 19, 20, 21]. The author in [18, 19, 20] assumes unexpected approach in which the non-gravitational interactions affect the spacetime structure. In this approach, and in a similar way as gravity, the non-gravitational interactions manifest themselves through a metric that is dependent on the different types of potential energies. Hence, the features as well as the nature of spacetime change from an interaction to another. By the way, this reminds us again of the deformed special relativity theory[2].

Despite it seems difficult to think about non-linear theory of electromagnetic field related to the curvature of spacetime, the unprecedented work [22] indicates an extended version of equivalence principle for the electromagnetic interaction. Also, by using a modified Lorentz gauge condition, it shows that the Maxwells' equations are derived from Einstein one in weak field limit, as well as the geodesic equation gives a form of Lorentz force.

In this thesis, based on the previously mentioned properties of nature, we will put emphasis on the geometrization of the electromagnetic interaction. The questions arising are: what is the manner in which electromagnetism affects the spacetime geometry? What is the right way to make electromagnetism dependent on the geometry of spacetime? How will be the transition from geometry to quantization?. Based on the aforementioned works [18, 19, 20, 22], we will discuss these questions. Then, we will attempt to provide another geometrical explanation for the electromagnetic interaction. Moreover, we will deal with Riemann geometry in four-dimensional spacetime where we can show how Einsteins' equations could describe an electromagnetic field, and furthermore,

[2]It should noting that in deformed special relativity, the curvature is zero, and the theory appears in cases of small scales.

the effect of the field on the metric, and how a trajectory of charged particle becomes a geodesic. The quantum effect of geometrization was expected and this effect obviously appears in the energy levels of quantum particles. Indeed, we will interpret this point clearly and with details in a quantum application of hydrogen atom. For someone start with the principle of solidarity (who is somewhat in agreement with this principle), he can ask this question, what is the manner in which the electromagnetism affects the spacetime properties?

Fundamentally, in order to distinguish between space-time structures of gravity and electromagnetism, we will use the term "universe". Any way, in general relativity, the geometrical theory of gravity is based on the so-called equivalence principle so that different objects follow the same worldline. By the same token, in electromagnetic universe, we urgently need a new concept for the equivalence principle to make a total geometrical description of electromagnetism. The problem is that different charged particles follow different worldlines according to the charge and mass properties. For that reason, it seems appropriate to consider a geometry related to the properties of particles. In a way, it is the interaction between charged particles that influences the spacetime geometry. From a founding principle of equivalence, a metric of electromagnetism can be inferred [23]. Moreover, the source of electromagnetism is described by the Einsteins' type equations. The transition towards a quantum mechanical theory, which is based on the geometry, may lead to thinking about quantum equivalence principle. The hydrogen atom is going to be the field of application where we will see how geometry have an effect on energy levels.

In the world of gravity, a convincing explanation for the cosmological constant is that it is the responsible for the cosmic expansion, while the proposition of a cosmological constant in the universe of electromagnetism does not seem impossible. Rather, we will be able, with its help, to give an explanation for the Lamb shift [23]. We call it the electromagnetic lambda term to distinguish between the two different constants.

This thesis is organized as follows: In chapter 2, we give a brief overview of the physics and mathematical tools of general relativity. In chapter 3, we exhibit the analogy between gravity and electromagnetism in weak field limit taking Barros approach into account. In chapter 4, we establish the metric of the electromagnetism with and without the electromagnetic lambda term, we study the motion of charged particles in both cases, and we show the effect of the geometrical theory on the energy levels of the hydrogen atom. In the chapter 5, we summarize the work and conclude.

Chapter 2

Spacetime Geometry

2.1 Introduction

Classically, the ”space” is the place where the physical object is located and where it moves. On the other hand, we have the mysterious notion ”time”, let’s say it comes to define the succession of different locations of the physical object. However, in relativity space and time are combined into one shape called spacetime and as the name suggests, we cannot distinguish between space and time because an observer unable to determine the location of the object separated from time, and thus there is no space without time and no time without space. Therefore, Einstein thought that time is a fourth dimension like the three other dimensions of space.

However, it is a long story to go from Newton era to the era of gravitational waves and Black holes. In this story, the spacetime plays a central role with a changed notion every time. At the beginning, the space is absolute with a one clock of time in this whole universe passing through relativistic spacetime to a dynamical spacetime (curve and expanded spacetime) and so on to arrive at a discrete and quantum spacetimes. For this reason, it is worth, here, to give an overview about the geometry of spacetime.

2.2 Why Einstein not Newton?

The “Luminiferous aether” hypothesis had arisen since the Galilei’s physics was unable to make a juxtaposition of the two well-defined theories of Newtonian dynamics and Maxwell’s electromagnetism. As it is known, the Galilean transformations did not keep the Maxwells’ equations invariant contrary to Newton’s dynamic equation. To save the Galileans’ transformations and overcome the problem of the non-invariant of Maxwells’ equations, the “aether” hypothesis was suggested. According to this hypothesis, the light (electromagnetic waves) does not have the same physical rules and must be travel in a medium called “aether” (as a note the air is the medium of sound’s waves propagation). Unfortunately, Michelson-Morley experiment disproved the existence of aether but the solution was the theory of relativity. (Openly, one can believe in the idea of aether[1] and with another concept and this does not require that relativity is wrong. Despite this, the equation $E = mc^2$ remains the smartest and most wonderful.)

2.2.1 from absolute to relative

Descartes viewed that space and matter are the same thing [24]. Indeed, there is no space without matter and that it mean the vacuum itself taken to be a matter. Going back to Newton, we find, according to him, that the absolute space is necessary result for the existence of anything, and hence in the absolute space, neither motion nor matter affect this space. In the bucket experiment, the rotation of a bucket fulled with water, makes the water take a concave shape relative to an observer at rest. Although the observer becomes rotating with the bucket, the water still in the concave shape. Newton[2] explain that as a result of water rotation relative to the surrounding absolute space. Even so, Mach argued that the water still taking a concave shape because it rotates relative to the distant stars and galaxies. The motion of a body according to Mach is the result of interaction between this body and all the matter in this universe. Therefore, there is no meaning of inertia in empty universe and consequently no motion [25].

However, who does assert that bodies with different amounts of energy occupy the same absolute space? Basically, something is moving, but what is meant exactly by saying that something is moving in relative to something. As Einstein reasoned [26], in this universe nothing is "absolute", and thus the water remains concave in the bucket experiment because it moves relative to the space time [25]. Taking another example a body moves at the ground relative to the earth, earth moves relative to the sun and the sun moves relative to the center of galaxy and so on. Consequently, one might give a physical size of something but this physical size stays real (exact) for its reference not for all references.The motion of the moon is circular relative to the earth while it has a spiral circular motion relative to the sun. This is what the word relative means. In special relativity, the term relative means everything move relative to everything, and the objects become significant, only, relative to other objects.

2.2.2 The spacetime-motion harmony

The shape of high speed rectilinear motion of a disk changes from a sphere to hyperbola relative to an observer at rest. Properly, the observer which is moving with the disk and with the same velocity see the disk in a spherical shape . In fact, both observers are right, the difference is that at the first reference the motion of the disk affects the surrounding

[1]We just keep the phenomenon of length contraction and time dilation taking place in oriented way that deviates from the direction of the ether (you can consult the Sagnac experiment).

[2]Newton show that the force is just an acceleration relative to the absolute space, which can be considered as a fixed frame. In addition, the definition of the inertial frames is related to this absolute space in which those frames are just move with a constant velocity relative to this absolute space. This means that the existence of absolute space is an inevitable necessity in order to define motion and inertial frame notion.

spacetime geometry, and therefore it is showing up in a deformed shape. This, obviously, impossible for an absolute space but it is convenient for special relativity by taking into account the length contraction and time dilation phenomena.

Further, if we returned to the disk example but in this case the disk rotates with high speed. According to special relativity, an observer in the center find that ($\frac{Circumference}{Diameter} = \pi\sqrt{1-\frac{(\omega R)^2}{c^2}} < \pi$) and this is something wrong (this called Ehrenfest paradox [27]). Einstein argument to handle this paradox was that the geometry is non-euclidean and the centrifugal force affects the spacetime.

On the other hand, in general relativity, matter (energy) affects the geometry of space-time. As a result, (for example) the orbit of Mercury, in fact, is not closed and this is confirmed by the curvature of space-time caused by the mass of the sun. More than that, the spacetime can be fluctuated and gives a gravitational waves.

From the previous, the spacetime becomes dynamical and that it means the spacetime is something material (acting and reacting), and therefore any thing occurs in this spacetime influence it, and reciprocally the influenced spacetime affects the motion.

2.3 Equivalence Principle

The classical theory shows that the gravity is just a force between masses, and its effect is given with

$$F_{12} = F_{21} = G\frac{M_1 M_2}{r_{12}^2} \tag{2.1}$$

According to Newton the gravitational mass plays two roles at the same time so that a mass m plays a role of an active mass, and acts a force on a passive mass M_p in which $F = G\frac{m_a M_p}{r^2}$. Simultaneously, the mass M plays a role of an active mass, and acts to m (in this case becomes passive mass) a force $F = G\frac{m_p M_a}{r^2}$. The two forces have the same strength leading to $\frac{m_a}{m_p} = \frac{M_a}{M_p}$ and this required that ($m_a = m_p$ and $M_a = M_p$). In another part, the inertial mass of some body appears when we apply a force to this body. We can see this type of masses as a resistance of bodies acceleration. Accidentally, in Newtonian dynamics, the gravitational mass cancels the inertial one and thus, we can approve this ($m_a = m_p = m_I$).

Otherwise, the Newtonian gravity theory is incompatible with the special theory of relativity [2, 4, 28] (non-covariant theory). This pushed Einstein after some thought Experiments to think about a geometrical theory of gravity in which he started by setting the equivalence principle.

Interestingly enough, the trajectories of particles in gravitational field are independent on the particle's properties, and hence they fall in the same way. Equivalently, an accelerated observers see objects (locally) fall in the same way independently from its properties. This reason is the consequence of the inertial and gravitational masses universality equality. Moreover, according to the equivalence principle, the gravitational field affect in a same way the motion of particles, and then the particles follow only one world-line called geodesic. Clearly, the observer and a test body have the same motion under the effect of gravitational field[3].

The motion of charged particle is dependent on its $(\frac{q}{m_I})$ term, and therefore for particles with different $(\frac{q}{m_I})$ terms it have a different world-lines. The background observer who is measuring the electromagnetic field is not a subject of that field [28]. In fact, this inertial observer follow a geodesic motion and show that the word-line of charged particle is just a deviation from its inertial motion.

The observer cannot measure the effect of gravitational field and we cannot isolate the observer from the gravity effect. Consequently, the gravity is not a force. Perfectly, the gravity is a geometrical phenomenon and its effect is given by the curvature of the spacetime metric while the observer follows the geodesic line of this metric (for more arguments lead to that gravity is geometry see [3]).

2.4 Mathematical tools of general relativity theory

In general relativity theory, the space time is four dimensional Pseudo-Riemannian Manifold [2, 3, 4] and the gravity is just an effect of the active mass on the spacetime geometry.

2.4.1 The metric tensor and the geodesic

The metric

In euclidean geometry, Pythagorean theorem plays a fundamental role. It states that in right triangle the area of the square whose side is the hypotenuse (c) is equal to the area of the squares of the other two sides (a and b) then we get the Pythagorean equation

$$a^2 + b^2 = c^2.$$

The content of the Pythagorean theorem supposed to stay valid in higher dimensions with infinitesimal displacement[29]. Thus, in four dimensional flat spacetime a distance between an event $(x^0,\ x^1,\ x^2,\ x^3)$ and a nearby one $(x^0 + \Delta x^0,\ x^1 + \Delta x^1,\ x^2 + \Delta x^2,\ x^3 + \Delta x^3)$ is given by

$$\begin{aligned}(\Delta s)^2 &= (\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 \\ &= \sum_{\mu,\nu=0}^{3} \delta_{\mu\nu} \Delta x^\mu \Delta x^\nu, \end{aligned} \tag{2.2}$$

then if we put $x^i \to ix^{i=1,2,3}$, we find

[3] Suppose that an observer and a body were stimulated by the effect of a gravitational field, and we suppose that the observer feels the effect of gravity only by the motion of the body relative to him. According to the equivalence principle the gravitational field affect the observer and the body in the same way. Both the observer and the body move under the effect of gravity but the observer cannot feels this effect because, relative to him, the body remains at rest.

$$(\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu. \tag{2.3}$$

Where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is the metric of the flat Minkowski spacetime and it is taken to be a constant. Whereas if we move from a flat to a curved space-time, the equation (2.3) must be generalized to give us a sort of infinitesimal Pythagorean equation in curved space-time, so that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.4}$$

and also in this case $g_{\mu\nu}$ is symmetric and it is not constant. In general relativity, the metric ($g_{\mu\nu}$) characterizes the gravity and its contents are the total properties of the gravitational source. We can deal with the metric as a geometrical description of potentials.

The relation between the metric and the connection was established through the metricity condition

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0, \tag{2.5}$$

where ∇_ρ is the covariant derivative. This operator writing as a partial derivative plus some corrections linear to the connection coefficients. Unlike the partial derivative, the covariant derivative [5] of a vector transforms as a tensor in arbitrary manifold but in Cartesian coordinates reduces to the partial derivative.

The permutations for the three indices in the equation (2.5) give

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}). \tag{2.6}$$

Knowing that this connection is metric compatible due to the metricity condition. This free-torsion connection called Christoffel connection (or symbols). The connection plays in some way a role of field but it is not properly transforms as a tensor.

Geodesics

In flat spacetime, the shortest distance between two points is a straight line segment but this fact is lost when we talking about curved spacetime. More accurately, in arbitrary manifold this definition of straight line is generalized to be a geodesic line (shortest distance in curved spacetime).

The geodesic equation or in other words the equation of motion is derived from least action. Least action means short distance and remembering that we talk about distance above in the part of metric (2.4). So, in a manifold the length between two fixed points λ_1 and λ_2 on a curve $x^\mu(\lambda)$ is given as

$$s = \int_{\lambda_1}^{\lambda_2} ds = \int_{\lambda_1}^{\lambda_2} \frac{ds}{d\lambda} d\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \tag{2.7}$$

Considering the infinitesimal variation in path $x^\mu(\lambda) = x^\mu(\lambda) + \delta x^\mu(\lambda)$, the geodesic requires $\frac{\delta s}{\delta x^\mu(\lambda)} = 0$, and by integration and some calculation [2, 3, 5] one can get

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{2.8}$$

In general relativity theory, the geodesic equations determine the effect of gravitational field on the motion of particles.

2.4.2 Parallel transport

The meaning of parallel transporting a vector is to keep it constant when we displace it from a point to another. This instrument denotes a sense of curvature and makes a difference between curved and flat manifolds. Thus, in order to figurate this sense, we parallel transport a vector on a triangle curve in flat manifold (see Figure 2.1). The transported vector returns to the started point with the same position of the started vector. In counterpart in curved manifold (the spherical one in Figure 2.1) the transported vector on the curved triangle return to the started point with a different direction. In addition, in curved manifold a parallel transport of a vector becomes path dependent.

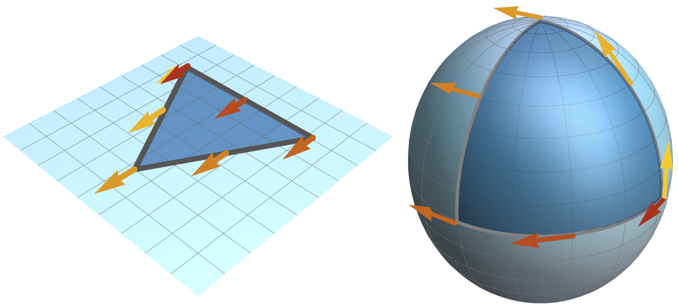

Figure 2.1: Parallel transport of a vector in flat and curved manifold. Source: http://physics.gmu.edu/~isatija/ExoticQW/Week9.pdf.

The partial derivative of a vector is a subtract of this vector in a point $x + dx$ from it in a point x, such as

$$\partial_\nu V^\mu(x) dx^\nu \equiv V^\mu(x + dx) - V^\mu(x). \tag{2.9}$$

With this definition the partial derivative of a vector does not transform as a tensor in arbitrary manifold. For that reason, this definition is not comprehensive, and therefor this definition should be generalized. In order to do that, we recall the parallel transport, and as we know the transported vector on a closed curve in curved manifold does not keep the same position (in the point $x + dx$ this vector becomes $V + \delta V$) . In order to define a covariant derivative which is transform as a tensor, we parallel transporting this quantity $V^\mu(x)$ from x to $x + dx$ and hence we obtain

$$\nabla_\nu V^\mu(x) dx^\nu \equiv V^\mu(x+dx) - [V^\mu(x) + \delta V^\mu(x)] . \tag{2.10}$$

The quantity $\delta V^\mu(x)$ should be proportional to the vector itself and it can be seen as a rotation of this vector, so this quantity is written as

$$\delta V^\mu(x) = -\Gamma^\mu_{\rho\nu}(x) V^\nu(x) dx^\rho, \tag{2.11}$$

then from this equation we can get

$$\nabla_\nu V^\mu(x) = \partial_\nu V^\mu(x) + \Gamma^\mu_{\rho\nu}(x) V^\rho(x). \tag{2.12}$$

The equation of parallel transport of a vector $V^\mu(x)$ is

$$\frac{dV^\mu(x)}{d\lambda} + \Gamma^\mu_{\rho\nu}(x) \frac{dx^\rho}{d\lambda} V^\nu(x) = 0. \tag{2.13}$$

Another equation that can be extracted from the parallel transport definition is the geodesic equation. It can be done if we parallel transporting the tangent vector $\frac{dx^\mu}{d\lambda}$ of a curve $x^\mu(\lambda)$.

Moreover, the most important thing that must arise clearly from the parallel transport is the Riemann tensor. Thenceforward, we parallel transporting a vector V^μ (see Figure 2.2) from a point A towards B along the two curves which pass through the two different points A_1 and A_2.

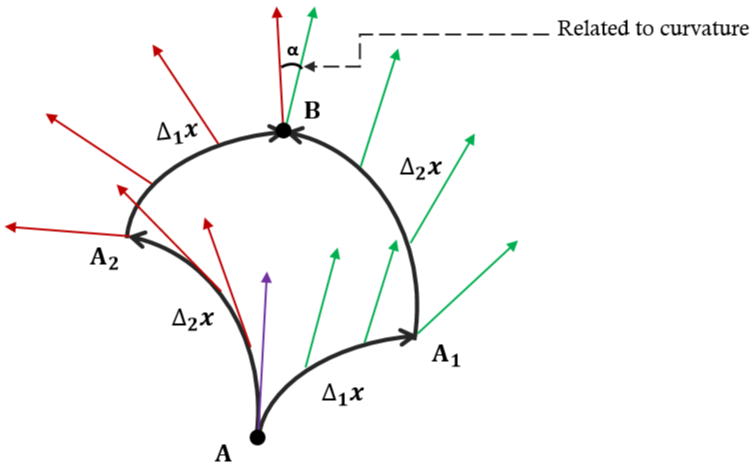

Figure 2.2: Transporting a vector from A to B parallel on the two curve, the first past trough A1 and A2.

The equation of parallel transporting a vector from A to A_1 is given by the following

$$V^\mu_{AA_1} = V^\mu - \Gamma^\mu_{\alpha\beta}(A) V^\alpha \Delta_1 x^\beta, \tag{2.14}$$

then we can consider the following expression to be right

$$\Gamma^{\mu}_{\alpha\beta}(A_1) = \Gamma^{\mu}_{\alpha\beta}(A + \Delta_1 x) \approx \Gamma^{\mu}_{\alpha\beta}(A) + \partial_\lambda \Gamma^{\mu}_{\alpha\beta}(A)\Delta_1 x^\lambda. \tag{2.15}$$

the equation of parallel transporting a vector from A trough $A_1$to B is

$$V^{\mu}_{AA_1B} = V^{\mu}_{AA_1} - \Gamma^{\mu}_{\alpha\beta}(A_1)V^{\alpha}_{AA_1}\Delta_2 x^\beta. \tag{2.16}$$

The replacement of the two equations (2.14) and (2.15) in the last one reach

$$\begin{aligned} V^{\mu}_{AA_1B} \approx & V^\mu - \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_1 x^\beta - \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_2 x^\beta + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\rho\sigma}V^\rho \Delta_1 x^\sigma \Delta_2 x^\beta \\ & -\partial_\lambda \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_1 x^\lambda \Delta_2 x^\beta. \end{aligned} \tag{2.17}$$

We do the same thing along the curve (AA_2B) and we get

$$\begin{aligned} V^{\mu}_{AA_2B} \approx & V^\mu - \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_2 x^\beta - \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_1 x^\beta + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\rho\sigma}V^\rho \Delta_2 x^\sigma \Delta_1 x^\beta \\ & -\partial_\lambda \Gamma^{\mu}_{\alpha\beta}V^\alpha \Delta_2 x^\lambda \Delta_1 x^\beta \end{aligned} \tag{2.18}$$

The subtraction between the last two equations gives

$$\begin{aligned} V^{\mu}_{AA_1B} - V^{\mu}_{AA_2B} &= \left[\partial_\beta \Gamma^{\mu}_{\rho\lambda} - \partial_\lambda \Gamma^{\mu}_{\rho\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\rho\lambda} - \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\alpha}_{\rho\beta}\right] V^\rho \Delta_1 x^\lambda \Delta_2 x^\beta \\ &= R^{\mu}_{\rho\beta\lambda} V^\rho \Delta_1 x^\lambda \Delta_2 x^\beta, \end{aligned} \tag{2.19}$$

where the term $R^{\mu}_{\rho\beta\lambda}$ is the so-called Riemann tensor. The latter can give the Ricci tensor in the case where $R^{\mu}_{\rho\beta\mu} \equiv R_{\rho\beta}$. Hence, the vanishing of all Riemann tensor components in every regions of a manifold is a satisfactory condition for this manifold to be flat.

Through the use of the Bianchi identity and the Riemann tensor properties, we can identify the Einstein's tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R, \tag{2.20}$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar. Thereafter, we can get the Einsteins' equations as

$$G_{\alpha\beta} = -\kappa T_{\alpha\beta}, \tag{2.21}$$

where $\kappa = \frac{8\pi G}{c^4}$, and $T_{\alpha\beta}$ is the stress-energy-momentum tensor which determines the source of gravity. In any case, we can considering the Einsteins' equations as a measurement of gravity source [28]. The equation (2.21) returns to be the Poisson's equation in weak field limit. Outside a source $T_{\alpha\beta} = 0$ and not necessary the gravity effect vanish and therefore the Einstein equation becomes

$$R_{\alpha\beta} = 0. \tag{2.22}$$

The solution of the last equation in the case of spherically symmetric gravitational source is given by the line element

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.23}$$

which represent the Schwarzschild metric.

2.5 Cosmology and cosmological constant

A purely (ordinary) massive universe falls into its center under the effect of its purely gravitational field. Einstein [30] think that the universe is static. For that reason, he introduce a cosmological constant in his equations in order to balance the gravitational effect in a sort of anti-gravity force. Thus, the Einstein's equations have been modified as follows

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = -\kappa T_{\alpha\beta}. \tag{2.24}$$

In cosmology [28, 35, 36, 37], the stress-energy tensor $T_{\alpha\beta}$ contain all types of matter contents (barionic matter, dark matter, radiations..) in our universe, and it is taken to be a stress-energy tensor of perfect fluid

$$T_{\alpha\beta} = \left(\rho + \frac{P}{c^2}\right) U_\alpha U_\beta - P g_{\alpha\beta}. \tag{2.25}$$

However, for a rest fluid $U_\alpha = (c, 0, 0, 0)$ with $U^\alpha U_\alpha = c^2$, the components of this tensor becomes

$$T_{\alpha\beta} = diag(\rho c^2 g_{00}, -P g_{11}, -P g_{22}, -P g_{33}). \tag{2.26}$$

The cosmological principle states that at large scale our universe is homogeneous (the same at every point) and isotropic (the same in all directions)[28, 32]. The isotropy has been confirmed in many experiments on the CMB[4] radiations (WMAP, COBE...). Homogeneity and isotropy are what make (2.26) to become

$$T_{\alpha\beta} = diag(\rho c^4, P, P, P). \tag{2.27}$$

[4]According to the Big Bang cosmological model the early universe seemed very dense with high temperature. This imply a non existence of atoms and all particles take a form of plasma with a thermal photons scatter with the electrons of this plasma. The universe expands and the temperature decreas. This led to a incorporation of an electron and proton to form a hydrogen atom. The thermal photons have a small cross section to scatter with hydrogen atoms and that makes them freely propagate in space formed what is called Cosmic Microwave Background[33]. As a result of these radiation, the sky appears to us black at night.

The other ways to written (2.24)

$$G_{\alpha\beta} = -\kappa \left(T_{\alpha\beta} + T^{\Lambda}_{\alpha\beta}\right) \tag{2.28}$$

$$R_{\alpha\beta} = -\kappa \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta}, \tag{2.29}$$

and thus we have

$$T^{\Lambda}_{\alpha\beta} = \rho_{\Lambda} c^2 g_{\alpha\beta} = \frac{\Lambda}{\kappa} g_{\alpha\beta}. \tag{2.30}$$

This means that the vacuum appearing the same for every observer. Moreover, from the equations (2.26), (2.27) and (2.30), we show that the contribution of the cosmological constant can be considered as a type of exotic energy has a negative pressure $P = -\rho_{\Lambda} c^2$. This energy called dark energy, the term "dark" because there is no direct detection of its effect. Furthermore, the equations (2.28) and (2.29) say that the empty spacetime (in the absence of matter $T_{\alpha\beta} = T = 0$ or any source of gravity) is not flat, indeed it is curved by a kind of energy ($\rho_{vac} = \frac{\Lambda c^2}{8\pi G} \neq 0$). Implying that the density must be $\rho \to \rho + \rho_{\Lambda}$and thus the Poison's equation can be modified as

$$\begin{aligned} \Delta \Phi &= 4\pi G \left(\rho + \rho_{\Lambda}\right) \\ &= 4\pi G \rho + \frac{\Lambda}{2} c^2 \end{aligned} \tag{2.31}$$

Looking to the sky, observations prove the opposite to Einstein's thought, and tell us that our universe is not static but expanding. Therefore, Einstein's belief seems to be wrong, but the form of the equations stays the same to explain the expansion of the universe (dark energy effect).

In 1929 Hubble showed with an astonishing observation that galaxies are moving away from each other [31]. This what makes us view our neighboring galaxies receding from us. Hubble measured the velocity υ of receding galaxies via its redshift (Doppler effect), and he find that

$$\upsilon = H_0 D. \tag{2.32}$$

This expression, linearly, relates the velocity recession υ of galaxies with its distance D from us, where H_0 is the Hubble constant at the present time. In addition, there are the type Ia supernovae [5] and CMB (anisotropies and polarization) observations support the expansion of the universe (for more information see [36]).

[5]In a binary system the white dwarf star ,according to its huge density, it can accrete the mass from its companion (red giant star). The mass of the white dwarf exceeds the Chandrasekhar limit (1.4 solar masses) and therefore the electron degeneracy pressure cannot hold out the gravitational force [28, 34]. The core of the white dwarf is unable to resist, and thus explodes with high brightness in the form of a supernova. Measuring the brightness via redshift enables to determine the density of dark energy and the other types of matter.

2.5.1 Friedmann-Robertson-Walker geometry

A misunderstanding arises if we really feel that the galaxies recede from us but, in fact, the space itself increases between galaxies. Therefore, all of the observers in different galaxies view us recede from them in the same way, and this completely agrees with what the cosmological principle states. The locations of galaxies are embedded into the space. The proper distance D between them is determined with respect to its comoving distance x (distance at rest)

$$D = a(t_0)x \tag{2.33}$$

where $a(t_0) \equiv a_0$ is the scale factor ,at the present time, function of cosmic time and thus we have

$$v = \frac{dD}{dt} = \frac{da(t_0)}{dt}x = \frac{\dot{a_0}}{a_0}D. \tag{2.34}$$

We let the Hubble constant to be $H_0 = \frac{\dot{a_0}}{a_0}$ then we recover the equation (2.32).

Making a side by side the cosmological principle and the comoving coordinate definition, one can set the form of the FRW metric [28, 32, 33, 37]

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{1}{1-kr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{2.35}$$

where k can be $0, -1$ or 1 represent a flat, opened or closed universe respectively. By inserting the FRW metric with the stress-energy tensor (2.26) of perfect fluid of homogeneous and isotropic universe in the equation (2.29), we get the well-known Friedmanns' equations [28, 32, 33, 37]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{1}{3}\Lambda c^2 \tag{2.36}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 - c^2k. \tag{2.37}$$

The other remaining equation is taken from the energy conservation equations $\nabla_\mu T^{\mu\nu} = 0$ which give

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \tag{2.38}$$

Besides, we should also show the so-called equation of state

$$P = w\rho c^2. \tag{2.39}$$

From (2.38) and the equation of state we can show that

$$\rho \propto a^{-3(1+w)}. \tag{2.40}$$

The different types of matter in our universe was classified as follows

- The matter or the non-relativistic matter with $w = 0$ includes all material items that have a zero pressure . For instance, the galaxies as well as gas and dust both are in interstellar space in addition to the dark matter. The density of matter is $\rho_m \propto a^{-3}$. Thus the variation of matter density $\rho_m(t)$ with the cosmic time is written terms of the present time density $\rho_{m,0}$ as

$$\rho_m(t) = \rho_{m,0} \left(\frac{a(t_0)}{a(t)} \right)^3 \tag{2.41}$$

- The radiation is the relativistic particle like photons. In the equation of state, this type of matter has $w = \frac{1}{3}$ and thus $\rho_r \propto a^{-4}$. We know that in early universe $a(t)$ is much smaller. This means that the early universe is a radiation-dominated universe.

- The vacuum or the dark energy is an exotic fluid that has a negative pressure with density $\rho_\Lambda \propto a^0$.

An important thing that we have to present here is the dimensionless density parameter of different types of matter

$$\Omega_i = \frac{\rho_i}{\rho_c}, \quad i = m, r, \Lambda. \tag{2.42}$$

Where the critical density $\rho_c = \frac{3H^2}{8\pi G}$ is defined in case of flat universe ($k = 0$) with a zero vacuum energy ($\Lambda = 0$). From the definition (2.42) and the Friedmann's equation (2.37), we get

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k. \tag{2.43}$$

Where $\Omega_k = -\frac{c^2 k}{H^2 R^2}$. However, based on the above results, the variation of Hubble constant can be given with [2]

$$H^2 = H_0^2 \left(\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2} \right) \tag{2.44}$$

The observations show that our universe is flat ($\Omega_k = 0$) with the following suggested present-time values [2]

$$H_0 \approx 70\, km\, s^{-1}\, Mpc^{-1}, \; \Omega_{r,0} \approx 0, \; \Omega_{m,0} \approx 0.3, \; \Omega_{\Lambda,0} \approx 0.7, \tag{2.45}$$

and thus

$$\rho_\Lambda = 5.96\, 10^{-27}\, kg\, m^{-1}, \; \Lambda = 1.11\, 10^{-52}\, m^{-2}. \tag{2.46}$$

2.5.2 Distance and redshift

It is well known that measuring the distances of objects in the whole sky depends on the analysis of the light emitted from these things. Hence, the light traveling in the expanding universe is redshifted. This shift called cosmological redshift (z). Therefore, the world line of a photon is null geodesic ($ds^2 = c^2dt^2 - a^2(t)dr^2 = 0$) if this photon emitted (in flat universe $k = 0$) at the moment $t_e + \delta t_e$ and observed at $t_{ob} + \delta t_{ob}$, it should travel along a distance

$$r = \int_{t_e}^{t_{ob}} \frac{cdt}{a(t)}. \tag{2.47}$$

From here we will have

$$\int_{t_e}^{t_e+\delta t_e} \frac{cdt}{a(t)} = \int_{t_{ob}}^{t_{ob}+\delta t_{ob}} \frac{cdt}{a(t)} \tag{2.48}$$

which gives

$$\frac{\delta t_e}{a(t_e)} = \frac{\delta t_{ob}}{a(t_{ob})}. \tag{2.49}$$

The photon have a frequency $\nu = \frac{1}{\delta t}$, then we are done with

$$1 + \mathrm{z} = \frac{\nu_e}{\nu_{ob}} = \frac{\delta t_{ob}}{\delta t_e} = \frac{a(t_{ob})}{a(t_e)}. \tag{2.50}$$

Thus, we have

$$\frac{d\mathrm{z}}{dt} = -(1 + \mathrm{z})H, \tag{2.51}$$

and by replacing in (2.47) in which $r = \int_t^{t_0} \frac{cdt}{a(t)}$ and hence we find

$$r = \frac{c}{a(t_0)} \int_0^{\mathrm{z}} \frac{d\bar{\mathrm{z}}}{H(\bar{\mathrm{z}})}. \tag{2.52}$$

It is obvious that the distance when the universe was young would be smaller than the distance in the present universe. Thus, the luminosity distance is given with

$$d_L(\mathrm{z}) = (1{+}\mathrm{z})a_0 r = \frac{c}{H_0} \int_0^{\mathrm{z}} \frac{d\bar{\mathrm{z}}}{\sqrt{\left(\Omega_{m,0}\left(1+\bar{\mathrm{z}}\right)^{-3} + \Omega_{r,0}\left(1+\bar{\mathrm{z}}\right)^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}\left(1+\bar{\mathrm{z}}\right)^{-2}\right)}}. \tag{2.53}$$

On the other hand, if the intrinsic luminosity L of an object is known and by measuring its flux of radiation F, we can determine the luminosity distance

$$F = \frac{L}{4\pi d_L^2}. \tag{2.54}$$

The type Ia supernovae are a standard candles (have the same absolute magnitude M). The equation (2.54) can be reformulated as

$$m = M + 5\log_{10}\left(\frac{d_L(\mathrm{z})}{1Mpc}\right) + 25 \tag{2.55}$$

and therefore a plot of magnitude vs redshift ($m-\mathrm{z}$) allows to determine the proportions of the physical components Ω_i of the universe. In addition to assert that our universe is in an accelerated expansion state.

2.6 Other types of geometry

It is known that gravity is well defined by Riemann's geometry. Since this geometry is unable to explain certain physical phenomena such as the beginning of the universe (dark matter and dark energy...), the problem of unification, and quantum gravity theory, so from that time it is indispensable to look for other theories that go beyond Riemann's geometry.

Generally, in geometry the connection is decomposed on the three parts [38, 39, 40] as

$$\overline{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}. \tag{2.56}$$

Where $\Gamma^{\lambda}_{\mu\nu}$ is the Christofel symbol which depends on the metric. The second term is the contortion that is defined by the torsion ($T^{\lambda}_{\mu\nu} = \overline{\Gamma}^{\lambda}_{\mu\nu} - \overline{\Gamma}^{\lambda}_{\nu\mu}$) in which

$$K^{\lambda}_{\mu\nu} = -K_{\nu\mu}^{\ \lambda} = \frac{1}{2}g^{\lambda\rho}(T_{\mu\rho\nu} + T_{\nu\mu\rho} + T_{\rho\nu\mu}). \tag{2.57}$$

Then the last term the disformation which is given through non-metricity ($Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$)

$$L^{\lambda}_{\mu\nu} = L^{\lambda}_{\nu\mu} = \frac{1}{2}g^{\lambda\rho}(-Q_{\mu\rho\nu} - Q_{\nu\mu\rho} + Q_{\rho\nu\mu}). \tag{2.58}$$

The types of geometries are classified according to the components of the connection. The vanishing of non-metricity gives Riemann-Cartan geometry, the vanishing of torsion gives the torsion free geometry, and taking the curvature $R^{\mu}_{\rho\beta\lambda}(\overline{\Gamma})$ to be zero gives teleparallel geometry (the path independent parallel transport). The Riemann geometry is just the vanishing of torsion and non-metricity. Moreover, the teleparallel geometry with zero torsion gives symmetric teleparallel geometry, and so on.

2.6.1 Weyl's theory

In 1918, the German mathematician Hermann Weyl in an unprecedented attempt, proposed a geometric theory [41, 42, 43, 44, 45] to unify general relativity and electromagnetism. To the unification to take place, Weyl argues that Riemann's geometry must be generalized in which the Riemann definition of parallel transport is changed. In addition to the vector changes of its direction, as saw in Riemann's geometry, the parallel transport of a vector from point A to the point B also changes the length of that vector. Therefor, the length of the vector becomes dependent on the path (history) traversed by the vector. The geometric meaning of that is this there is another kind of curvature that should appear side by side to Riemann curvature in the equation (2.19). More accurately, the non-metricity vanish.

In Weyl theory the metricity condition takes the following form

$$\widetilde{\nabla}_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \widetilde{\Gamma}^\beta_{\alpha\mu} g_{\beta\nu} - \widetilde{\Gamma}^\beta_{\alpha\nu} g_{\mu\beta} = 2A_\alpha g_{\mu\nu}. \tag{2.59}$$

In this case the Weyl connection $\widetilde{\Gamma}^\beta_{\alpha\mu}$ is not metric compatible since this connection depends on the gravitational metric $g_{\mu\nu}$ and the recurrence one-form[6] A_α which plays the role of electromagnetic potential. To show this clearly, we permute the three indices in the last equation

$$\begin{array}{c} \partial_\nu g_{\alpha\mu} - \widetilde{\Gamma}^\beta_{\nu\alpha} g_{\beta\mu} - \widetilde{\Gamma}^\beta_{\mu\alpha} g_{\nu\beta} = 2A_\nu g_{\alpha\mu} \\ \partial_\mu g_{\nu\alpha} - \widetilde{\Gamma}^\beta_{\mu\nu} g_{\alpha\beta} - \widetilde{\Gamma}^\beta_{\nu\mu} g_{\alpha\beta} = 2A_\mu g_{\nu\alpha} \end{array}. \tag{2.60}$$

We don't care about the order of indices because both the metric $g_{\mu\nu} = g_{\nu\mu}$ and the connection $\widetilde{\Gamma}^\beta_{\alpha\mu} = \widetilde{\Gamma}^\beta_{\mu\alpha}$ are symmetric. Hence, the subtraction of the equation (2.59) from the sum of the equation (2.60) gives

$$\begin{aligned} \widetilde{\Gamma}^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) + g^{\rho\alpha}(A_\alpha g_{\mu\nu} - A_\mu g_{\nu\alpha} - A_\nu g_{\alpha\mu}) \\ &= \Gamma^\rho_{\mu\nu} + g^{\rho\alpha}(A_\alpha g_{\mu\nu} - A_\mu g_{\nu\alpha} - A_\nu g_{\alpha\mu}) \qquad , \end{aligned} \tag{2.61}$$

where $\Gamma^\rho_{\mu\nu}$ is Christoffel symbols of Riemann geometry and we note that in the case $A_\alpha = 0$ the Weyl's theory return to be general relativity.

The important thing in Weyl's theory is that Weyl connection and consequently the curvature tensor are invariant under the following transformations

$$\begin{array}{c} \bar{g} = e^f g \\ \bar{A} = A + df \end{array}, \tag{2.62}$$

where f is some scalar function. These transformations represent a new concept of gauge theory. For that reason Weyl believe that his theory is so effective.

The equations of motion are derived from the action [46]

$$S = \int d^4x \sqrt{-g}(R^2 + \kappa F_{\mu\nu} F^{\mu\nu}), \tag{2.63}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and g is the determinant of the metric. Thence, the variation of the action with respect to the metric tensor and the potential vector respectively gives a type of Einstein-Maxwell equations

$$R\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = -\kappa T_{\mu\nu} \tag{2.64}$$

[6]For reasons of the units we can make $A_\alpha \equiv \epsilon A_\alpha$, where ϵ is taken to be equal to 1.

and a type of Maxwells' equations in curved spacetime

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = -\frac{3}{2}g^{\mu\nu}(A_\mu R + \partial_\mu R), \tag{2.65}$$

where $T_{\mu\nu} = -\frac{1}{\mu_0}\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$ is the energy-momentum tensor of electromagnetic field.

The physical problem in Weyl's theory is the dependent of the length of the vector V^μ on its history so that we have

$$L^2 = g_{\mu\nu}V^\mu V^\nu \tag{2.66}$$

and then

$$2LdL = \partial_\alpha g_{\mu\nu}dx^\alpha V^\mu V^\nu + g_{\mu\nu}dV^\mu V^\nu + g_{\mu\nu}V^\mu dV^\nu$$

and by using the equation (2.11) we get

$$2LdL = \widetilde{\nabla}_\alpha g_{\mu\nu}V^\mu V^\nu = 2A_\alpha g_{\mu\nu}V^\mu V^\nu dx^\alpha = 2A_\alpha L^2 dx^\alpha$$

after integration we obtain

$$L = L_0 e^{\int A_\alpha dx^\alpha}, \tag{2.67}$$

where L_0 is the initial length of the vector.

Therefore, Einstein did not accept Weyl's theory because it contradicts with the physical observations, and it gives an arbitrarily physical quantities [47] . The problem comes from non-invariant length of the vector dx^μ and its derivatives the velocity $u^\mu = \frac{dx^\mu}{ds}$, momentum p^μ, and the proper time. Consequently, the mass of particles and the "atomic clocks"[7] will become related to its history and this is something impractical.

Weyl's theory is dead and the rest are some attempts to revive this theory such as Dirac and Eddington extended theories and some recent researches [48, 49] . The important theories that try to save Weyl's theory are those involving quantum gravity and cosmology.

Another attempt to unify general relativity and electromagnetism came with Schrödinger's theory [8, 47]. Besides to the vanishing of the non-metricity, Schrödinger added a cyclic condition to the metric and the most important result is the relation of the potential

$$A_\mu = \frac{q}{mc}g_{\mu\nu}\frac{dx^\nu}{ds}. \tag{2.68}$$

The problem in this relation is that the physical meaning is not well defined, and we can see it as an accidental, also the theory is not invariant under the transformations (2.62).

[7]The atomic clock is the number of oscillations of the atom per unit of time. The caesium atom, for example, has about nine billion vibration per second.

2.6.2 Kaluza-Klein theory

Kaluza's contribution was the extension of general relativity in five dimensions in order to let the four dimensional Einstein's theory cover the electromagnetism [9]. Thence, The five dimensional metric has been computed with the two conditions[51]: i) The cylinder condition in which all of the metric components are written independently from the fifth dimension. ii) The fifth dimension is orthogonal on the four dimensional spacetime after we get the form of line element

$$ds^2 = \left(g_{\mu\nu} - \lambda^2\phi^2 A_\mu A_\nu\right) dx^\mu dx^\nu - \lambda\phi^2 A_\mu dx^\mu dy - \lambda\phi^2 A_\nu dy dx^\nu - \phi^2 dy^2, \tag{2.69}$$

and hence the metric

$$\gamma_{AB} = \begin{pmatrix} g_{\mu\nu} - \lambda^2\phi^2 A_\mu A_\nu & -\lambda\phi^2 A_\mu \\ -\lambda\phi^2 A_\nu & -\phi^2 \end{pmatrix} \tag{2.70}$$

and its inverse matrix

$$\gamma^{AB} = \begin{pmatrix} g^{\mu\nu} & -\lambda A^\nu \\ -\lambda A^\mu & -\frac{1}{\phi^2} + \lambda^2 A^\mu A_\mu \end{pmatrix} A = 0...4, \mu = 0...3. \tag{2.71}$$

Where $\lambda = \sqrt{2\kappa}$ is some constant and ϕ is a scalar function and with the indices ($A = 0...4, \mu = 0...3$).

It can be assumed that there is no received energy from the fifth dimension or, in other words, the extra-dimensions decrease the energy. This means the vanishing of Einstein's tensor in five dimensions $G_{AB} = 0$ or identically $R_{AB} = 0$ and that give, in the case where $\phi = 1$, the Einstein-Maxwell and Maxwell equations [50, 52]

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \tag{2.72}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{2.73}$$

The equation of motion is just a five dimensional geodesic equation

$$\frac{d^2x^A}{ds^2} + \Gamma^A_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0, \tag{2.74}$$

which gives

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = \lambda\frac{dy}{ds}g^{\mu\rho}F_{\nu\rho}\frac{dx^\nu}{ds}, \tag{2.75}$$

and if we put $\lambda\frac{dy}{ds} = \frac{q}{m}$, we get the equation of motion in curved spacetime of charged particle in electromagnetic field.

In Kaluza-Klein theory, the vector $A_\mu$transforms as an abelian gauge vector. To show that, we just applying the following transformations

$$y \to \acute{y} = y + \lambda\varepsilon(x) \tag{2.76}$$

on the equation $g_{\mu 4} = \lambda A_\mu g_{44}$ to getting

$$g_{\mu 4}(x, y) = \acute{g}_{\mu 4}(x, \acute{y}(y, x))\frac{\partial \acute{x}^\mu}{\partial x^A}\frac{\partial \acute{y}}{\partial x^B}$$

$$\acute{A}_\mu = A_\mu + \partial_\mu \varepsilon(x) \tag{2.77}$$

In order to improve the theory and explain the invisibility of the fifth dimension as well as the cylinder condition, Klein [10] assumed a compactified dimension where it is taken to be a circle with smaller radius r. Therefore, all of the coordinates become periodic $\psi(x^\mu, y) = \psi(x^\mu, y + 2\pi)$ and then the wave field becomes Fourier expansion

$$\psi(x, y) = \sum_{-\infty}^{+\infty} \psi^{(n)}(x) e^{i\frac{n}{r}y}. \tag{2.78}$$

By assuming that the fifth dimension is length-like, we can get the Klein-Gordon five dimensional equation of that massless field

$$\left(\Box_x - \frac{1}{r^2}\frac{\partial^2}{\partial y^2}\right)\psi(x, y) = 0, \tag{2.79}$$

where $\Box_x = g^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembert operator. By replacing the equation (2.78) in the last one we get

$$\left(\Box_x + m_n^2\right)\psi^{(n)}(x) = 0. \tag{2.80}$$

We can show that a massless field in five dimensions is just a four dimensional field carries a mass $m_n = \frac{n}{r}$.

More than that the Fourier mode it can be carry a quantized charge, and this occur if we apply the transformations (2.76) in which

$$\psi^{(n)}(x) \to e^{i\frac{n\lambda}{r}\varepsilon(x)}\psi^{(n)}(x).$$

Inserting the transformed mode in the equation (2.80) and by comparing to the covariant derivative in $U(1)$ local transformations $(\partial_\mu + iqA_\mu)$ we obtain

$$q_n = n\frac{\lambda}{r} \tag{2.81}$$

With this result, Klein attempt to explain the proportionality of all particles charges with the electronic charge e.

Even so, the theory remains incomplete and suffers from several problems like the non-included strong and weak interactions. In addition, it is incompatible with experiments and the quantized charge and mass stay invalid. Moreover, the existence of fifth dimension and its effects are undetectable. In another way, theories more than five dimensions, for example super gravity, string and, super string theories have been, theoretically, accepted but are not yet experimentally confirmed.

2.7 Conclusion

In this chapter, we hastily mentioned the story of spacetime starting with an absolute space and time arriving at a dynamical (curved) relativistic spacetime with extra dimensions. In addition, we showed that the most developments in physics are directly related to our view of the spacetime concept.

The story, here, is just a story of spacetime nothing more nothing less and then if we develop, and in the right way, our view of the concept of spacetime, we will inevitably reach the right and comprehensive physics.

Chapter 3

On the analogy between gravitation and electromagnetism

3.1 Introduction

In the previous chapter, we saw that it is very difficult to find a single geometric theory that matches the two theories, general relativity and electromagnetism. Where we also deduced that the deviation from Riemann's geometry creates problems and puts us in front of non-physical meaning of the physical grandeur. On the other hand, the further problem that faces us is the absence of experimental confirmation on theories with extra dimensions. For these reasons, we will try to keep Riemann's geometry, and in opposite reasoning we will not thinking about a single theory based on a gauge gravity theory. Contrary to that, we think towards an established metric theory for electromagnetism. In addition, we focus on a geometrical theory for electromagnetism and not for the unification.

For this purpose, we can pay attention to Einstein's saying [53] *"The idea that there are two structures of space independent of each other, the metric-gravitational and the electromagnetic, is intolerable to the theoretical spirit"*. Thence, it is not seems forbidden for someone assume that gravity and electromagnetism are two different universes and they are not necessarily independent. The two interactions defined in an equivalent geometrical way (for a similar idea see [18, 20, 54]), and that means, again, the similarity of nature. The authors in [54] used the concept "proper space-time" rather than the term "electromagnetic universe".

We should taking a note that the observer is a subject of gravity universe and not a subject of electromagnetism. Because in this case it is not affected by particle properties like charge, mass, spin.... For that reason, he does not see charged particles falling free under the effect of electromagnetism. On the other hand, if this observer is attached to a charged particle which has a ratio $\frac{q}{m}$ equal to $\frac{q}{m}$ of some other charged particles, then he cannot distinguish (locally) between the effect of the electromagnetic field for him and its effect on that charged particles.

In this chapter, we first showing up the analogy between gravity and electromagnetic universes in weak field limit and we shed light, quickly, on the ideas of Barros approach. From now and on we will use the term $\varrho = \frac{q}{m}$ denotes charge to mass ratio of charged test particle.

3.2 The weak field limit in gravity universe

There are many versions [55, 56, 57, 58, 59, 60, 61, 62, 63] to linearize Einsteins' equations, and that according to the suggested gauge conditions and the components of the metric. At first, the similarity appeared between Coulomb's and Newton's laws. After that, this similarity has been shown in linearized general relativity theory when the weak gravitational field can split in an electrogravitational and magnetogravitational fields [57, 64, 65]. Moreover, and as known in Maxwell's theory, the movement of charged particles creates a magnetic field also the same thing happens in gravity so that the movement of matter creates a magnetogravitational field [58, 59]. We can take the proper rotation of the sun as an example to this field.

3.2.1 Stationary case

In this section, we follow [2] and we will be interested in the stationary case when all the metric components are time independent. The Minkowski spacetime is perturbed by a weak gravitational field and then the metric becomes written on the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ |h_{\mu\nu}| \ll 1. \tag{3.1}$$

In the first order approximation and by using the following transformation (trace reverse of $h_{\mu\nu}$)

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{3.2}$$

where $h = h^{\alpha}_{\alpha} = -\bar{h}$, and by taking into account the harmonic de Donder gauge condition

$$\partial_\alpha \bar{h}^{\mu\alpha} = 0, \tag{3.3}$$

thus, the Einsteins' equations

$$G^{\mu\nu} = -\frac{8\pi G}{c_g^4}T^{\mu\nu}$$

$$\frac{1}{2}\left(\partial_\mu\partial_\nu h + \Box h_{\mu\nu} - \partial_\mu\partial_\rho h^{\rho}_{\nu} - \partial_\nu\partial_\rho h^{\rho}_{\mu} - \eta_{\mu\nu}(\Box h - \partial_\sigma\partial_\rho h^{\sigma\rho})\right) = -\frac{8\pi G}{c_g^4}, \tag{3.4}$$

become simply

$$\Box\bar{h}^{\mu\nu} = -\frac{16\pi G}{c_g^4}T^{\mu\nu}. \tag{3.5}$$

Where c_g is the propagation velocity of the gravitational waves, $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian operator, and $T_{\mu\nu}$ is chosen to be

$$T^{00} = \rho c_g^2, \quad T^{0i} = c_g \rho u^i, \quad T^{ij} = \rho u^i u^j \approx 0, \quad i,j = 1...3. \tag{3.6}$$

It should be noted that the minus sign [2] placed in the tensor is only to indicate that in gravity universe there is solely one type of mass and therefore gravity is attractive and not repulsive.

The solution of (3.5) is given by

$$\bar{h}^{00} = \frac{4A^0}{c_g} = \frac{4\Phi}{c_g^2}, \quad \bar{h}^{0i} = \frac{A^i}{c_g}, \quad \bar{h}^{ij} = 0. \tag{3.7}$$

Where A^μ, in this case, represents the four vector gravitational potential. Hence, the equation (3.2) implies

$$h^{00} = h^{ii} = \frac{2A^0}{c_g} = \frac{2\Phi}{c_g^2}, \quad h^{0i} = \frac{A^i}{c_g}. \tag{3.8}$$

meaning that

$$ds^2 = (1 + \frac{2\Phi}{c_g^2})c_g^2 dt^2 + 2A_i dt dx^i - (1 - \frac{2\Phi}{c_g^2})\delta_{ij} dx^i dx^j. \tag{3.9}$$

By substitution of the equations (3.7) and (3.6) in (3.5), we obtain the Maxwells' equations of gravitational fields

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \quad \textit{and} \quad \nabla^2 \vec{A} = -\mu_0 \vec{j}, \tag{3.10}$$

with the usual definition $\vec{E_g} = -\vec{\nabla}\Phi$ and $\vec{B_g} = \vec{\nabla} \times \vec{A}$ we get

$$\vec{\nabla}.\vec{E_g} = \tfrac{\rho}{\varepsilon_0}, \qquad \vec{\nabla}.\vec{B_g} = 0$$

$$\vec{\nabla}\times\vec{E_g} = 0, \qquad \vec{\nabla}\times\vec{B_g} = \mu_0 \vec{j}$$

with $\mu_0 = -\frac{16\pi G}{c_g^2}$ and $\varepsilon_0 = -\frac{1}{4\pi G}$.

In addition, the geodesic equations in slow motion cases give us a form of Lorentz's gravitational force

$$\frac{d^2 x^i}{dt^2} \approx -\left(c^2 \Gamma^i_{00} + 2c\Gamma^i_{0j}\nu^j\right) = \vec{E_g} + \vec{\nu} \times \vec{B_g}. \tag{3.11}$$

It is worth noting that the effect of the gravitational magnetic part can be neglected. This is can be explored in the case where the rest energy of the matter distribution in the energy-momentum tensor (see §17 of [2]) becomes the dominated part and, thus, the equation (3.9) becomes

$$ds^2 = (1 + \frac{2\Phi}{c_g^2})c_g^2 dt^2 - (1 - \frac{2\Phi}{c_g^2})\delta_{ij} dx^i dx^j. \tag{3.12}$$

Roughly speaking, we can put that $\mu_0\varepsilon_0 = \frac{4}{c_g^2}$. The problem, here, is the four factor and we can solve this problem with a simple change $\bar{h}^{0i} = h^{0i} = \frac{4A^i}{c_g}$ but this factor [22] return and appears again in the magnetic term of the gravitational Lorentz's force.

Furthermore, an additional magnetic term in Newton's law plays an important role in cosmology, and it may also be able to give an explanation to dark matter and precise the motion of planets without the need of relativistic theory [64].

It is important to showing that the terms $h^{0i} = \frac{A^i}{c_g}$ denote a rotating spacetime and in the same time are the responsible of the magnetic part. Thus, the rotation is equivalent to magnetism. This equivalence between rotation and magnetism has been established by Larmor [66, 59] via the relation $\omega = \frac{qB}{2mc}$, where ω is the angular velocity.

3.2.2 Non-stationary case

We follow [63], and for more details one can see [55, 56]. The idea of the author in [63] is to introduce a generalized tensor, analogous to the electromagnetic tensor

$$\mathcal{G}^{\mu\nu\lambda} = \frac{1}{4}\left(\partial^\lambda \bar{h}^{\mu\nu} - \partial^\nu \bar{h}^{\mu\lambda} + \eta^{\mu\nu}\partial_\alpha \bar{h}^{\lambda\alpha} - \eta^{\mu\alpha}\partial_\alpha \bar{h}^{\nu\lambda}\right). \tag{3.13}$$

The linearized harmonic gauge condition (3.3), leads to

$$\mathcal{G}^{\mu\nu\lambda} = \frac{1}{4}\left(\partial^\lambda \bar{h}^{\mu\nu} - \partial^\nu \bar{h}^{\mu\lambda}\right), \tag{3.14}$$

and we can note that this tensor is antisymmetric in the two last indices, and this lead to the following cyclic properties

$$G^{\mu\nu\lambda} + G^{\lambda\mu\nu} + G^{\nu\lambda\mu} = 0 \tag{3.15}$$

$$\partial^\lambda G^{\rho\mu\nu} + \partial^\nu G^{\rho\lambda\mu} + \partial^\mu G^{\rho\nu\lambda} = 0. \tag{3.16}$$

Hence, through the use of the harmonic gauge condition (3.3) and the metric trace reverse (3.2) both of them in equation (3.4) they lead to the linearized Einsteins' equations to be on this form

$$\partial_\lambda \mathcal{G}^{\mu\nu\lambda} = -\frac{16\pi G}{c_g^4} T^{\mu\nu}. \tag{3.17}$$

By putting $h^{0\mu} = \frac{4A^\mu}{c_g}$, the gravitational Maxwells' equations can be extracted from the (0-0) and (0-i) components of the equation (3.17)

$$\vec{\nabla}.\vec{E}_g = \vec{\nabla}\left(-\vec{\nabla}\Phi - \frac{1}{c_g}\frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{B}_g = \mu_0 \vec{j} + \mu_0\varepsilon_0 \frac{\partial \vec{E}_g}{\partial t}. \tag{3.18}$$

Besides, the cyclic condition (3.16) gives

$$\vec{\nabla}\times\vec{E}_g = -\frac{\partial \vec{B}_g}{\partial t}, \quad \vec{\nabla}.\vec{B}_g = 0. \tag{3.19}$$

In this case $\mu_0 = -\frac{4\pi G}{c_g^2}$ and $\varepsilon_0 = -\frac{1}{4\pi G}$ and they lead to $\varepsilon_0\mu_0 = -\frac{1}{c_g^2}$ but the problem, here, is the four factor in magnetic part of Lorentz's gravitational force in (3.11). The other problem is the dependent of the fields definitions on the condition (3.3). To deal with those problems, the authors in [22] and [55] proposed another approach based on the dependence of the field directly with h and not $\bar{h}$. Moreover, a subtle gauge has been used. In fact, the three spacial components ($\nu = 1, 2, 3$) of the harmonic condition (3.3) has been preserved. While the zero component ($\partial_\alpha h^{\alpha 0} - \frac{1}{2}\eta^{\alpha 0}\partial_\alpha h = 0$) has been replaced instead by the alternative condition $h_i^i = 0$. Hence, the components of the metric are given as

$$h^{00} = \frac{2A^0}{c_g}, \quad h^{0i} = \frac{A^i}{c_g}, \tag{3.20}$$

and with this form of the metric, the linearized Einsteins' equations become

$$\frac{1}{2c_g}\partial_\mu F^{\mu\nu} = -\frac{8\pi G}{c_g^4}T^{0\nu}. \tag{3.21}$$

Therefore, the Maxwells' equations for gravity were obtained based on these Einsteins' equations. Additionally, the geodesic equation gave a Lorentz's force without the four factor [22].

3.2.3 Weak gravitational field with cosmological constant

At yet the role of cosmological constant and its effects remain strange and unclear. This constant may refer to an additional gravitational field comes from vacuum (anti-gravity force), perhaps it is an extra energy or it is just another type of matter. More than that, we are not sure that this constant can be considered as a first order weak gravitational field. It can be showed as a second order approximations.

If we deal with this constant, on the basis, as a form of energy arises from vacuum, in this case we will be unable to set the correct formula for the condition (3.3). Does this constant change the condition or not?

Despite that, we will find in the reference [67] a use of the following coordinate condition

$$\partial_\alpha \bar{h}_\mu^\alpha = \partial_\alpha \left(h_\mu^\alpha - \frac{1}{2}\delta_\mu^\alpha h \right) = \Lambda \eta_{\mu\alpha} x^\alpha \tag{3.22}$$

we can see this condition as a breakdown of gauge invariance. Accordingly, the Ricci tensor can be written as

$$\begin{aligned}
R_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\nu h + \Box h_{\mu\nu} - \partial_\mu\partial_\rho h^\rho_\nu - \partial_\nu\partial_\rho h^\rho_\mu\right)\\
&= \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\rho\left(h^\rho_\nu - \frac{1}{2}\delta^\rho_\nu h\right) - \frac{1}{2}\partial_\nu\partial_\rho\left(h^\rho_\mu - \frac{1}{2}\delta^\rho_\mu h\right)\\
&= \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\Lambda\eta_{\nu\alpha}\delta^\alpha_\mu - \frac{1}{2}\Lambda\eta_{\mu\alpha}\delta^\alpha_\nu\\
&= \frac{1}{2}\Box h_{\mu\nu} - \Lambda\eta_{\mu\nu}.
\end{aligned} \tag{3.23}$$

However, the Einsteins' equations are taken to be

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{3.24}$$

and we should note that the author in [67] used Einsteins' equations with negative sign in the term of the cosmological constant. If we multiply by $g^{\mu\nu}$ then we get

$$R = \kappa T - 4\Lambda \tag{3.25}$$

we replace the last equation in (3.24) to obtain

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} + \frac{1}{2}\kappa T g_{\mu\nu}. \tag{3.26}$$

The substitution of (3.23) in (3.26) lead to

$$\Box h_{\mu\nu} + 2\Lambda h_{\mu\nu} = -2\kappa T_{\mu\nu} + \kappa T g_{\mu\nu} \tag{3.27}$$

without a source of matter we have $T_{\mu\nu} = T = 0$ and this means [67]

$$\Box h_{\mu\nu} + 2\Lambda h_{\mu\nu} = 0. \tag{3.28}$$

The treatment of this equation in a stationary case leads to a modified Maxwells' gravitational equations by the cosmological constant

$$\begin{aligned}
\vec{\nabla}.\vec{E}_g &= -2\Lambda\Phi\\
\vec{\nabla}\times\vec{B}_g &= -2\Lambda\vec{A}
\end{aligned} \quad . \tag{3.29}$$

Whereas, if we work in the non-stationary case, we can find

$$\Box\Phi + 2\Lambda\Phi = 0 \tag{3.30}$$

and this equation can be considered as a Klein-Gordon equation of graviton with a mass $m_g = \frac{\hbar}{c}\sqrt{2\Lambda}$. Thus, the propagation velocity of the gravitational field, or in other words the graviton speed is not exactly equal to the speed of light, because its energy depends on the cosmological constant [67]. There are many research topics on the massive gravity and massive graviton and for more knowledge see [68, 69, 70, 71, 72, 73]. Moreover, by comparing the equation (3.30) with the equation (2.80) from the Kaluza-Klein theory, the cosmological term can be shown ,as Wesson suggested, as a matter comes from the fifth dimension [52].

3.3 The weak field limit in electromagnetic Universe

3.3.1 The weak field limit without electromagnetic lambda term

In the electromagnetic universe, charged particles live and exist, and the term $\varrho = \frac{q}{m}$ of those particles it can be the responsible of the existence of this universe. For a deeper thinking, we should go back to early universe and ask this question: what is the origin of charge? There are many different explications to the origin of charge. One can see the charge as a topology or some physical aspect of spacetime [74, 75]. Even today, we only know the charge from the electromagnetic phenomena, but we do not really know its true origin.

From Newton's dynamics, we can establish the equation of motion of a charged particle which moves under the effect of uniform electric field, as follows

$$a^i = \frac{d^2x^i}{dt^2} = \varrho E^i \tag{3.31}$$

and then

$$x^i(t) = \frac{1}{2}\varrho E^i t^2. \tag{3.32}$$

We can make this coordinate transformation $\acute{x} = x - \frac{1}{2}\varrho E t^2$ to cancel the effect of the field. To make this physically available, we have to admit that all of the points in the space in the new coordinates must be indicated by ϱ term. This it means that the ϱ term becomes dependent on spacetime geometry and thus we can naively show that

$$\begin{aligned} x &\to \tfrac{1}{\varrho}\bar{x} \\ t &\to \tfrac{1}{\varrho}\bar{t} \end{aligned} \tag{3.33}$$

and with this simple transformations, the equation (3.32) becomes

$$\bar{x}^i(\bar{t}) = \frac{1}{2}E^i\bar{t}^2. \tag{3.34}$$

The problem of dimensions in (3.33) can be solved if we assume in the electromagnetic universe that the charge and mass have the same dimensions.The other assumption to solve this problem is that $[\bar{x}] = \frac{[C]}{[Kg]}[x]$ and we do the same thing with $\bar{t}$. More easily we can do this $x \to \frac{\zeta}{\varrho}\bar{x}$, where $\zeta = 1Kg/C$. We should dealing with (3.33) as a transitions from gravity universe to elecromagnetic and not as a set of coordinate transfomations.

Therefore, and by following the same line of thought, in electromagnetic universe and if we have passed from the gravity to the electromagnetic universes in weak field limit the metric h has to transform as

$$h_{\mu\nu} \to \varrho h_{\mu\nu} \tag{3.35}$$

and also the energy-momentum tensor [76, 22, 55]

$$-T_{\mu\nu} \to \varrho T_{\mu\nu}. \tag{3.36}$$

We eliminate the minus sign because in the electromagnetic universe, we have two types of charged particles and therefore we must have the two effects attractive and repulsive. Moreover, from the comparison between Newton's and Coulomb's laws, we should replace the constant G by K. For those reasons, in electromagnetic universe the Einsteins' equations have to write on this form [76, 22, 55, 54]

$$G_{\mu\nu} = \chi_e \varrho T_{\mu\nu}. \tag{3.37}$$

Where χ_e is the coupling constant in electromagnetic universe. According to the authors in [22] the linear approximation is sufficient to describe the electromagnetism.

In this case, we will be interested in studying the non-stationary case and of course by following the same steps in [22]. By applying the transformation (3.35) in the equation (3.37) and thus we will show that the equation (3.21) becomes at first order in the following form

$$\frac{\varrho}{2c}\partial_\mu F^{\mu\nu} \approx \chi_e \varrho T^{0\nu}. \tag{3.38}$$

In the linear regime, $T^{0\nu} \sim j^\nu$ which lead the above equation to be

$$\partial_\mu F^{\mu\nu} = 2c^2 \chi_e j^\nu. \tag{3.39}$$

As known the usual Maxwells' equations are $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$ and thus leading to $\chi_e = \frac{2\pi K}{c^4}$. More than that, the geodesic equations in first order approximations and in slow motion give us the electromagnetic Lorentz's force as we can see here

$$\frac{d^2x^i}{dt^2} \approx -\left(c^2\Gamma^i_{00} + 2c\Gamma^i_{0j}\nu^j\right) = \varrho\left(E + \vec{\nu} \times \vec{B}\right)^i. \tag{3.40}$$

It is more interest to note, from the equation (3.37), that the curvature of spacetime is the result of the interaction of the particle's charge with the field, and for this we see that the metric formula is related to the charge ϱ of that particle. Otherwise, we can see in [77] that, in the electromagnetic universe, the metric should be independent from the particle properties, but besides this a modified formula for the geodesic equations must be used as follows

$$\frac{d^2x^\mu}{d\tau^2} + \varrho\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{3.41}$$

Thus, particles with different ϱ terms they will take a different geodesic lines. Furthermore, the equation (3.39) clearly means that in electromagnetic universe and similar to gravity, the electromagnetism is geometry.

3.3.2 The weak field limit with electromagnetic lambda term Λ_e

In a similar way in gravity universe, we assume the existence of a constant we call it the electromagnetic lambda Λ_e. This constant maybe play the same role of the cosmological constant maybe it is a type of energy or matter (we will discuss the role of that constant precisely in the next chapter). Therefore, the Einsteins' equations become

$$G_{\mu\nu} - \Lambda_e g_{\mu\nu} = \frac{2\pi K}{c^4} \varrho T_{\mu\nu}. \tag{3.42}$$

By applying on this equation an approximation of the first order outside a source, and by using the condition (3.22), we will obtain in stationary case a modified form of Maxwells' equations

$$\begin{aligned} \vec{\nabla}.\vec{E} &= -2\Lambda_e \varPhi \\ \vec{\nabla}\times\vec{B} &= -2\Lambda_e \vec{A} \end{aligned} . \tag{3.43}$$

As we saw in the gravity universe, here, also we can see that the photon carries a mass related to the electromagnetic lambda constant, and we set out the formula of the photon mass as it follows

$$m_\gamma = \frac{\hbar}{c}\sqrt{2\Lambda_e}. \tag{3.44}$$

The possibility of obtaining a massive photon has been proposed by many researchers [68, 78, 79, 80, 81, 82] but, in fact, the search of massive vector field was suggested at first by Proca [78]. After that, Stueckelberg [79, 80] introduced a new scalar field in order to generate a mass to the Abelian gauge theory. Interestingly enough, and as shown in [82] the mass of the photon is a consequence of dark energy.

Additionally, in the case where $\Lambda \neq \Lambda_e$ then the velocity of gravitational waves propagation is not the same for the electromagnetism. It should be noted that there are new ideas and results in this chapter that have not yet been published.

3.4 The analogy between interactions in Barros approach

3.4.1 Barros approach

C. C. Barros in an unexpected idea [18, 20, 54], based on general relativity theory, he attempts to describe the non-gravitational interactions and similarly to gravity, in a geometrical framework (we shall focus here on the method of finding the metric). Accordingly, the effects of non-gravitational potentials should manifest through the spacetime metric. The idea is based on the dynamics of Schwarzschild, and by starting from the metric form

$$ds^2 = \varepsilon(r)c^2dt^2 - \varepsilon(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.45}$$

and it can be easy to find

$$d\tau = \frac{1}{c}\sqrt{ds^2} = dt\sqrt{\varepsilon(r) - \varepsilon(r)^{-1}\beta_r^2 - r^2(\beta_\theta^2 + \sin^2(\theta)\beta_\phi^2)} = \frac{dt}{\gamma}, \tag{3.46}$$

where $\beta = \frac{dx^i}{dt}$. By the use of the energy-momentum vector definition, one can show that (for more details see [55])

$$p_0 = g_{00}p^0 = \frac{E}{c} = \varepsilon(r)mc\frac{dt}{d\tau}$$
$$E = \varepsilon(r)mc^2\gamma. \tag{3.47}$$

If $\beta = 0$ then

$$E = \frac{\varepsilon(r)mc^2}{\sqrt{\varepsilon(r)}} = mc^2 + U(r) \tag{3.48}$$

and thus [18, 20]

$$\varepsilon(r) = \left(1 + \frac{U(r)}{mc^2}\right)^2. \tag{3.49}$$

The last equation is an important relation where we see that the potential energy $U(r)$ can be gravitational or non-gravitational. Moreover, we can consider this result as a generalized Schwarzschild solution to the strong field. While, in weak field limit $c \to \infty$ leading to $\frac{U(r)}{mc^2} \ll 1$ and therefore the equation (3.49) brings $\varepsilon(r) \approx \left(1 + 2\frac{U(r)}{mc^2}\right)$.

3.4.2 Application to the hydrogen atom

Barros, in his applications to the hydrogen atom, did not use the tetrade formalism, but rather he started from the principle of correspondence in a curved spacetime. We have new results here and we are looking forward to publish them.

According to our understanding, Barros gave a total geometric description to the hydrogen atom so that the old view of the atom has been replaced by a new one. Thus, an electron moves around the nucleus under the effect of its interaction with the electric field produces from nucleus. This vision can be viewed as follows, the electron falls freely (follows geodesic line) in a spacetime curved by the interaction between the electron and the nucleus. Therefore, instead of using minimal coupling in the usual Dirac equation

$$(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\psi = 0,$$

we will, in fact, use the Dirac equation of free electron in curved spacetime

$$(i\gamma^\mu(\partial_\mu - \Gamma_\mu) - m)\psi = 0. \tag{3.50}$$

Where $\Gamma_\mu = -\frac{1}{8}g_{\lambda\alpha}\Gamma^\alpha_{\mu\rho}\left[\gamma^\lambda, \gamma^\rho\right]$ is an additional term, which defines the covariant derivative of a spinor field. In other words, it can be considered as a corrections to the spinor field [83].

In the treatment of Dirac equation in curved spacetime [85, 86, 87, 88] it is convenient to use the tetrade fields [84] (for all the details calculus in this subsection see appendix A). Let us using equation (3.49) in the like Schwarzschild metric (3.45) to describe the hydrogen atom in electromagnetic universe

$$ds^2 = \left(1-\frac{\alpha}{r}\right)^2 dt^2 - \left(1-\frac{\alpha}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.51}$$

where $\alpha = \frac{Ke^2}{mc^2}$ is the classical electron radius. Thereafter, the Dirac equation (3.50) in this metric is written as (see appendix A)

$$\begin{aligned}&[i\gamma^{(0)}\left(1-\frac{\alpha}{r}\right)^{-1}\partial_0 + i\gamma^{(1)}\left(1-\frac{\alpha}{r}\right)\partial_1 + i\gamma^{(2)}\frac{1}{r}\partial_2 + i\gamma^{(3)}\frac{1}{r\sin\theta}\partial_3 \\ &+ \frac{i\gamma^{(1)}}{r}\left(1-\frac{\alpha}{2r}\right) + \frac{i\gamma^{(2)}}{2}\frac{1}{r}\cot\theta - m]\psi = 0,\end{aligned} \tag{3.52}$$

we introduce the new ansatz

$$\psi(t,r,\theta,\phi) = \left(1-\frac{\alpha}{r}\right)^{\frac{1}{2}} e^{-iEt}\Upsilon(r,\theta,\phi). \tag{3.53}$$

and again by following [86]

$$\Upsilon(r,\theta,\phi) = e^{im_\phi\phi}\begin{pmatrix} g(r)\varepsilon(\theta) \\ -ih(r)\sigma^3\varepsilon(\theta)\end{pmatrix}. \tag{3.54}$$

The angular momentum operator is defined as [86]

$$\hat{K}\varepsilon(\theta) = \left[-\sigma^2\left(\partial_2 + \frac{1}{2}\cot\theta\right) + i\sigma^1\frac{m_\phi}{\sin\theta}\right]\varepsilon(\theta) = i\kappa\varepsilon(\theta) \tag{3.55}$$

where σ^i are the Pauli matrices. After some calculations and manipulations (see appendix A), we managed to show that

$$\begin{array}{l}\left(1-\frac{\alpha}{r}\right)^2\frac{dg(r)}{dr} + \frac{(1+\kappa)}{r}\left(1-\frac{\alpha}{r}\right)g(r) - \left(\frac{E}{\hbar c} + \frac{mc}{\hbar}\left(1-\frac{\alpha}{r}\right)\right)h(r) = 0 \\ \left(1-\frac{\alpha}{r}\right)^2\frac{dh(r)}{dr} + \frac{(1-\kappa)}{r}\left(1-\frac{\alpha}{r}\right)h(r) + \left(\frac{E}{\hbar c} - \frac{mc}{\hbar}\left(1-\frac{\alpha}{r}\right)\right)g(r) = 0\end{array}. \tag{3.56}$$

Obviously, this system of equations which are obtained here, by following new method are the same obtained by Barros [18, 20, 54] while the author in [55] did not reach the same result.

By making those transformations

$$g = e^{-\rho}F,\ \ h = e^{-\rho}G,\ \ \rho = \beta r,\ \ \alpha = \frac{\gamma}{mc^2},$$

we then get

$$\left(1+\tfrac{\gamma^2\beta^2}{m^2c^4\rho}-\tfrac{2\gamma\beta}{mc^2\rho}\right)\acute{F}-\left(1+\tfrac{\gamma^2\beta^2}{m^2c^4\rho}-\tfrac{2\gamma\beta}{mc^2\rho}\right)F+\tfrac{(1+\kappa)}{\rho}\left(1-\tfrac{\beta\gamma}{mc^2\rho}\right)F$$
$$-\left(\tfrac{E}{\beta\hbar c}+\tfrac{mc}{\beta\hbar}-\tfrac{\gamma}{\hbar c}\tfrac{1}{\rho}\right)G=0$$

$$\left(1+\tfrac{\gamma^2\beta^2}{m^2c^4\rho}-\tfrac{2\gamma\beta}{mc^2\rho}\right)\acute{G}-\left(1+\tfrac{\gamma^2\beta^2}{m^2c^4\rho}-\tfrac{2\gamma\beta}{mc^2\rho}\right)G+\tfrac{(1-\kappa)}{\rho}\left(1-\tfrac{\beta\gamma}{mc^2\rho}\right)G$$
$$+\left(\tfrac{E}{\beta\hbar c}-\tfrac{mc}{\beta\hbar}+\tfrac{\gamma}{\hbar c}\tfrac{1}{\rho}\right)F=0 .$$

Let the solution to be in series form

$$F=\sum_{n=0}^{N}a_n\rho^{n+s},\;\; G=\sum_{n=0}^{N}b_n\rho^{n+s}. \tag{3.57}$$

and we obtain the following coefficients recurrence relations

$$\left[\frac{\left(mc^2-E\right)}{\beta\hbar c}(N+1+\kappa+\frac{2\gamma\beta}{mc^2})-\frac{\gamma}{\hbar c}\right]a_N=\left[N+1-\kappa+\frac{2\gamma\beta}{mc^2}-\frac{\gamma}{(\hbar c)^2}\frac{\left(mc^2-E\right)}{\beta}\right]b_N \tag{3.58}$$

again

$$\frac{\gamma^2\beta^2}{m^2c^4}\left(n+3\right)a_{n+3}-\frac{\gamma\beta}{mc^2}(2n+5+\kappa+\frac{\gamma\beta}{mc^2})a_{n+2}+(n+2+\kappa+\frac{2\gamma\beta}{mc^2})a_{n+1}$$
$$-a_n-\frac{\left(E+mc^2\right)}{\beta\hbar c}b_n+\frac{\gamma}{\hbar c}b_{n+1}=0 \tag{3.59}$$

$$\frac{\gamma^2\beta^2}{m^2c^4}\left(n+3\right)b_{n+3}-\frac{\gamma\beta}{mc^2}(2n+5-\kappa+\frac{\gamma\beta}{mc^2})b_{n+2}+(n+1-\kappa+\frac{2\gamma\beta}{mc^2})b_{n+1}$$
$$-b_n+\frac{\left(E-mc^2\right)}{\beta\hbar c}a_n+\frac{\gamma}{\hbar c}a_{n+1}=0. \tag{3.60}$$

Next, we shall work to find the energy spectrum of the hydrogen atom. Accordingly, if we make $n\to N$ in the last two equations, then all coefficients they have an order higher than N vanish. Thus, the use of the recurrence relations in the equations above gives

$$\frac{2\gamma}{mc^2}\beta^2+(N+1)\beta-\frac{\gamma}{(\hbar c)^2}mc^2=0 \tag{3.61}$$

and we get

$$\beta = \sqrt{\frac{(m^2c^4 - E^2)}{(\hbar c)^2}} = \frac{mc^2}{4\gamma}\left[-(N+1) \pm \sqrt{(N+1)^2 + \frac{8\gamma^2}{(\hbar c)^2}}\right]. \tag{3.62}$$

As consequence, we find

$$E_N = mc^2\sqrt{1 - \left(\frac{\hbar c}{4\gamma}\right)^2\left[-(N+1) \pm \sqrt{(N+1)^2 + \frac{8\gamma^2}{(\hbar c)^2}}\right]^2}. \tag{3.63}$$

Clearly, this specterum is different from the one obtained by Barros. Whereas Barros assert that the spectrum is very close to the experimental data, both our spectrum and Barros result are independent from the κ eigenvalues. This it mean that maybe this is not the right way to solve the Dirac equation.

The interesting thing is that the work in weak field limit $\frac{U(r)}{mc^2} \ll 1$ allows to recover the usual Dirac system of equations for the hydrogen atom [18, 20]. Therefore, the approximation $\frac{\alpha}{r} \ll 1$ in (3.56) leads to

$$\begin{aligned} \frac{dg(r)}{dr} + \frac{(1+\kappa)}{r}g(r) &\approx \left(\frac{E}{\hbar c}\left(1+\frac{\alpha}{r}\right) + \frac{mc}{\hbar}\right)h(r) \\ \frac{dh(r)}{dr} + \frac{(1-\kappa)}{r}h(r) &\approx -\left(\frac{E}{\hbar c}\left(1+\frac{\alpha}{r}\right) - \frac{mc}{\hbar}\right)g(r) \end{aligned}. \tag{3.64}$$

In the case of lower moments $\frac{E}{mc^2}$ is taken, nearly, to be 1 [18, 20]. This makes the solution of the above equations exactly coincide with the Dirac spectrum of the hydrogen atom.

3.5 Conclusion

In this chapter, we shown another viewpoint to the definition of spacetime geometry so that the spacetime not only affected by the properties of active particles. More than that, it is interacting with the properties of passive particles. Furthermore, the four interactions should take place in a different spacetimes.

In general relativity, precisely in the Reissner–Nordström solution one can see how an electromagnetic energy affects on the mass ($m_{eff} = m - \frac{Q^2}{r^2}$), and therefore contribute on the curvature of the spacetime [89, 90]. On the other hand, we have the Melvin universe in which a purely magnetic field affects the spacetime geometry [91]. This is the manner how the electromagnetism affect the spacetime geometry in gravity universe. Along the same line, the thinking about a direct effect of electromagnetism on the geometry, in electromagnetic universe, is not seemed to be impossible.

However, if we think about the electromagnetic curvature of spacetime in classical mechanics. It might seem a little easy but if we follow the same thinking in quantum mechanics this description becomes more complicated. We will find ourselves in the case where we can think about a quantum free fall of the electron in hydrogen atom. Moreover, the introduce of the electromagnetic lambda appears as correction term to the Maxwells' equations, but what about the quantum effect of that constant? What are the consequences of this introducing constant on the energy levels of atoms?

The extent to which the observer can measure precisely the geodesic motion of a particle is related to his manner of interaction with the spacetime of the interaction. In fact, the idea that the electromagnetism can affect the same spacetime in which we live is not makes a sense, because we know that there is no effects of an electromagnetic field on the clocks in our spacetime, and consequently no bending of light by the electromagnetic field. In order to deal with this problem, we have already proposed the existence of the electromagnetic universe and we will discuss its characteristic in the next chapter.

Chapter 4

The geometry of electromagnetic universe

4.1 Introduction

In general relativity, the differences between the standard definition of the field and the geometrical description of the field itself lies in the rate of time and the geodesic (equivalence principle). Due to the equality $m_I = m_g$ for each particle, we have the same dynamic scene for all particles. On the other hand, in the electromagnetic case each particle characterized by a specific ratio $\frac{q}{m}$, so we will have many scenes for different particles. Therefore, one from the points of view that could be pursued is that all of charged particles occupy another universe, we call it the electromagnetic universe. The space and time of that universe could be completely different (different does not necessarily mean separate) from our spacetime. Working on several particles will be extremely complicated, and this is what will lead us in this work to focus on the electron proper spacetime (the state of one particle).

From a philosophical perspective, we humans and all neutral objects are not a subject to the electromagnetic universe but our geometrical spacetime is the real one because, according to our awareness, the majority of matter in this universe are neutral, and the dominant interaction in this visual universe is gravity. In addition, our realization and interaction with the universe around us seem to be very clear and strongly related.

In this chapter, we investigate the possibility of making a geometrical description to the electromagnetic universe based on Riemann geometry. This chapter is divided in two parts the classical part where we will show the advantages that geometry added to the classical motion of charged particle. In the other part, which is the quantum mechanical part, we will exhibit the effect of the geometry on the electron energy levels. The likelihood to explain the Lamb shift and the possibility of the existence of a new type of vacuum energy will be discussed in this part.

The classical approach

4.2 Towards an equivalence principle of the electromagnetic universe

In general relativity, the inertial mass equal to gravitational one, and hence the motion becomes independent from the particle properties. This what the equivalence principle was stated. Therefore, in small regions we cannot distinguish between the gravitational force and the acceleration (inertial force). In electromagnetism, the ratio $\frac{q}{m}$ often does not equal to 1 but this does not necessarily mean that we can distinguish between acceleration and the effect of an electromagnetic field in all cases. However, the equation (3.34) makes no distinction between electromagnetism and acceleration with respect to an observer attached to the charged particle.

As known, the accelerated charged particle radiates but these radiations have been detected by an observer which is a subject of gravity universe. However, the interaction between charged test particle q and the electric field E undistinguished from the acceleration of that particle. If this is truly right, then a charge $q1$ fixed in the electric field should be radiate in electromagnetic universe. In reality, no one can know exactly what happen when we fix a charged particle in an electric field. In order to show this, we need more informations from the observer in electromagnetic universe.

The problem of the electromagnetic interaction is the non universality of $\varrho = \frac{q}{m}$ ratio. Consequently, the metric becomes dependent on the particle properties and we will reserve a different metrics to a particles with different properties [76, 92]. The same problem appears in the case where $m_I \neq m_g$ and this can happen if we take into consideration the contributions of the self force of a body [93, 94] .

4.3 The metric of the electromagnetic universe

4.3.1 Description of the method

Basically, the idea is just an electromagnetic field affect the spacetime geometry, in electromagnetic universe, in a similar way as gravity. This is what the observer sees but in this case the observer must be included in the electromagnetic universe. Therefore, in spherically symmetric spacetime of the electromagnetic universe and in regions outside the source ($T_{\mu\nu}^{Matter} = 0$) of an electric field, the metric is taken to be

$$ds^2 = A(r)c^2dt^2 - D(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.1}$$

Hence, the field is a radial electrostatic produces from a spherical charge Q, which is takes the following form

$$F_{\mu\nu} = -F_{\nu\mu} = \frac{E(r)}{c} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

The observer which is not belongs to the electromagnetic universe deal with this system as Maxwell's theory in curved spacetime, and hence we can employ the Maxwells' equations in curved spacetime outside the source

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}F^{\mu\nu}\right) = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right) = 0, \tag{4.3}$$

and then

$$\partial_r \left[r^2\sqrt{A(r)D(r)}\sin\theta g^{11}g^{00}F_{10}\right] = 0, \tag{4.4}$$

which leads to

$$\partial_r \left(\frac{r^2}{\sqrt{A(r)D(r)}}E(r)\right) = 0 \tag{4.5}$$

and the integration yields

$$E(r) = \frac{\sigma}{r^2}\sqrt{A(r)D(r)}, \tag{4.6}$$

where σ is the integration constant. Generally, σ carries the source informations so that in a distant point from the source the spacetime becomes flat. The effect of the curvature vanish and the electric field will be written in the usual form, and then we put $\sigma = \frac{Q}{4\pi\varepsilon_0} = KQ$.

In equivalent way, an observer belongs to the electromagnetic universe sees that spacetime is curved by the electric field. Therefore, the equation of state with respect to that observer is the Einsteins' type equations

$$G^{EM}_{\mu\nu} = -\chi_e T^{EM}_{\mu\nu}, \tag{4.7}$$

which play the same role of Maxwells' equations and this is what was mentioned in chapter 3. In addition to that we already show in [22] that the Einsteins' equations in weak field limit turn to be Maxwells' equations.

Now, let us give some comments about the electromagnetic energy–momentum tensor $T^{EM}_{\mu\nu}$. At first, we can consider this tensor as a high order approximations of the metric perturbation ($h^{0\mu} \sim A^\mu$) and this can be explained as follows

$$T^{EM}_{\mu\nu} = -\frac{1}{\mu_0}\left(F_{\mu\rho}F_{\nu}^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{4.8}$$

$$= -\frac{1}{\mu_0}((\partial_\mu h_{0\rho} - \partial_\rho h_{0\mu})(\partial_\nu h_0^\rho - \partial^\rho h_{0\nu}) - \frac{1}{4}\eta_{\mu\nu}(\partial_\rho h_{0\sigma} - \partial_\rho h_{0\mu})(\partial^\rho h_0^\sigma - \partial^\sigma h_0^\rho) \tag{4.9}$$

$$-\frac{1}{4}f(h_{\mu\nu})(\partial_\rho h_{0\sigma} - \partial_\rho h_{0\mu})(\partial^\rho h_0^\sigma - \partial^\sigma h_0^\rho), \tag{4.10}$$

where $f(h_{\mu\nu})$ represents all the necessary orders of the metric perturbation. In addition, the tensor $T^{EM}_{\mu\nu}$ is traceless and that what makes the equation (4.7) taken to be

$$R^{EM}_{\mu\nu} = -\chi_e T^{EM}_{\mu\nu}. \tag{4.11}$$

In weak field limit (at first order approximations) this equation is just $R^{EM}_{\mu\nu} \approx 0$.

More than that and as we saw in gravity (see section 17.11 of [2]) the electromagnetic energy–momentum tensor $T^{EM}_{\mu\nu}$ produces from the electromagnetic field itself. In our case, this means that the existence of electromagnetic field gives an additional electromagnetic field of the field itself and so on.

Nevertheless, the Einstein equations (4.7) denote that all the types of energy ($T_{\mu\nu}$) can contribute to produce an electromagnetic field. Interestingly enough, one can reach an electromagnetic field from the gravitational energy–momentum, and thus the equation (4.7) it can be rewrite as

$$G^{EM}_{\mu\nu} = -\chi_e T^{EM}_{\mu\nu} + \chi_e T^{GR}_{\mu\nu}. \tag{4.12}$$

Accordingly, in higher order approximation or in strong gravitational field, an observer in electromagnetic universe can observe the contributions of the gravitational field to generate electromagnetic waves.

4.3.2 The solution of Einsteins' type equations

We shall work here to find the solution of the equation (4.11) but firstly we calculate the components of the tensor (4.8), and make them directly depends on spacetime geometry (4.1), and then we have

$$T_{00} = -\frac{1}{\mu_0}\left(F_{0\rho}F_0^{\ \rho} - \frac{1}{4}g_{00}F_{\rho\sigma}F^{\rho\sigma}\right),$$

and by using (4.2) and (4.1) we get

$$
\begin{aligned}
T_{00} &= -\frac{1}{\mu_0}\left(F_{01}F_0{}^1 - \frac{1}{4}g_{00}\left(F_{01}F^{01} + F_{10}F^{10}\right)\right) \\
&= -\frac{1}{\mu_0}\left(F_{01}F_0{}^1 - \frac{1}{4}g_{00}\left(F_{01}F^{01} + F_{10}F^{10}\right)\right) \\
&= -\frac{1}{2\mu_0}g^{11}\left(F_{01}\right)^2 \\
&= \frac{1}{2\mu_0}\frac{E(r)^2}{D(r)c^2}.
\end{aligned}
$$

Taking into a count (4.6) we obtain

$$
T_{00} = \frac{1}{2\mu_0}\left(\frac{KQ}{c}\right)^2\frac{A(r)}{r^4} \tag{4.13}
$$

and we do the same thing with T_{11}, T_{22}, and T_{33} and thus the Einsteins' equations yield [23]

$$
\begin{aligned}
&-\frac{A''}{2D} + \frac{A'}{4D}\left(\frac{A'}{A} + \frac{D'}{D}\right) - \frac{A'}{rD} = -\frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2\frac{A(r)}{r^4} && (4.14)\\
&\frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{D'}{D}\right) - \frac{D'}{rD} = \frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2\frac{D(r)}{r^4} && (4.15)\\
&\frac{1}{D} - 1 + \frac{r}{2D}\left(\frac{A'}{A} - \frac{D'}{D}\right) = -\frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2\frac{1}{r^2} && (4.16)\\
&R_{33} = R_{22}\sin^2\theta = T_{22}\sin^2\theta && (4.17)
\end{aligned}
$$

with a simple manipulation the sum of equation (4.14) multiply by $\frac{D}{A}$ with (4.15) gives

$$
\frac{d}{dr}(AD) = 0. \tag{4.18}
$$

We integrate and the constant of integration is determined in which the spacetime is Minkowski in the case where $r \to \infty$, and that implies

$$
A(r) = \frac{1}{D(r)}. \tag{4.19}
$$

The substitution of (4.19) in (4.16) lead up to

$$
\frac{d}{dr}(rA) = 1 + \frac{a_1}{r}, \tag{4.20}
$$

and then the integration gives us

$$
A(r) = 1 + \frac{a_1}{r^2} + \frac{a_2}{r}, \tag{4.21}
$$

where $a_1 = \frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2$ and a_2 is a constant of integration.

If we return to the chapter 3 exactly to Barros approach [18, 20] and as shown in the equation (3.49) the component of the metric is just

$$\varepsilon(r) = \left(1 + \frac{U(r)}{mc^2}\right)^2. \tag{4.22}$$

According to the equations (4.6), (4.19) and $E = -\frac{dV(r)}{dr}$ we can find, in our case, this

$$U(r) = qV(r) = q\frac{KQ}{r} \tag{4.23}$$

and that leads to

$$\varepsilon(r) = 1 + \left(\frac{q}{mc^2}\frac{KQ}{r}\right)^2 + 2\frac{q}{mc^2}\frac{KQ}{r}. \tag{4.24}$$

By comparing the last equation with (4.21), we get

$$a_1 = \left(\frac{KqQ}{mc^2}\right)^2, \qquad a_2 = 2\left(\frac{KqQ}{mc^2}\right), \tag{4.25}$$

in order to do that we have to take

$$\chi_e = \frac{2}{\varepsilon_0}\left(\frac{q}{mc^2}\right)^2. \tag{4.26}$$

Thus, we get [23]

$$A(r) = \left(1 + \frac{KqQ}{mc^2 r}\right)^2, \tag{4.27}$$

and the surprising thing is that the same result was obtained by [54] in completely different way. Actually, this is the only method which we have for determining the constants a_2 and χ_e , but in reality we are not certain if it is the correct method or not. What we have to do is wait until the end and through the obtained results we will know whether our proposal is correct or not. The constant χ_e can be determined in condition that the geodesic equations in weak field limit give a form of Coulomb's law.

However, the term $\left(\frac{q}{m}\right)^2$ is not necessarily an amount comes from the coupling constant χ_e but it is a part of the energy-momentum tensor $T^{EM}_{\mu\nu}$. The square in the term comes from the second order approximations in the tensor. More than that the term $\frac{a_1}{r^2}$ is a second order comes from the contributions of $T^{EM}_{\mu\nu}$ tensor. Therefore, in weak field limit $T^{EM}_{\mu\nu} = 0$ and the Einstein equations (4.11) becomes $R^{EM}_{\mu\nu} = 0$, and its solution is given with

$$A(r) = \frac{1}{D(r)} = \left(1 + \frac{a_2}{r}\right), \tag{4.28}$$

and this is the Schwarzschild metric of the electromagnetic universe.

Furthermore, we can do all of that with gravity universe in which the Einsteins' equations of the gravitational field become

$$R^{GR}_{\mu\nu} = \kappa T^{GR}_{\mu\nu} = -\frac{\kappa}{\mu_0}\left(F_{\mu\rho}F_{\nu}^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{4.29}$$

where $\mu_0 = \frac{4\pi G}{c^2}$ and $F_{\mu\nu}$ denote the gravity tensor field, which is taken as

$$E_g(r) = \frac{GM}{r^2}\sqrt{A(r)D(r)}. \tag{4.30}$$

Thence, we can find the exact Schwarzschild solution of Einsteins' equations as follows

$$A(r) = \frac{1}{D(r)} = \left(1 - \frac{GM}{c^2}\frac{1}{r}\right)^2, \tag{4.31}$$

but the problem with gravity is that the second order term is very weak and can be neglected, so the equation (4.31) is reduced to $A(r) = \left(1 - 2\frac{GM}{c^2}\frac{1}{r}\right)$.

4.4 Classical applications

The objective of this applications is to show the advantages and the importance of the geometrical descriptions of the electromagnetic field, but these applications need to be confirmed experimentally.

4.4.1 Equations of motion

In gravity universe, the earth moves around the sun under the effect of the gravitational field (spacetime curvature equivalent to gravitational field effect) produces by the sun itself. In a similar way, we choice the system of the classical hydrogen atom where an electron (with a charge $q = -e$) moves around the proton ($Q = +e$) under the effect of a radial electrostatic field on the geometry of the spacetime in the electromagnetic universe. In the attractive case with the use of (4.27), we can characterize this system by the following metric

$$ds^2 = \left(1 - \frac{\alpha}{r}\right)^2 (cdt)^2 - \left(1 - \frac{\alpha}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.32}$$

where $\alpha = \frac{Ke^2}{mc^2}$.

Commonly, The equations of motion have been established by an observer which is not belongs to the electromagnetic universe, and those equations are given by

$$\frac{d^2x^\mu}{d\tau^2} = \left(\frac{q}{m}\right)F^{\mu}_{\ \sigma}\upsilon^\sigma, \tag{4.33}$$

where $\upsilon^\sigma = \frac{dx^\sigma}{d\tau} = \dot{x}^\sigma$. In the electromagnetic universe, the observer sees instead that the electron follows the geodesic line represent by the geodesic equations

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\rho\sigma}\upsilon^\rho\upsilon^\sigma. \tag{4.34}$$

In order to give the solution to this equation in the metric (4.32) we use, equivalent to that, the Lagrange procedure [2] in which $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ and then we have

$$L = \left(1-\frac{\alpha}{r}\right)^2 c^2\dot{t}^2 - \left(1-\frac{\alpha}{r}\right)^{-2}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{4.35}$$

and we substitute in the Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{4.36}$$

we find [23]

$$\left(1-\frac{\alpha}{r}\right)^2 \dot{t} = k \tag{4.37}$$

$$\left(1-\frac{\alpha}{r}\right)^{-2}\ddot{r} - \frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)^{-3}\dot{r}^2 + \frac{\alpha c^2}{r^2}\left(1-\frac{\alpha}{r}\right)\dot{t}^2 - r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2) = 0 \tag{4.38}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{4.39}$$

$$r^2\sin^2\theta\dot{\phi} = b. \tag{4.40}$$

Where k and b are constants and they respectively play a role of energy and angular momentum. It should be known that L is not dependent on ϕ and t. The equation (4.38) is more complicated and for a non-null geodesic[1] we can replace it by the first integral of the geodesic equations

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2. \tag{4.41}$$

In addition, we interest to the electron motion in the equatorial plane where $\theta = \frac{\pi}{2}$ and thus we have

$$\left(1-\frac{\alpha}{r}\right)^2 \dot{t} = k \tag{4.42}$$

$$\left(1-\frac{\alpha}{r}\right)^2 c^2\dot{t}^2 - \left(1-\frac{\alpha}{r}\right)^{-2}\dot{r}^2 - r^2\dot{\phi}^2 = c^2 \tag{4.43}$$

$$r^2\dot{\phi} = b \tag{4.44}$$

The substitution of (4.42) and (4.44) in (4.43) gives [23]

$$\dot{r}^2 + \frac{b^2}{r^2}\left(1-\frac{\alpha}{r}\right)^2 - \frac{c^2\alpha}{r}\left(2-\frac{\alpha}{r}\right) = c^2(k^2-1). \tag{4.45}$$

[1]For a null geodesic we have $ds^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$

This equation represents the dynamics of a relativistic charged particle in Coulomb's field in more accurate form. Thus, in the case where $b = 0$ (movement in one direction) we get

$$m\ddot{r} = -\frac{Ke^2}{r^2} + \frac{K^2e^4}{mc^2}\frac{1}{r^3}. \tag{4.46}$$

In order to show the shape of the electron orbit we use the equation $r^2\dot{\phi} = b$ which gives us

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{b}{r^2}\frac{dr}{d\phi} \tag{4.47}$$

and thus we get

$$\left(\frac{du}{d\phi}\right)^2 + u^2(1-\alpha u)^2 - \frac{c^2\alpha}{b^2}u(2-\alpha u) = \frac{c^2}{b^2}(k^2-1), \tag{4.48}$$

where $u = \frac{1}{r}$, and the differentiation on ϕ gives

$$\frac{d^2u}{d\phi^2} + u(1-\alpha u)^2 - \alpha u^2(1-\alpha u) - \frac{c^2\alpha}{b^2} + \left(\frac{c\alpha}{b}\right)^2 u = 0. \tag{4.49}$$

This equation gives the electron orbit more complicated shape than the classical motion.

4.4.2 The perihelion advance in the classical electron orbit

An electron moves in curved spacetime in electromagnetic universe does not return to the same starting point and thus its orbit is not closed. Obviously, the electron orbit shows an advance of perihelion. In order to illustrate that we start from equation (4.49) in weak field limit ($\alpha \ll 1$)

$$\frac{d^2u}{d\phi^2} + u = \frac{c^2\alpha}{b^2} + 3\alpha u^2 \tag{4.50}$$

as usual [2] the solution of this equation is given with the following form

$$u = \frac{c^2\alpha}{b^2}(1 + e\cos\phi) + \Delta u. \tag{4.51}$$

Where e is the eccentricity of the orbit and Δu is some perturbation of the elliptical solution. We make a substitution in (4.50) and this implies that the solution must be written as follows

$$u \approx \frac{c^2\alpha}{b^2}\left(1 + e\cos\left[\phi\left(1-\beta\right)\right]\right) \tag{4.52}$$

where $\beta = 3\frac{\alpha^2c^2}{b^2}$. From (4.52) we have to note that the period is larger than 2π and thus we can prove this [23]

$$\delta\phi = \frac{2\pi}{(1-\beta)} - 2\pi \approx 2\pi\beta = 6\pi\left(\frac{Ke^2}{bmc}\right)^2. \tag{4.53}$$

From this expression, we must comment on the obtained perihelion shift in the electromagnetic universe. This shift is written terms of the ratio $\frac{q}{m}$ of the test particle and the angular momentum $b = \frac{L}{m}$, which have an important role in quantum application, and we will show this later.

More interestingly, a possible of an atomic black hole comes from the singularity $r = -\frac{KqQ}{mc^2}$ and for an electron-proton interaction this singularity takes value $r \sim 10^{-15}m$, and this is the scale where strong interaction takes place. Two attractive charges give us a positive singularity, but in fact one can see this singularity as an atomic hole to escape the electromagnetic universe.

4.4.3 Clocks slow down in electromagnetic universe

The central problem faced the geometric description of electromagnetic interaction is the problem of time, and we have seen a problem like that in Weyl theory. If we return to our work, on reality, there is no effect of the electromagnetic field on the clocks rate. On the other hand, an observer in the electromagnetic universe, his clock slowing down (or speeding up depending on the charges sign) under the influence of the electromagnetic field so that his clock rate is determined from the metric (4.32) for an observer at rest as follows

$$\Delta\tau = \left(1 + \frac{q}{m}\frac{\varphi_e}{c^2}\right)\Delta t \tag{4.54}$$

where φ_e is the electric potential, $\Delta\tau$ and Δt are the rates of time for an observer in electromagnetic universe and the other in gravity universe respectively (for a same result (4.54) see [21]).

For person who has the necessary capability to give his opinion in order to explain this phenomenon, he can say the following

For an observer in gravity universe he is not belong to the electromagnetic universe. For that reason, he cannot see the electromagnetic field as a geometric phenomenon and thus he cannot observe the slow down of his clock. In the other way, an observer in electromagnetic universe see his clock slow down. In equivalent way the observer in gravity universe see a charge (Q) which is the source of the electromagnetic field.

Instead of the observer seeing the clock slowing down and the electromagnetic field is a geometric phenomenon, he will see rather than that a source (Q) of electromagnetic field (for more informations see [95]).

As known, an accelerated charged particle emit an electromagnetic field. This means (for an observer in electromagnetic universe) the acceleration affect the spacetime geometry of electromagnetic universe.

The rate of time and the perihelion advance, were reached by many researches (see for example [21, 96]) in the context of the analogy between gravity and electromagnetic but it stay a mathematical result without any precise physical explanation.

4.5 The metric with electromagnetic lambda term Λ_e

In electromagnetic universe, the lambda term is a new attempt to provide a new definition to the vacuum. The importance of this new constant will be clearly shown in the quantum applications in the next section.

Now, let us first look for a solution to Einsteins' equations

$$G_{\mu\nu}^{EM} = -\chi_e T_{\mu\nu}^{EM} - \Lambda_e g_{\mu\nu}, \tag{4.55}$$

in another way

$$G_{\mu\nu}^{EM} = -\chi_e \left(T_{\mu\nu}^{EM} + T_{\mu\nu}^{\Lambda_e} \right). \tag{4.56}$$

Before we look for a solution we must talk a little bit about the new constant. For that, we have the term $T_{\mu\nu}^{\Lambda_e} = \frac{\Lambda_e}{\chi_e} g_{\mu\nu}$ which is an additional electromagnetic energy-momentum tensor. If we assume that $T_{\mu\nu}^{\Lambda_e}$ has the same nature of $T_{\mu\nu}^{EM}$ that consequently means an additional electromagnetic field comes from the contributions of lambda term. In gravity universe, the cosmological constant may arise as an additional gravitational potential [97, 98, 99]

$$\Phi(r) \equiv \Phi_{Newton} + \Phi_\Lambda = -\frac{GM}{r} - \frac{1}{6}\Lambda r^2. \tag{4.57}$$

Thus we have [97, 98]

$$\Delta\Phi(r) = 4\pi G\rho - \frac{\Lambda}{3}, \tag{4.58}$$

Accordingly, in the electromagnetic universe the presence of the electromagnetic lambda term leads to the following splitting of the potential $A_\mu \to A_\mu + A_\mu^{\Lambda_e}$, which produces $F_{\mu\nu} \to F_{\mu\nu} + F_{\mu\nu}^{\Lambda_e}$. Hence, the Maxwells' equations become

$$\nabla_\mu \left(F^{\mu\nu} + F_{\Lambda_e}^{\mu\nu} \right) = \mu_0 j^\nu \tag{4.59}$$

and then

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu - \nabla_\mu F_{\Lambda_e}^{\mu\nu}. \tag{4.60}$$

We put down

$$\nabla_\mu F_{\Lambda_e}^{\mu\nu} = \eta j_{\Lambda_e}^\nu, \tag{4.61}$$

and we know that there is no physical meaning of negative density and that is confirmed by the definition of $T_{\mu\nu}^{\Lambda_e}$. Hence, the vacuum permeability η of vacuum must be negative and this leads to the permittivity to be also negative. Therefore, if we let $\eta \to -\mu_0$ then the equation (4.60) is

$$\nabla_\mu F^{\mu\nu} = \mu_0 \left(j^\nu + j_{\Lambda_e}^\nu \right) \tag{4.62}$$

From what we mentioned previously, and by following the same line of thinking of [100, 101]. the vacuum can be considered as a material with a negative permittivity and permeability with a negative refractive index. This type of matter called metamaterials.

We return to the solution of Einsteins' type equations, so a traceless electromagnetic energy-momentum tensor $T^{\mu}_{\mu} = 0$ gives us a Ricci scalar $R = 4\Lambda_e$ and that leads

$$R_{\mu\nu} = -\chi_e T^{EM}_{\mu\nu} + \Lambda_e g_{\mu\nu} \tag{4.63}$$

and then we get [23]

$$-\frac{A''}{2D} + \frac{A'}{4D}\left(\frac{A'}{A} + \frac{D'}{D}\right) - \frac{A'}{rD} = -\frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2 \frac{A(r)}{r^4} + \Lambda_e A(r) \tag{4.64}$$

$$\frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{D'}{D}\right) - \frac{D'}{rD} = \frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2 \frac{D(r)}{r^4} - \Lambda_e D(r) \tag{4.65}$$

$$\frac{1}{D} - 1 + \frac{r}{2D}\left(\frac{A'}{A} - \frac{D'}{D}\right) = -\frac{\chi_e}{2\mu_0}\left(\frac{KQ}{c}\right)^2 \frac{1}{r^2} - \Lambda_e r^2 \tag{4.66}$$

$$R_{33} = R_{22}\sin^2\theta = T_{22}\sin^2\theta. \tag{4.67}$$

By following the same procedure used in the previous we find [23]

$$ds^2 = \left[\left(1 + \frac{q}{mc^2}\frac{KQ}{r}\right)^2 - \frac{\Lambda_e}{3}r^2\right](cdt)^2 - \left[\left(1 + \frac{q}{mc^2}\frac{KQ}{r}\right)^2 - \frac{\Lambda_e}{3}r^2\right]^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.68}$$

In order to show the influence of the lambda term on the dynamics, we take the Lagrange $L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$of the metric (4.68) of classical hydrogen atom, and we substitute in the Euler-Lagrange equations (4.36). The steps involved in solving equations (4.36) (in the case of zero lambda term) they have been used here, so as to get the following [23]

$$\dot{r}^2 + \frac{b^2}{r^2}\left[\left(1 - \frac{\alpha}{r}\right)^2 - \frac{\Lambda_e}{3}r^2\right] - \frac{c^2\alpha}{r}\left(2 - \frac{\alpha}{r}\right) - \frac{c^2\Lambda_e}{3}r^2 = c^2(k^2 - 1) \tag{4.69}$$

$$\frac{d^2u}{d\phi^2} + u(1 - \alpha u)^2 - \alpha u^2(1 - \alpha u) + \left(\frac{c\alpha}{b}\right)^2 u = \frac{c^2\alpha}{b^2} - \frac{\Lambda_e}{3}\frac{1}{b^2u^3}. \tag{4.70}$$

From the first equation, it can be noted that the form of the electron energy is corrected by an attractive kind of energy from lambda term. Moreover, from the differentiating of (4.69) with respect to τ, we can deduce the form of lambda field as follows

$$m\frac{d\dot{r}^2}{d\dot{r}}\frac{d\dot{r}^2}{d\tau} + m\frac{d}{dr}\left[\frac{b^2}{r^2}\left[\left(1 - \frac{\alpha}{r}\right)^2 - \frac{\Lambda_e}{3}r^2\right] - \frac{c^2\alpha}{r}\left(2 - \frac{\alpha}{r}\right) - \frac{c^2\Lambda_e}{3}r^2\right]\frac{dr}{d\tau} = 0, \tag{4.71}$$

then we get

$$m\ddot{r} = m\frac{b^2}{r^3}\left(1-\frac{\alpha}{r}\right)^2 - m\frac{b^2\alpha}{r^4}\left(1-\frac{\alpha}{r}\right) - m\frac{c^2\alpha}{r^2} + m\frac{c^2\alpha^2}{r^3} + \frac{mc^2\Lambda_e}{3}r. \tag{4.72}$$

Clearly, the lambda field is

$$F(\Lambda_e, r) = \frac{m}{q}\frac{c^2\Lambda_e}{3}r. \tag{4.73}$$

It is obvious that

$$m\dot{\phi} = L = mb, \tag{4.74}$$

thereby (4.73) becomes

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2}\left[\left(1-\frac{\alpha}{r}\right)^2 - \frac{\Lambda_e}{3}r^2\right] - \frac{mc^2\alpha}{2r}\left(2-\frac{\alpha}{r}\right) - \frac{mc^2\Lambda_e}{6}r^2 = \frac{mc^2}{2}(k^2-1) \tag{4.75}$$

This may mean that the nature of this field is a magnetism because it acts on the angular momentum in the energy equation (4.69). If we assume that a part of this field is electric, then we can get

$$\varphi = \varphi + \frac{1}{6}\frac{m}{q}c^2\Lambda_e r^2, \tag{4.76}$$

and we can offset it in the Maxwells' equation $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$ in order to extract the vacuum density expression

$$\rho_{vac}(\Lambda_e) = \frac{1}{3}\frac{m}{q}\frac{\Lambda_e}{\mu_0}. \tag{4.77}$$

In general, and if equation (4.61) is taken into account, we will find

$$\rho_{vac}(\Lambda_e) = \frac{1}{3}\frac{m}{q}\frac{\Lambda_e}{\eta}. \tag{4.78}$$

As it is well known, the negative density is meaningless in gravity universe but it is possible in electromagnetic case. The significant thing here is the dependent of the vacuum density on the particles properties. This means that we see a different vacuum densities for particles with different $\frac{m}{q}$ terms.

On the other part, the high order equation(4.70) is difficult to solve. It represents a complicated shape of the electron's circular orbit.

Quantum approach

4.6 Quantum applications

In this part of the work, we will try to give a different interpretation to the well-known Lamb shift. As known, this shift was explained in quantum electrodynamics (QED) by the vacuum fluctuations. For that reason, and before we go further, we would like first to give a simple introduction to the so-called vacuum.

4.6.1 The vacuum

- Empty space is really empty?
- Can Our mind really realize the meaning of vacuum?

4.6.1.1 Vacuum fluctuations

A vacuum with energy zero is not compatible with the uncertainty principle $\Delta E \Delta t \geqslant \frac{\hbar}{2}$. Therefore, in quantum mechanics [102, 103, 104, 105, 106, 107, 108, 109, 110], the empty space is not empty but in fact it is full with virtual particles which manifest themselves as a pairs of matter and anti matter. These pairs appear (create) as an energy ΔE, and spontaneously disappear (annihilate) after a period of time Δt, provided that they not violate the uncertainty principle. This means that one can borrow any amount of energy from the vacuum and then return it, but at a time when it does not break the uncertainty principle. This what means by vacuum fluctuations.

The existence of virtual particles effect seems to be convenient to explain some experiences such as

Casimir effect This effect appears when we put a two uncharged plats of a metal in vacuum at short distance ($10^{-6}m$) between them, we notice that they are close each other [28, 114]. In quantum field theory, this can be explained by the quantum vacuum effect. The virtual photons created between plates are less than the others created outside. In addition, the photons wavelength between plates is limited. The virtual photons outside hit the plates, which causes a pressure applied to the plates and push them together.

Lamb shift is a splitting between $2s_{1/2}$and $2p_{1/2}$ energy levels of the hydrogen atom with a difference $\Delta E = E_{s_{1/2}} - E_{p_{1/2}} = 4.372 \times 10^{-6} eV$ it was experimentally measured by Lamb and Rutherford [108]. The shift is not predicted by Dirac's theory, due to the absence of the vacuum definition in this theory. In the other hand, in quantum electrodynamics (QED) [102, 105, 107, 109], this shift was interpreted based on the fact that the electron of the hydrogen atom interact with the vacuum fluctuations (non-zero field). The quantized electromagnetic fields of the vacuum fluctuations affect the position of the electron and not the proton because the last is so heavy. There are two sources of the vacuum that contribute to the Lamb shift. The first one is the well-known vacuum polarization (photon self-energy) in which a photon (electromagnetic field, the Coulomb field in hydrogen atom) creates a virtual electron-positron pairs. The vacuum polarization (the pair production) affects the propagation of the photon which means a modification in Coulomb's field [109] where the charge of the electron replaced by a renormalized charge (for the details of calculus see [103, 104, 105]). In fact, the contribution of vacuum polarization to the Lamb shift is much smaller. The second contribution type is the electron self-energy where an electron emits and reabsorbs a virtual photon. This affects the propagation of the electron [105] and leads to the renormalization of the electron mass ($m_R = m_e + \delta m$). The interaction of the electron with its own quantized field changes its energy states[110]. It is worth noting that the mass term (δm) is different from the electric mass [111] of the classical electron.

Hawking radiations In the surface of the black hole, there are many pairs of virtual particles when a pair created one of them escape the event horizon inside the black hole [112]. The other which is outside the black hole cannot annihilate and becomes a real particle. (we should noting that there is no direct experiment which verify the characteristics of the vacuum).

4.6.1.2 Vacuum state

By definition the vacuum state is the state of the lowest energy $\mid n = 0 >$ of the quantum field or in other words the state of no particles. A good example is the harmonic oscillator [106] where the zero-point energy takes this value $E_0 = \frac{1}{2}\hbar\omega \neq 0$. This is totally different from the classical mechanics where the vacuum is defined by the absence of the field and, as a result, the zero value for energy equal to zero.

Let us now determine the energy of vacuum in the quantized Maxwell field [113, 114, 115, 116]. Initially, we start with the expression of the electromagnetic Hamiltonian which is given by

$$H = \int d^3\mathbf{x}(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2), \tag{4.79}$$

the electric and the magnetic fields, by following [113], are written in quantized form

$$\mathbf{E} = \mathrm{i}\sqrt{\frac{\hbar\mu_0 \mathrm{c}^2}{(2\pi)^3}}\int d^3\mathbf{k}\sqrt{\frac{\omega}{2}}\sum_{\lambda=1}^{2}\epsilon(\mathbf{k},\lambda)\left(a(\mathbf{k},\lambda)e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - a^+(\mathbf{k},\lambda)e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\right) \tag{4.80}$$

$$\mathbf{B} = \mathrm{i}\sqrt{\frac{\hbar\mu_0 \mathrm{c}^2}{(2\pi)^3}}\int d^3\mathbf{k}\sqrt{\frac{1}{2\omega}}\sum_{\lambda=1}^{2}\mathbf{k}\mathrm{x}\epsilon(\mathbf{k},\lambda)\left(a(\mathbf{k},\lambda)e^{i(\mathbf{k}.\mathbf{x}-\omega t)} - a^+(\mathbf{k},\lambda)e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\right), \tag{4.81}$$

where $\omega = c\,|\mathbf{k}| = ck$ and $\epsilon(\mathbf{k},\lambda)$ is the vector polarization with $\lambda = 1, 2$. The substitution of the fields expressions in (4.79) with the use of the following relations

$$\delta^3(\mathbf{k}-\acute{\mathbf{k}}) = \tfrac{1}{(2\pi)^3}\int d^3\mathbf{x}e^{i(\mathbf{k}-\acute{\mathbf{k}}).\mathbf{x}}, \qquad \epsilon(\mathbf{k},\lambda).\epsilon(\mathbf{k},\rho) = \delta_{\lambda\rho}, \qquad \mathbf{k}.\epsilon(\mathbf{k},\lambda) = 0$$

lead to

$$H = \int d^3\mathbf{k}\frac{\hbar\omega}{2}\sum_{\lambda=1}^{2}\big(a^{+}(\mathbf{k},\lambda)a(\mathbf{k},\lambda) + a(\mathbf{k},\lambda)a^{+}(\mathbf{k},\lambda)\big) \tag{4.82}$$

and we also have

$$\Big[a(\mathbf{k},\lambda), a^{+}(\acute{\mathbf{k}},\rho)\Big] = \delta_{\lambda\rho}\delta^3(\mathbf{k}-\acute{\mathbf{k}}). \tag{4.83}$$

The last equation in (4.82) gives

$$H = \int d^3\mathbf{k}\hbar\omega\sum_{\lambda}\left(a^{+}(\mathbf{k},\lambda)a(\mathbf{k},\lambda) + \frac{1}{2}\delta^3(0)\right). \tag{4.84}$$

The zero-point energy of the Hamiltonian (4.84) is given by

$$E =< 0 \mid H \mid 0 >= \delta^3(0)\int d^3\mathbf{k}\sum_{\lambda}\frac{\hbar\omega}{2}, \tag{4.85}$$

and it is obvious this expression diverge denoting an infinite vacuum. The introducing of box renormalization [115, 116] in which $\delta^3(\mathbf{k}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{x}e^{i\mathbf{k}.\mathbf{x}} \to \frac{V}{(2\pi)^3}$ for $\mathbf{k}$ tends to zero, and thence we obtain

$$\begin{aligned}\rho_{vac} =< 0 \mid \rho \mid 0 >=< \rho >_{vac} &= \frac{E}{V} = \frac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\hbar\omega \\ &= \frac{1}{2}\int\frac{4\pi k^2 dk}{(2\pi)^3}\hbar\omega = \frac{\hbar}{(2\pi)^2 c^3}\int\omega^3 d\omega.\end{aligned} \tag{4.86}$$

In order to avoid the divergence, practically, a cut-off can be imposed

$$\rho_{vac} = \frac{\hbar}{(2\pi)^2 c^3}\int_0^{\omega_{max}}\omega^3 d\omega = \frac{\hbar\omega_{max}^4}{16\pi^2 c^3}. \tag{4.87}$$

At the Plank scale $E_P = \hbar\omega_{max} = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{-19}GeV$, and therefore we get the value of the vacuum density in natural units as [116]

$$\rho_{vac} \sim 10^{76}GeV^4 \tag{4.88}$$

4.6.1.3 The cosmological constant problem

The problem of cosmological constant is the high discrepancy between the observed value of cosmological constant and its estimated value in quantum mechanics [117]. In other words, the zero energy vacuum cannot give a large cosmological constant. As an illustrative example (see [28]), in semiclassical approach in quantum field theory the gravity is treated as classical field and the zero energy vacuum as scalar field. Thus in Einsteins' equations (2.24), one can see that [28]

$$\Lambda \to \Lambda + \frac{8\pi G}{c^4} < \rho >_{vac} . \tag{4.89}$$

Naively, in case where Λ in the right equal to zero we get

$$\Lambda = \frac{8\pi G}{c^2} < \rho >_{vac} . \tag{4.90}$$

In contrary, and unfortunately, if we compare the observed value of cosmological constant density $\rho_\Lambda = \frac{3H_0^2}{8\pi G}\Omega_\Lambda \sim 10^{-48} GeV^4$ with (4.88) we find

$$\frac{\rho_{vac}}{\rho_\Lambda} \sim 10^{120}. \tag{4.91}$$

This conflicting with (4.90). Hence, the cosmological constant problem remains the biggest challenge in modern physics.

4.6.2 Fluctuations vs electromagnetic lambda term

As mentioned above, the Lamb-shift and other phenomena can be explained by the vacuum fluctuations. Despite this, here we are going to give another way of seeing things in which the geometric description plus the introducing of electromagnetic lambda together give a new explanation for the Lamb shift. As usual in quantum mechanics, we should look for a Hamiltonian to the system (hydrogen atom in our study). The first thing we think about it is the equation of motion. For that, the transition from classical form to an operator form in the equation (4.75) allows us to write this

$$\begin{array}{c} \left[\frac{\hat{p}^2(r)}{2m} + \frac{\hat{L}^2}{2m\hat{r}^2}\left[\left(1-\frac{\alpha}{\hat{r}}\right)^2 - \frac{\Lambda_e}{3}\hat{r}^2\right] - \frac{mc^2\alpha}{2\hat{r}}\left(2-\frac{\alpha}{\hat{r}}\right) - \frac{mc^2\Lambda_e}{6}\hat{r}^2\right]\psi(r,\theta,\phi) \\ = E\psi(r,\theta,\phi) \end{array}, \tag{4.92}$$

where the energy can be taken as

$$E = \frac{mc^2}{2}(k^2 - 1). \tag{4.93}$$

The equation (4.92) is a Schrödinger type. Properly, The comparison between (4.92) and the ordinary Schrödinger's equation of the hydrogen atom leads to [23]

$$\hat{H} = \hat{H}_0 + \hat{H}_p \tag{4.94}$$

where $\hat{H}_0 = \frac{\hat{p}^2(r)}{2m} + \frac{\hat{L}^2}{2m\hat{r}^2} - \frac{mc^2\alpha}{\hat{r}}$ is the usual Hamiltonian of the hydrogen atom and the additional terms are taken to be a perturbed Hamiltonian to the Schrödinger one [23]

$$\hat{H}_p = -\frac{\alpha\hat{L}^2}{2m\hat{r}^3}\left(2 - \frac{\alpha}{\hat{r}}\right) - \frac{\Lambda_e\hat{L}^2}{6m} + \frac{mc^2\alpha^2}{2\hat{r}^2} - \frac{mc^2\Lambda_e}{6}\hat{r}^2. \tag{4.95}$$

If we use the following eigenvalue expression

$$\hat{L}^2 Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\theta,\phi), \tag{4.96}$$

we get another form of $\hat{H}_p$ as

$$\hat{H}_{pr\ell} = -\frac{\alpha\ell(\ell+1)\hbar^2}{2m\hat{r}^3}\left(2 - \frac{\alpha}{\hat{r}}\right) - \frac{\Lambda_e\ell(\ell+1)\hbar^2}{6m} + \frac{mc^2\alpha^2}{2\hat{r}^2} - \frac{mc^2\Lambda_e}{6}\hat{r}^2. \tag{4.97}$$

It should note that the conditions on parameters r, θ and ϕ in the wave function are independent from the classical motion. Maybe the conditions stay the same for the s orbital.

So the geodesic equation and the vacuum field enabled us to find a corrective Schrodinger's Hamiltonian. The rest is to find the corrected energies from this Hamiltonian. Based on perturbation theory [118], we find the correction terms of the hydrogen atom energy levels as follows

For the ground state (1s: n=1,ℓ=0, m=0)

At first we have to take into consideration that the perturbed Hamiltonian in (4.97) is purely radial and it is not affect the angular part of the bound states and also we have

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m Y_{\acute{\ell}}^{\acute{m}*} \sin\theta d\theta d\phi = \delta_{\ell\acute{\ell}}\delta_{m\acute{m}}. \tag{4.98}$$

Besides, the degeneracy of the hydrogen atom bound stats is $g_n = n^2$, and thus the ground state is non-degenerate. Therefore, at first order the energy correction is calculated as follows

$$\begin{aligned}
E^{(1)}_{p(1s)} &=< 100 \mid H_p \mid 100 >= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |R_{10}|^2\, H_p Y_0^0 Y_0^{0*} r^2 dr \sin\theta d\theta d\phi \\
&= \int_0^{\infty} |R_{10}|^2\, H_{pr\ell=0} r^2 dr = \int_0^{\infty} \left|2a_0^{-\frac{3}{2}} e^{-\frac{r}{a_0}}\right|^2 \left(\frac{mc^2\alpha^2}{2r^2} - \frac{mc^2\Lambda_e}{6}r^2\right) r^2 dr \\
&= 2mc^2 a_0^{-3}\int_0^{\infty} e^{-2\frac{r}{a_0}}\left(\alpha^2 - \frac{\Lambda_e}{3}r^4\right) dr,
\end{aligned} \tag{4.99}$$

and with the help of the integral $\int_0^\infty r^n e^{-\beta r} dr = \frac{n!}{\beta^{n+1}}$ we can get

$$
\begin{aligned}
E^{(1)}_{p(1s)} &= 2mc^2 a_0^{-3}\left[\alpha^2\frac{a_0}{2} - \frac{\Lambda_e}{3}\left(24\left(\frac{a_0}{2}\right)^5\right)\right] \\
&= 2mc^2 a_0^{-3}\left(\alpha^2\frac{a_0}{2} - \Lambda_e\frac{a_0^5}{4}\right) \\
&= \frac{mc^2\alpha^2}{a_0^2} - \frac{mc^2\Lambda_e a_0^2}{2}.
\end{aligned} \tag{4.100}
$$

From that the ground state energy becomes [23]

$$
E = E^{(0)}_{1s} + E^{(1)}_{p(1s)} = E_0 + E^{(1)}_{p(1s)} = -13.6\,eV + \frac{mc^2\alpha^2}{a_0^2} - \frac{mc^2\Lambda_e a_0^2}{2}. \tag{4.101}
$$

Additionally, the eigenfunction of the ground state, at the first order correction it is given with

$$
\mid \psi_{1s} >=\mid 100 > + \sum_{n\ell m\neq 100} \frac{< n\ell m \mid H_p \mid 100 >}{E^{(0)}_{1s} - E^{(0)}_{n\ell m}} \mid n\ell m >=\mid 100 >, \tag{4.102}
$$

and thus there is no first order corrections on the eigenfunction of the ground state.

For state n=2

This state is degenerate with $g_n = 4$. In this case the correction terms are given by the following matrix

$$
E_p = \begin{array}{c|cccc}
 & \mid 200 > & \mid 211 > & \mid 210 > & \mid 21-1 > \\
< 200 \mid & H_p & H_p & H_p & H_p \\
< 211 \mid & H_p & H_p & H_p & H_p \\
< 210 \mid & H_p & H_p & H_p & H_p \\
< 21-1 \mid & H_p & H_p & H_p & H_p
\end{array} . \tag{4.103}
$$

The condition (4.98) asserts that all of the elements of this matrix are zero except the diagonals' elements. The radial perturbed Hamiltonian (4.97) imply that for a states with different n and ℓ have a different corresponding energies and that leads to $E^{(1)}_{p(2p_x)} = E^{(1)}_{p(2p_y)} = E^{(1)}_{p(2p_z)}$.

For the state $n = 2, \ell = 0$ and by following the same steps in (4.99) we obtain

$$
\begin{aligned}
E^{(1)}_{p(2s)} &=< 200 \mid H_p \mid 200 >= \frac{mc^2}{2a_0^3}\int_0^\infty \left(1 - \frac{r}{2a_0}\right)^2 e^{-\frac{r}{a_0}}\left(\frac{\alpha^2}{2r^2} - \frac{\Lambda_e}{6}r^2\right) r^2 dr \\
&= \frac{mc^2\alpha^2}{8a_0^2} - 7mc^2\Lambda_e a_0^2.
\end{aligned} \tag{4.104}
$$

Thus we get [23]

$$E_{2s} = \frac{E_0}{4} + \frac{mc^2\alpha^2}{8a_0^2} - 7mc^2\Lambda_e a_0^2. \tag{4.105}$$

For the state $n = 2, \ell = 1$ we find this

$$\begin{aligned} E^{(1)}_{p(2p)} &=< 21 \mid H_{pr\ell=1} \mid 21 > \\ &= \frac{1}{24a_0^5}\int_0^\infty \left(-\frac{\alpha\hbar^2}{mr^3}\left(2-\frac{\alpha}{r}\right) - \frac{\Lambda_e\hbar^2}{3m} + \frac{mc^2\alpha^2}{2r^2} - \frac{mc^2\Lambda_e}{6}r^2\right) r^4 e^{-\frac{r}{a_0}} dr \\ &= -\frac{\alpha\hbar^2}{12ma_0^3} + \frac{\alpha^2\hbar^2}{24ma_0^4} + \frac{mc^2\alpha^2}{24a_0^2} - \frac{\Lambda_e\hbar^2}{3m} - 5mc^2\Lambda_e a_0^2 \end{aligned} \tag{4.106}$$

and that leads to [23]

$$E_{2p} = \frac{E_0}{4} - \frac{\alpha\hbar^2}{12ma_0^3} + \frac{\alpha^2\hbar^2}{24ma_0^4} + \frac{mc^2\alpha^2}{24a_0^2} - \frac{\Lambda_e\hbar^2}{3m} - 5mc^2\Lambda_e a_0^2. \tag{4.107}$$

Then, we work to reach the Lamb shift experimental value ($\Delta E_{Lamb} = 4.372 \times 10^{-6}\, eV$), and this is can be done by determining the cosmological constant Λ_e of the electromagnetic universe. Hence, if we impose that

$$E_{2s} - E_{2p} = \frac{\alpha\hbar^2}{12ma_0^3} - \frac{\alpha^2\hbar^2}{24ma_0^4} + \frac{mc^2\alpha^2}{12a_0^2} + \frac{\Lambda_e\hbar^2}{3m} - 2mc^2\Lambda_e a_0^2 = \Delta E_{Lamb}. \tag{4.108}$$

Then we can obviously get this

$$\Lambda_e = \frac{3m}{(\hbar^2 - 6(mca_0)^2)}\left[\Delta E_{Lamb} - \frac{\alpha\hbar^2}{12ma_0^3} + \frac{\alpha^2\hbar^2}{24ma_0^4} - \frac{mc^2\alpha^2}{12a_0^2}\right] \tag{4.109}$$

and after numerical applications we find [23]

$$\Lambda_e = 8.4413239602 \times 10^{10}\, m^{-2}. \tag{4.110}$$

From this, we can drawn a constant with length characteristic

$$l_e = \frac{1}{\sqrt{\Lambda_e}} = 3.4418720060 \times 10^{-6}\, m. \tag{4.111}$$

Because we talk here about the existence of two different vacuums of two universes. there is no point in comparing the observed cosmological constant of gravity and the constant Λ_e. We now return to the length l_e which represent the horizon of the hydrogen atom. This constant can indicate the quantum limit of the electron of a hydrogen atom so that when the electron exceeds this distance, it becomes free and unbound. We can also consider this distance as the necessary distance for the Casimir effect to appear.

For all different values of n and ℓ

In order to give a general energy expression, we use the following results [118]

$$< n\ell \mid r^2 \mid n\ell >= \frac{n^2}{2}\left(5n^2+1-3\ell(\ell+1)\right)a_0^2 \tag{4.112}$$

$$< n\ell \mid r^{-2} \mid n\ell >= \frac{2}{n^3\,(2\ell+1)\,a_0^2} \tag{4.113}$$

$$< n\ell \mid r^{-3} \mid n\ell >= \frac{2}{n^3\ell\,(\ell+1)\,(2\ell+1)\,a_0^3} \tag{4.114}$$

$$< n\ell \mid r^{-4} \mid n\ell >= \frac{4\left[3n^2-\ell\,(\ell+1)\right]}{n^5\ell\,(\ell+1)\,(2\ell+1)\left[(2\ell+1)^2-4\right]a_0^4} \tag{4.115}$$

to obtain from them [23]

$$\begin{aligned} E_{n\ell}(\Lambda_e) = {} & \frac{E_0}{n^2} - \frac{2\alpha\hbar^2}{m}\frac{1}{n^3\,(2\ell+1)\,a_0^3} + \frac{2\alpha^2\hbar^2}{m}\frac{\left[3n^2-\ell\,(\ell+1)\right]}{n^5\,(2\ell+1)\left[(2\ell+1)^2-4\right]a_0^4} \\ & - \frac{\Lambda_e\ell(\ell+1)\hbar^2}{6m} + \frac{mc^2\alpha^2}{n^3\,(2\ell+1)\,a_0^2} - \frac{mc^2\Lambda_e}{12}n^2\left(5n^2+1-3\ell(\ell+1)\right)a_0^2. \end{aligned} \tag{4.116}$$

The energies numerical values given by the formula (4.116) for $n = 1...7$ levels are presented in the table below (Table 4.1)

n	$\frac{E_0}{n^2}$	ℓ	$E_p^{(1)}$	$E_{n\ell}(\Lambda_e)$	$\bar{E}_p^{(1)}$	$\left(\bar{E}_p^{(1)}/\frac{E_0}{n^2}\right)$ X 100
1	-13.600	0	-0.0015030	-13.6002	0.0015030	0.011051
2	-3.400	0	-0.0010308	-3.4001	8.4923e-4	0.024977
		1	-6.6762e-04	-3.4007		
3	-1.5111	0	-0.0042454	-1.5154	0.0034903	0.23097
		1	-0.0036630	-1.5148		
		2	-0.0025623	-1.5137		
4	-0.8500	0	-0.013145	-0.86315	0.010702	1.2591
		1	-0.012158	-0.86216		
		2	-0.010211	-0.86021		
		3	-0.0072937	-0.85729		
5	-0.54400	0	-0.031907	-0.57591	0.025824	4.7471
		1	-0.030381	-0.57438		
		2	-0.027342	-0.57134		
		3	-0.022784	-0.56678		
		4	-0.016709	-0.56071		
6	-0.37778	0	-0.065985	-0.44376	0.053222	14.088
		1	-0.063794	-0.44157		
		2	-0.059419	-0.43720		
		3	-0.052857	-0.43063		
		4	-0.044108	-0.42189		
		5	-0.033172	-0.41095		
7	-0.27755	0	-0.12206	-0.39961	0.098240	35.395
		1	-0.11908	-0.39663		
		2	-0.11312	-0.39068		
		3	-0.10419	-0.38174		
		4	-0.092285	-0.36984		
		5	-0.077401	-0.35495		
		6	-0.059539	-0.33709		

Table 4.1. The first order corrections of Bohr's energies levels for $n = 1...7$ and its corresponding sub-levels in electron-Volt.

Were $\bar{E}_p^{(1)} = \frac{1}{n}\sum_0^{n-1} E_p^{(1)}$ in the table denotes the average of the sublevels perturbed energies.

However, it is well to note that in the case where $\Lambda_e = 0$, the equation (4.108) becomes

$$E_{2s} - E_{2p} = \frac{\alpha\hbar^2}{12ma_0^3} - \frac{\alpha^2\hbar^2}{24ma_0^4} + \frac{mc^2\alpha^2}{12a_0^2}. \tag{4.117}$$

This results in a shift of $\Delta E = 2.4595087102\,10^{-4} < \Delta E_{Lamb}$, and for that reason, we need some contributions to reach the exact value of the shift. In this way, the electromagnetic lambda constant has been proposed.

Judging by the title (Fluctuations vs electromagnetic lambda term Λ_e), we should add some comments here. In gravity one can accept that the cosmological constant is the zero-energy arises from the fluctuations of the gravitational field [119, 120]. By the same token, the vacuum energy arises from the fluctuations of electromagnetic field can be considered as an additional energy comes from contributions of the electromagnetic lambda term. Moreover, in some cases the effect of the vacuum fluctuations on the hydrogen atom is taken as a corrections to the Coulomb's potential $V(r+\delta r) = V(r) + \delta r \frac{dV(r)}{dr} + ...$(see [109, 121]), we saw a same procedure in the equation (4.76). In the same way and as the results of this work stated, the contributions of the geometric description plus the electromagnetic lambda term are together participate to modify the motion of charged particles.

4.6.3 Energy divergence and the cut-off

It is obvious that the energy expression (4.116) is divergent when $n \to \infty$. This, in fact, is not a biggest problem and a same situation has been encountered in harmonic oscillator energy expression. When the solution was that the series have to stop at $n = N_{max}$ to avoid the divergence of both the eigenfunctions and as consequent the energy. By following a similar process but in this case we use a cut-off technique instead of N_{max}. This makes us think on the existence of a relationship between the cut-off (the horizon of hydrogen atom) and N_{max} . The source of the divergence in the terms of perturbation is the infinity. For that reason, the cut-off is used through this expression

$$f(\Lambda_e) = \int_0^{\frac{1}{\sqrt{\Lambda_e}}} \left[H_{pr\ell=0} \left|R_{20}\right|^2 - H_{pr\ell=1} \left|R_{21}\right|^2 \right] r^2 dr - \Delta E_{Lamb}. \tag{4.118}$$

One can use a software to compute this integral. Anyway, the plot of this function is given in Figure 4.1. From this plot we can determine graphically (by finding solutions of $f(\Lambda_e) = 0$) the new value of the constant Λ_e so that we get [23]

$$\Lambda_{e_{graph}} \approx 6.16658 \times 10^{10} m^{-2}. \tag{4.119}$$

Hence, we can determine the cut-off value with

$$l_{e_{graph}} = \frac{1}{\sqrt{\Lambda_{e_{graph}}}} \approx 4.02696 \times 10^{-6} m. \tag{4.120}$$

These values are not far from the previously obtained values. Now we will try to follow our intuition to develop an attempt to find the relationship between the cut-off and N_{max}. The preliminary thinking lead us to think about Bohr's relation $r_n = a_0 n^2$ which allows us to find

$$N_{max} = \sqrt{\frac{l_{e_{graph}}}{a_0}} \approx 276. \tag{4.121}$$

Also, there is another problem in which the values of perturbed energies (at first order) are close to the values of the unperturbed energies (zero order). These are clearly shown

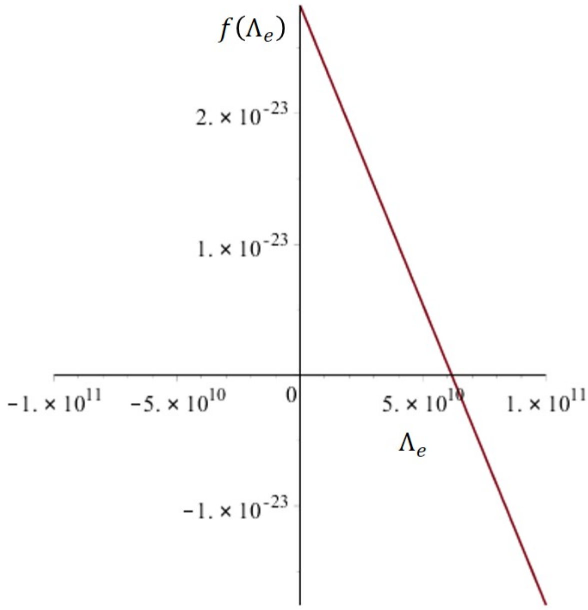

Figure 4.1: The plot representation of $f(\Lambda_e)$

in the percentage $\bar{E}_p^{(1)}/\frac{E_0}{n^2}$ of the table and for a much higher number of n. This maybe due to the fact that the additional field from vacuum breaks the spherical symmetry. In the same line of thought, one can think about cylindrical symmetry of hydrogen atom rather than the spherical one.

4.7 Attempt to a geometrical description of magnetic field

We start with the electromagnetic metric in weak field limit where we saw that the magnetic terms are implicit in the components g_{0i}. In addition to that there is an equivalence between rotation and the magnetic field effect [59, 66] shown with this expression

$$\omega = \frac{qB}{2mc}. \tag{4.122}$$

A similar relation can be derived in the case of classical circular motion of a charged particle under the effect of a uniform magnetic field and we have to write

$$F_c = \frac{mv^2}{r} = qvB, \tag{4.123}$$

where $v = r\omega$ and that gives

$$\omega = \frac{qB}{m}. \tag{4.124}$$

For those reasons the magnetic field can be represented geometrically by a rotating spacetime. Therefore, we chose the rotating cylindrical coordinates as an example, so we have

$$ds^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2 \tag{4.125}$$

after the rotation transformation $d\phi = d\phi + \omega dt$ we get [122]

$$ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2dt^2 - 2\omega r^2 dtd\phi - dr^2 - r^2d\phi^2 - dz^2. \tag{4.126}$$

In equivalent way we can obtain

$$ds^2 = \left(1 - \left(\frac{qB}{mc}r\right)^2\right) c^2dt^2 - 2\left(\frac{q}{mc}\right) Br^2cdtd\phi - dr^2 - r^2d\phi^2 - dz^2. \tag{4.127}$$

According to the square of the term $\frac{q}{m}$ in the component g_{00}, the term $\left(\frac{qB}{mc}r\right)^2$ it can be considered as an approximation of second order. The problem is that the use of equation (4.124) is it correct or not? For a check, we have to search for some compatibility between the geodesic equation of the metric (4.127) and the ordinary Lorentz equation of motion. When we talk about the geometrical magnetic field, the difficulty arises in choosing the correct symmetry.

4.8 Discussion of the results

The new phenomena shown here are the perihelion advance for a charged particle in classical motion, and the corresponding Lamb shift in quantum mechanics. However, this should not be seen as a quantification of the charged particle trajectories [123]. Properly speaking, the trajectory determines exact the position of, for example, the electron in classical motion. In the other hand the atomic orbital (a state represented in Hilbert space) is used to give the probability region of the existence of that electron. The difference appears clearly in the classical approach where the solution leads to $\theta = \frac{\pi}{2}$. This condition was abandoned in the quantum application.

In our work, it is true that we succeeded in determining the value of the term lambda electromagnetic in quantum motion, but what about its value in classical? Our expectations state that the value of classical lambda (Λ_e) is different from the quantum one. Because the classical Λ_e just denote a classical vacuum but the quantum Λ_e represent a vacuum with a quantum effect. In addition to that and as we have noted, the constant concerns the limit of the quantum effect. Thereby, one can expect a presence of quantized electromagnetic lambda [124].

Going back to chapter three, where we briefly talked about the photon mass. There it can be said that the photon mass is equivalent to

$$m_\gamma = \frac{\hbar}{c}\sqrt{2\Lambda_e} \sim 10^{-37} Kg. \tag{4.128}$$

And as we discussed in the chapter, the inequality of the cosmological constant and lambda lead to a tiny difference on the propagation speed between gravitational and electromagnetic waves. In other words perhaps this difference explain why the electromagnetism is much stronger than gravity.

Obviously, from the equation (4.78), we can give the value of the vacuum density to the hydrogen atom as

$$\rho_{vac}(\Lambda_e) = \frac{1}{3}\frac{m_e}{e}\frac{\Lambda_{e_{graph}}}{\mu_0} = 9.3002 \times 10^4 C\cdot m^{-3}. \tag{4.129}$$

We should note that an induced charge density can be generated from the vacuum polarization [125].

The importance of the electromagnetic lambda term appears in the early universe. Where it is possible to ask about the amount of matter Ω_{Λ_e} for this constant, and how the contribution of this matter affects the geometry of the universe? While an effective cosmological constant comes from an electromagnetic field is possible in cosmic scale [126]. On the other hand, massive photons maybe could explain the effect of dark energy. In this case, we can abandon the cosmological constant. Furthermore, All of this motivates us to search for a unifying theory between gravity and electromagnetism.

For a toy model to the unification in the electromagnetism universe, one can think about a five dimensional spacetime. The four dimension is the electromagnetism metric and the fifth dimension is for the gravitation vector potential. In this case we can get rid of the charge quantization problem in Kaluza-Klein theory. A bi-metric theory can also be a way of thinking towards the unification.

Chapter 5

General Conclusion

In this thesis, we presented a novel approach in the light of Barros idea, which states that the electromagnetic interaction affects the spacetime structure, as general relativity did to describe gravity. In this regard, we can see the electromagnetic field geometrically depends on the spacetime curvature. In which the full analogy between the two interactions in weak field limit is taken into consideration.

As we have said before, the main reason for most developments in theoretical physics is the change of our vision about spacetime concept. This what leads us to think about a new spacetime with a different properties, and we called it the electromagnetic universe. This spacetime has a direct relation with the observer properties which are, in our work, the charge and mass, or in short, the ratio $\frac{q}{m}$. In addition, the spacetime type changes with the interaction characteristics. Therefore, in electromagnetic universe all points in its spacetime become marked with the report $\frac{q}{m}$, and this what leads us to establish another concept of the equivalence principle. In this case, the observer and all the particles which have the same $\frac{q}{m}$ ratio move, by the same way, under the influence of the electromagnetic field on the geometry. This means that those particles follow the same worldline called geodesic in a spacetime curved by the electromagnetic field. Subsequently, the observer itself becomes a part of this geometry or, in other words, a subject of this interaction. Then, we cannot disconnect that observer from the electromagnetic effect.

The equivalence principle means a type of Einsteins' equations for the electromagnetism. This means a metric, and the geodesic of that metric means an equation of motion for charged particles. This is what we specifically detailed in this work so that in Einsteins' equations of the electromagnetic universe, we equated the tensor of electromagnetic energy with geometry (Einstein's tensor). For the Coulomb's field, the advantages of spherically symmetric allow to give an exact metric-solution of these equations. Of course, the thing that we gain from geometrization is the precision of particles dynamics. This is really what happened with the equation of motion for charged particles, and more than that, we recorded an advance of perihelion in the case of classical circular motion for those particles. However, the price to be paid in the case of curved spacetime is time. In this occasion, a distinction must be made between an observer who belongs to the universe of electromagnetism and the one who does not. The first (who belongs) is attached with the charged particle might see its spacetime marked by the $\frac{q}{m}$ ratios, and he will record a time-slowdown in its clock. In quantum mechanics, the correction terms appear in the Hamiltonian of the system. In Schrödinger's hydrogen atom, these terms lead to the energy level splitting and accordingly, we set out a shift with the order $10^{-4}\,eV$ between the orbitals $2s$ and $2p$.

The inserting of the vacuum notion was the solution to make the shift reach the experimental Lamb value . In general relativity, the cosmological constant denotes the vacuum of gravity interaction, and it is considered as a fluid with negative pressure . Accordingly, in order to define geometrically the vacuum of electromagnetism, we need to another constant and we called it the electromagnetic lambda term. This constant is not considered as a fluid in this case but as an induced field from vacuum. In classical treatment, we set some properties for this vacuum in which the vacuum density becomes dependent on the particle properties. Besides this, we have noticed a possibility of a negative permeability of that induced field. The corrective terms to the Coulomb's potential arise from the geometry of spacetime, in addition to the term Λ_e are taken as a perturbation to the Schrödinger's hydrogen atom. Those terms allowing us to explain the Lamb shift which is not predicted by Dirac theory and furthermore without the need of QED explanation.

Lastly, we refer to some difficulties that we have encountered in this research. First, we determine the metric by using both Einstein and Maxwell equations. One can ask why the Maxwells' equations are used in curved spacetime. This means that a part of the field described geometrically and the other part keeps the ordinary definition. Second, the problem of the depending of the metric with particle properties, and for this we should link a different metrics to a different particles. The third problem is the unification between the two different spacetimes gravity and electromagnetism. In addition to another problems such as

- The absence of a good explanation for a magnetic metric.
- The problem of Heisenberg uncertainty in curved spacetime.
- The ambiguous definition of vacuum in our quantum applications.
- The electromagnetism is a gauge theory and indeed, we need to a geometrical gauge theory.

Therefore, our perspectives will be to look for a solutions of those problems.

Appendix A

A.1 Tetrad fields

If we deal with the dynamics of massive spin-half particles in curved spacetime it is better to define at every points of that spacetime a local inertial frames [84]. Accordingly, to define a vector field $V^\mu(x)$ (or tensor field) of general coordinates with respect to a scalar field $V^a(x)$ in local inertial frames, we have to go through the tetrad fields definition[84]

$$V^\mu(x) = e^\mu_a(x) V^a(x) \tag{A.1}$$

in the other side

$$V^a(x) = e^a_\mu(x) V^\mu(x). \tag{A.2}$$

Where $e^\mu_a(x)$ is the so-called tetrad fields and $e^a_\mu(x)$ is its inverse, so that they fulfill the orthonormality conditions

$$\begin{array}{c} e^\mu_a e^a_\nu = \delta^\mu_\nu \\ e^a_\mu e^\mu_b = \delta^a_b \end{array} . \tag{A.3}$$

The Latin indices $a, b...$ denote the local frames and the Greece indices $\rho, \mu, \nu...$ refer to the general coordinates. The tetrad $e^\mu_a(x) \equiv e^\mu_a$ is a four vector field forms an orthonormal basis in general coordinates. Generally, the tetrad caries all informations about the spacetime curvature and then we have

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x) \tag{A.4}$$

and vice versa

$$\eta_{ab} = g_{\mu\nu}(x) e^\mu_a(x) e^\nu_b(x) \tag{A.5}$$

We should note from the equation (A.2) that $V^a(x)$ is a scalar field in general coordinates and transforms as a vector in local Lorentz frames [84]

$$\widetilde{V}^a(\widetilde{x}) = \Lambda^a_b(x) V^b(x) \tag{A.6}$$

A.2 Dirac equation in curved spacetime

In Dirac equation ψ is a spinor field and it is not a tensor field, and the problem is the undefined behavior of this spinor field in curved spacetime [84]. This problem arises because we need to describe this spinor in general coordinates and there is no meaning to do that. Consequently, to deal with this problem the spinors field should be observed from the local inertial frames described by tetrad fields. Therefore, the form of Dirac's equation in curved spacetime becomes

$$(i\gamma^{\mu} D_{\mu} - m)\psi = 0 \tag{A.7}$$

where $\gamma^{\mu} = e^{\mu}_{a}\gamma^{a}$ are Dirac matrices in curved spacetime expressed terms of ones in Minkowski spacetime γ^{a} with

$$\begin{array}{c} \gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a} = -2\eta^{ab} \\ \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = -2g^{\mu\nu} \end{array} . \tag{A.8}$$

The covariant derivative is taken to be $D_{\mu} = \partial_{\mu} - \Gamma_{\mu}$in which the term Γ_{μ} is the spin connection

$$\Gamma_{\mu} = -\frac{1}{4}\gamma^{a}\gamma^{b}e^{\nu}_{a}\nabla_{\mu}e_{b\nu}. \tag{A.9}$$

From that it is clear to prove this

$$\Gamma_{\mu} = -\frac{1}{8}g_{\lambda\alpha}\Gamma^{\alpha}_{\mu\rho}\left[\gamma^{\lambda}, \gamma^{\rho}\right]. \tag{A.10}$$

In fact

$$\begin{aligned} \Gamma_{\mu} &= -\frac{1}{4}\gamma^{a}\gamma^{b}e^{\nu}_{a}\nabla_{\mu}e_{b\nu} \\ &= -\frac{1}{4}\gamma^{\nu}\gamma^{\lambda}e^{b}_{\lambda}\left(\partial_{\mu}e_{b\nu} - \Gamma^{\rho}_{\mu\nu}e_{b\rho}\right) \end{aligned} \tag{A.11}$$

we have $g_{\mu\nu} = e^{b}_{\mu}e_{b\nu}$ and that leads

$$\nabla_{\mu}g_{\lambda\nu} = \left(\nabla_{\mu}e^{b}_{\lambda}\right)e_{b\nu} - e^{b}_{\lambda}\left(\nabla_{\mu}e_{b\nu}\right) = 0 \tag{A.12}$$

and then

$$e^{b}_{\lambda}\partial_{\mu}e_{b\nu} = \Gamma^{\rho}_{\mu\lambda}g_{\rho\nu} + \Gamma^{\rho}_{\mu\nu}g_{\rho\lambda} - e_{b\nu}\partial_{\mu}e^{b}_{\lambda}. \tag{A.13}$$

The equation (A.13) in (A.11) gives

$$\Gamma_{\mu} = -\frac{1}{4}\gamma^{\nu}\gamma^{\lambda}\left(\Gamma^{\rho}_{\mu\lambda}g_{\rho\nu} - e_{b\nu}\partial_{\mu}e^{b}_{\lambda}\right). \tag{A.14}$$

The sum of the last equation with (A.11) yields

$$2\Gamma_\mu = -\frac{1}{4}\left(\gamma^\nu\gamma^\lambda\Gamma^\rho_{\mu\lambda}g_{\rho\nu} - \gamma^\nu\gamma^\lambda\Gamma^\rho_{\mu\nu}g_{\rho\lambda}\right) \tag{A.15}$$

The permutation in the indices ν and λ gives

$$\Gamma_\mu = -\frac{1}{8}g_{\lambda\alpha}\Gamma^\alpha_{\mu\rho}\left[\gamma^\lambda,\gamma^\rho\right] \tag{A.16}$$

A.3 Applications

As we discussed in the chapter three the hydrogen atom becomes geometrically described by the following metric

$$ds^2 = \left(1-\frac{\alpha}{r}\right)^2 dt^2 - \left(1-\frac{\alpha}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.17}$$

where $\alpha = \frac{Ke^2}{mc^2}$. The tetrad of this metric and its inverse are given as follows

$$e^a_\mu = diag\left(\left(1-\frac{\alpha}{r}\right), \left(1-\frac{\alpha}{r}\right)^{-1}, r, r\sin\theta\right) \tag{A.18}$$

$$e^\mu_a = diag\left(\left(1-\frac{\alpha}{r}\right)^{-1}, \left(1-\frac{\alpha}{r}\right), \frac{1}{r}, \frac{1}{r\sin\theta}\right). \tag{A.19}$$

Besides that from the connection formula

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}),$$

we can give all the non-zero connections of that metric

$$\begin{aligned}
&\Gamma^0_{01} = \frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)^{-1},\ \Gamma^1_{00} = \frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)^3,\ \Gamma^1_{11} = -\frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right),\\
&\Gamma^1_{22} = -r\left(1-\frac{\alpha}{r}\right)^2,\ \Gamma^1_{33} = -r\sin^2\theta\left(1-\frac{\alpha}{r}\right)^2,\ \Gamma^2_{21} = \frac{1}{r},\\
&\Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{31} = \frac{1}{r},\ \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}.
\end{aligned}$$

From that we can also compute the spin connections

$$\Gamma_\lambda = -\frac{1}{8}g_{\mu\alpha}\Gamma^\alpha_{\nu\lambda}\left[\gamma^\mu,\gamma^\nu\right],$$

and then

$$\begin{aligned}
\Gamma_0 &= -\frac{1}{8}g_{\mu\alpha}\Gamma^\alpha_{\nu0}\left[\gamma^\mu,\gamma^\nu\right] = -\frac{1}{8}g_{00}\Gamma^0_{10}\left[\gamma^0,\gamma^1\right] - \frac{1}{8}g_{11}\Gamma^1_{00}\left[\gamma^1,\gamma^0\right]\\
&= -\frac{1}{8}\frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)^2\left(1-\frac{\alpha}{r}\right)^{-1}\left[\gamma^0,\gamma^1\right] + \frac{1}{8}\frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)^{-2}\left(1-\frac{\alpha}{r}\right)^3\left[\gamma^1,\gamma^0\right]\\
&= \frac{1}{4}\frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)\left[\gamma^1,\gamma^0\right],
\end{aligned}$$

$$\Gamma_1 = -\frac{1}{8}g_{00}\Gamma^1_{01}\left[\gamma^0,\gamma^0\right] - \frac{1}{8}g_{11}\Gamma^1_{11}\left[\gamma^1,\gamma^1\right] - \frac{1}{8}g_{22}\Gamma^2_{21}\left[\gamma^2,\gamma^2\right] - \frac{1}{8}g_{33}\Gamma^3_{31}\left[\gamma^3,\gamma^3\right]$$

we have $[\gamma^\mu, \gamma^\nu] = 0$ in the case where $\mu = \nu$ and thence

$$\Gamma_1 = 0,$$

$$\begin{aligned}
\Gamma_2 &= -\frac{1}{8}g_{11}\Gamma^1_{22}\left[\gamma^1,\gamma^2\right] - \frac{1}{8}g_{22}\Gamma^2_{12}\left[\gamma^2,\gamma^1\right] - \frac{1}{8}g_{33}\Gamma^3_{32}\left[\gamma^3,\gamma^3\right] \\
&= -\frac{1}{8}\left(-\left(1-\frac{\alpha}{r}\right)^2\right)\left(-r\left(1-\frac{\alpha}{r}\right)^2\right)\left[\gamma^1,\gamma^2\right] + \frac{1}{8}r^2\left(\frac{1}{r}\right)\left[\gamma^2,\gamma^1\right] \\
&= \frac{r}{4}\left[\gamma^2,\gamma^1\right],
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_3 &= -\frac{1}{8}g_{11}\Gamma^1_{33}\left[\gamma^1,\gamma^3\right] - \frac{1}{8}g_{22}\Gamma^2_{33}\left[\gamma^2,\gamma^3\right] - \frac{1}{8}g_{33}\Gamma^3_{13}\left[\gamma^3,\gamma^1\right] - \frac{1}{8}g_{33}\Gamma^3_{23}\left[\gamma^3,\gamma^2\right] \\
&= -\frac{1}{8}r\sin^2\theta\left(1-\frac{\alpha}{r}\right)^2\left(1-\frac{\alpha}{r}\right)^{-2}\left[\gamma^1,\gamma^3\right] - r^2\sin\theta\cos\theta\left[\gamma^2,\gamma^3\right] \\
&+ \frac{1}{8}r^2\sin^2\theta\frac{1}{r}\left[\gamma^3,\gamma^1\right] + \frac{1}{8}r^2\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)\left[\gamma^3,\gamma^2\right] \\
&= \frac{r}{4}\sin^2\theta\left[\gamma^3,\gamma^1\right] + \frac{r^2}{4}\sin\theta\cos\theta\left[\gamma^3,\gamma^2\right].
\end{aligned}$$

We have to take $1, 2..$ refer to the general coordinates and $(1), (2)...$to the inertial local frames and we use all of this to find the Dirac equation in that case

$$\begin{aligned}
(i\gamma^\mu D_\mu - m)\psi = 0 \\
(i\gamma^\mu\left(\partial_\mu - \Gamma_\mu\right) - m)\psi = 0 \\
(i\gamma^a e^\mu_a\left(\partial_\mu - \Gamma_\mu\right) - m)\psi = 0,
\end{aligned} \tag{A.20}$$

then we get

$$\begin{aligned}
&[i\gamma^{(0)}e^0_{(0)}\partial_0 + i\gamma^{(1)}e^1_{(1)}\partial_1 + i\gamma^{(2)}e^2_{(2)}\partial_2 + i\gamma^{(3)}e^3_{(3)}\partial_3 - i\gamma^{(0)}e^0_{(0)}\Gamma_0 \\
&- i\gamma^{(1)}e^1_{(1)}\Gamma_1 - i\gamma^{(2)}e^2_{(2)}\Gamma_2 - i\gamma^{(3)}e^3_{(3)}\Gamma_3 - m]\psi = 0
\end{aligned}$$

$$[i\gamma^{(0)}\left(1-\frac{\alpha}{r}\right)^{-1}\partial_0 + i\gamma^{(1)}\left(1-\frac{\alpha}{r}\right)\partial_1 + i\gamma^{(2)}\frac{1}{r}\partial_2 + i\gamma^{(3)}\frac{1}{r\sin\theta}\partial_3$$
$$+ i\gamma^{(0)}e^0_{(0)}\frac{1}{4}\frac{\alpha}{r^2}\left(1-\frac{\alpha}{r}\right)\left[\gamma^{(0)},\gamma^{(1)}\right]e^0_{(0)}e^1_{(1)} + i\gamma^{(2)}e^2_{(2)}\frac{r}{4}\left[\gamma^{(1)},\gamma^{(2)}\right]e^1_{(1)}e^2_{(2)}$$
$$+ i\gamma^{(3)}e^3_{(3)}\frac{r}{4}\sin^2\theta\left[\gamma^{(1)},\gamma^{(3)}\right]e^1_{(1)}e^3_{(3)} + i\gamma^{(3)}e^3_{(3)}\frac{r^2}{4}\sin\theta\cos\theta\left[\gamma^{(2)},\gamma^{(3)}\right]e^2_{(2)}e^3_{(3)} - m]\psi = 0. \tag{A.21}$$

We have $\left(\gamma^{(0)}\right)^2 = 1$ and $\left(\gamma^{(i)}\gamma^{(j)}\right)\delta_{ij} = -1$, so we can show that

$$\gamma^{(0)}\left[\gamma^{(0)},\gamma^{(1)}\right] = \gamma^{(0)}\gamma^{(0)}\gamma^{(1)} - \gamma^{(0)}\gamma^{(1)}\gamma^{(0)} \tag{A.22}$$

and we obtain from (A.8) that

$$\gamma^{(0)}\gamma^{(1)} = -\gamma^{(1)}\gamma^{(0)}$$

and therefore

$$\gamma^{(0)}\left[\gamma^{(0)},\gamma^{(1)}\right] = \gamma^{(0)}\gamma^{(0)}\gamma^{(1)} + \gamma^{(0)}\gamma^{(0)}\gamma^{(1)} = 2\gamma^{(1)}$$

By taking this into account the equation (A.21) becomes

$$(i\gamma^{(0)}\left(1-\frac{\alpha}{r}\right)^{-1}\partial_0 + i\gamma^{(1)}\left(1-\frac{\alpha}{r}\right)\partial_1 + i\gamma^{(2)}\frac{1}{r}\partial_2 + i\gamma^{(3)}\frac{1}{r\sin\theta}\partial_3$$
$$+ \frac{i\gamma^{(1)}}{2}\frac{\alpha}{r^2} + \frac{i\gamma^{(1)}}{r}\left(1-\frac{\alpha}{r}\right) + \frac{i\gamma^{(2)}}{2}\frac{1}{r}\cot\theta - m)\psi = 0 \tag{A.23}$$

$$[i\gamma^{(0)}\left(1-\frac{\alpha}{r}\right)^{-1}\partial_0 + i\gamma^{(1)}\left(1-\frac{\alpha}{r}\right)\partial_1 + i\gamma^{(2)}\frac{1}{r}\partial_2 + i\gamma^{(3)}\frac{1}{r\sin\theta}\partial_3$$
$$+ \frac{i\gamma^{(1)}}{r}\left(1-\frac{\alpha}{2r}\right) + \frac{i\gamma^{(2)}}{2}\frac{1}{r}\cot\theta - m]\psi = 0 \tag{A.24}$$

we adopt the following ansatz

$$\psi(t,r,\theta,\phi) = \left(1-\frac{\alpha}{r}\right)^{\frac{1}{2}} e^{i(m_\phi\phi - Et)}\chi(r,\theta). \tag{A.25}$$

By substitution in (A.24)

$$[\gamma^{(0)}\left(1-\frac{\alpha}{r}\right)^{-1}E + i\gamma^{(1)}\left(1-\frac{\alpha}{r}\right)\partial_1 + i\gamma^{(2)}\frac{1}{r}\left(\partial_2 + \frac{1}{2}\cot\theta\right) - \gamma^{(3)}\frac{m_\phi}{r\sin\theta}$$
$$+ \frac{i\gamma^{(1)}}{r} - m]\chi(r,\theta) = 0 \tag{A.26}$$

We have the right to do that [86, 88]

$$\gamma^{(1)} \to \gamma^{(3)}, \;\; \gamma^{(3)} \to \gamma^{(2)}, \;\; \gamma^{(2)} \to \gamma^{(1)}$$

then we get

$$[\gamma^{(0)} \left(1-\frac{\alpha}{r}\right)^{-1} E + i\gamma^{(3)} \left(1-\frac{\alpha}{r}\right) \partial_1 + i\gamma^{(1)} \frac{1}{r} \left(\partial_2 + \frac{1}{2}\cot\theta\right) - \gamma^{(2)} \frac{m_\phi}{r\sin\theta}$$
$$+ \frac{i\gamma^{(3)}}{r} - m]\chi(r,\theta) = 0 \tag{A.27}$$

and we put [86, 88]

$$\chi(r,\theta) = \begin{pmatrix} g(r)\varepsilon(\theta) \\ \\ -ih(r)\sigma^3\varepsilon(\theta) \end{pmatrix} \tag{A.28}$$

where $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the z-Pauli matrix. In addition to that we should define the $\hat{K}$ operator with [86, 88]

$$\hat{K}\varepsilon(\theta) = \left[-\sigma^2 \left(\partial_2 + \frac{1}{2}\cot\theta\right) + i\sigma^1 \frac{m_\phi}{\sin\theta}\right] \varepsilon(\theta) = i\kappa\varepsilon(\theta) \tag{A.29}$$

and also we have

$$\gamma^{(0)} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \;\; \gamma^{(i)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{A.30}$$

By replacing (A.30) and (A.28) in (A.27) we get

$$\begin{pmatrix} \left(1-\frac{\alpha}{r}\right)^{-1} E - m & \begin{array}{c} i\sigma^3 \left(1-\frac{\alpha}{r}\right)\partial_1 + \frac{i\sigma^3}{r} \\ +\frac{i\sigma^1}{r}\left(\partial_2 + \frac{1}{2}\cot\theta\right) - \sigma^2 \frac{m_\phi}{r\sin\theta} \end{array} \\ \\ -i\sigma^3 \left(1-\frac{\alpha}{r}\right)\partial_1 - \frac{i\sigma^3}{r} - \frac{i\sigma^1}{r}\left(\partial_2 + \frac{1}{2}\cot\theta\right) + \sigma^2 \frac{m_\phi}{r\sin\theta} & -\left(1-\frac{\alpha}{r}\right)^{-1} E - m \end{pmatrix}$$
$$\times \begin{pmatrix} g(r)\varepsilon(\theta) \\ \\ -ih(r)\sigma^3\varepsilon(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}$$

and this gives

$$\left(\left(1-\frac{\alpha}{r}\right)^{-1} E - m\right) g(r)\varepsilon(\theta) + \left(\sigma^3\right)^2 \left(1-\frac{\alpha}{r}\right) \frac{dh(r)}{dr}\varepsilon(\theta) + \frac{\left(\sigma^3\right)^2}{r} h(r)\varepsilon(\theta)$$
$$+ \left[\frac{\sigma^1\sigma^3}{r}\left(\partial_2 + \frac{1}{2}\cot\theta\right) + i\sigma^2\sigma^3 \frac{m_\phi}{r\sin\theta}\right] h(r)\varepsilon(\theta) = 0 \tag{A.31}$$

$$\left(\left(1-\frac{\alpha}{r}\right)^{-1}E+m\right)h(r)\sigma^3\varepsilon(\theta)-\sigma^3\left(1-\frac{\alpha}{r}\right)\frac{dg(r)}{dr}\varepsilon(\theta)-\frac{\sigma^3}{r}g(r)\varepsilon(\theta)$$
$$-\left[\frac{\sigma^1}{r}\left(\partial_2+\frac{1}{2}\cot\theta\right)+i\sigma^2\frac{m_\phi}{r\sin\theta}\right]g(r)\varepsilon(\theta)=0. \tag{A.32}$$

We have

$$\sigma^1\sigma^3=-\sigma^3\sigma^1=-i\sigma^2,\ \ \sigma^2\sigma^3=-\sigma^3\sigma^2=i\sigma^1$$
$$\left(\sigma^3\right)^2=\mathbf{1}.$$

From there, the equation (A.31) is simplified as

$$\left(1-\frac{\alpha}{r}\right)\frac{dh(r)}{dr}+\frac{(1-\kappa)}{r}h(r)+\left(\left(1-\frac{\alpha}{r}\right)^{-1}E-m\right)g(r)=0. \tag{A.33}$$

and then the equation σ^3x(A.32) leads to

$$\left(\left(1-\frac{\alpha}{r}\right)^{-1}E+m\right)h(r)\varepsilon(\theta)-\left(1-\frac{\alpha}{r}\right)\frac{dg(r)}{dr}\varepsilon(\theta)-\frac{1}{r}g(r)\varepsilon(\theta)$$
$$-\left[\frac{i\sigma^2}{r}\left(\partial_2+\frac{1}{2}\cot\theta\right)+\sigma^1\frac{m_\phi}{r\sin\theta}\right]g(r)\varepsilon(\theta)=0. \tag{A.34}$$

By simplifications using (A.29) we get

$$\left(1-\frac{\alpha}{r}\right)\frac{dg(r)}{dr}+\frac{(1+\kappa)}{r}g(r)-\left(\left(1-\frac{\alpha}{r}\right)^{-1}E+m\right)h(r)=0. \tag{A.35}$$

At the end, we finish up to the following system of differential equations

$$\begin{aligned}&\left(1-\tfrac{\alpha}{r}\right)\tfrac{dg(r)}{dr}+\tfrac{(1+\kappa)}{r}g(r)-\left(\left(1-\tfrac{\alpha}{r}\right)^{-1}E+m\right)h(r)=0\\&\left(1-\tfrac{\alpha}{r}\right)\tfrac{dh(r)}{dr}+\tfrac{(1-\kappa)}{r}h(r)+\left(\left(1-\tfrac{\alpha}{r}\right)^{-1}E-m\right)g(r)=0\end{aligned} \tag{A.36}$$

A.4 The solution in series form

If we work in system of unites where c and $\hbar$ not equal to one in this case the equation (A.36) becomes

$$\begin{aligned}&\left(1-\tfrac{\alpha}{r}\right)^2\tfrac{dg(r)}{dr}+\tfrac{(1+\kappa)}{r}\left(1-\tfrac{\alpha}{r}\right)g(r)-\left(\tfrac{E}{\hbar c}+\tfrac{mc}{\hbar}\left(1-\tfrac{\alpha}{r}\right)\right)h(r)=0\\&\left(1-\tfrac{\alpha}{r}\right)^2\tfrac{dh(r)}{dr}+\tfrac{(1-\kappa)}{r}\left(1-\tfrac{\alpha}{r}\right)h(r)+\left(\tfrac{E}{\hbar c}-\tfrac{mc}{\hbar}\left(1-\tfrac{\alpha}{r}\right)\right)g(r)=0\end{aligned} \tag{A.37}$$

we pose that [18, 20]

$$g = e^{-\rho}F, \;\; h = e^{-\rho}G, \;\; \rho = \beta r, \;\; \alpha = \frac{\gamma}{mc^2}.$$

The substitutions give a system of differential equations as follows

$$\begin{aligned}
&\left(1 + \tfrac{\gamma^2\beta^2}{m^2c^4\rho} - \tfrac{2\gamma\beta}{mc^2\rho}\right)\acute{F} - \left(1 + \tfrac{\gamma^2\beta^2}{m^2c^4\rho} - \tfrac{2\gamma\beta}{mc^2\rho}\right)F + \tfrac{(1+\kappa)}{\rho}\left(1 - \tfrac{\beta\gamma}{mc^2\rho}\right)F - \left(\tfrac{E}{\beta\hbar c} + \tfrac{mc}{\beta\hbar} - \tfrac{\gamma}{\hbar c}\tfrac{1}{\rho}\right)G = 0 \\
&\left(1 + \tfrac{\gamma^2\beta^2}{m^2c^4\rho} - \tfrac{2\gamma\beta}{mc^2\rho}\right)\acute{G} - \left(1 + \tfrac{\gamma^2\beta^2}{m^2c^4\rho} - \tfrac{2\gamma\beta}{mc^2\rho}\right)G + \tfrac{(1-\kappa)}{\rho}\left(1 - \tfrac{\beta\gamma}{mc^2\rho}\right)G + \left(\tfrac{E}{\beta\hbar c} - \tfrac{mc}{\beta\hbar} + \tfrac{\gamma}{\hbar c}\tfrac{1}{\rho}\right)F = 0
\end{aligned}.$$

The functions in series form are given with

$$F = \sum_{n=0}^{N} a_n\rho^{n+s}, \;\; G = \sum_{n=0}^{N} b_n\rho^{n+s}. \tag{A.38}$$

the substitution in the above system leads to

$$\begin{aligned}
&\sum_{n=0}^{N}(n+s)a_n\rho^{n+s-1} + \sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}(n+s)\,a_n\rho^{n+s-3} - \frac{2\gamma\beta}{mc^2}\sum_{n=0}^{N}(n+s)a_n\rho^{n+s-2} - \sum_{n=0}^{N}a_n\rho^{n+s} \\
&-\sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}a_n\rho^{n+s-2} + \frac{2\gamma\beta}{mc^2}\sum_{n=0}^{N}a_n\rho^{n+s-1} + (1+\kappa)\sum_{n=0}^{N}a_n\rho^{n+s-1} - (1+\kappa)\frac{\gamma\beta}{mc^2}\sum_{n=0}^{N}a_n\rho^{n+s-2} \\
&-\frac{(E+mc^2)}{\beta\hbar c}\sum_{n=0}^{N}b_n\rho^{n+s} + \frac{\gamma}{\hbar c}\sum_{n=0}^{N}b_n\rho^{n+s-1} = 0
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{N}(n+s)b_n\rho^{n+s-1} + \sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}(n+s)\,b_n\rho^{n+s-3} - \frac{2\gamma\beta}{mc^2}\sum_{n=0}^{N}(n+s)b_n\rho^{n+s-2} - \sum_{n=0}^{N}b_n\rho^{n+s} \\
&-\sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}b_n\rho^{n+s-2} + \frac{2\gamma\beta}{mc^2}\sum_{n=0}^{N}b_n\rho^{n+s-1} + (1-\kappa)\sum_{n=0}^{N}b_n\rho^{n+s-1} - (1-\kappa)\frac{\gamma\beta}{mc^2}\sum_{n=0}^{N}b_n\rho^{n+s-2} \\
&+\frac{(E-mc^2)}{\beta\hbar c}\sum_{n=0}^{N}a_n\rho^{n+s} + \frac{\gamma}{\hbar c}\sum_{n=0}^{N}a_n\rho^{n+s-1} = 0.
\end{aligned}$$

This can be simplified as follows

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}(n+s)\,a_n\rho^{n+s-3} - \frac{\gamma\beta}{mc^2}\sum_{n=0}^{N}(2n+2s+1+\kappa+\frac{\gamma\beta}{mc^2})a_n\rho^{n+s-2} \\
&+\sum_{n=0}^{N}(n+s+1+\kappa+\frac{2\gamma\beta}{mc^2})a_n\rho^{n+s-1} - \sum_{n=0}^{N}a_n\rho^{n+s} - \frac{(E+mc^2)}{\beta\hbar c}\sum_{n=0}^{N}b_n\rho^{n+s} + \frac{\gamma}{\hbar c}\sum_{n=0}^{N}b_n\rho^{n+s-1} = 0
\end{aligned}$$

$$\sum_{n=0}^{N}\frac{\gamma^2\beta^2}{m^2c^4}\,(n+s)\,b_n\rho^{n+s-3}-\frac{\gamma\beta}{mc^2}\sum_{n=0}^{N}(2n+2s+1-\kappa+\frac{\gamma\beta}{mc^2})b_n\rho^{n+s-2}$$
$$+\sum_{n=0}^{N}(n+s+1-\kappa+\frac{2\gamma\beta}{mc^2})b_n\rho^{n+s-1}-\sum_{n=0}^{N}b_n\rho^{n+s}+\frac{\left(E-mc^2\right)}{\beta\hbar c}\sum_{n=0}^{N}a_n\rho^{n+s}+\frac{\gamma}{\hbar c}\sum_{n=0}^{N}a_n\rho^{n+s-1}=0$$

with a rearrangement in the boundaries of the series

$$\sum_{n=-3}^{N-3}\frac{\gamma^2\beta^2}{m^2c^4}\,(n+3+s)\,a_{n+3}\rho^{n+s}-\frac{\gamma\beta}{mc^2}\sum_{n=-2}^{N-2}(2n+2s+5+\kappa+\frac{\gamma\beta}{mc^2})a_{n+2}\rho^{n+s}$$
$$+\sum_{n=-1}^{N-1}(n+s+2+\kappa+\frac{2\gamma\beta}{mc^2})a_{n+1}\rho^{n+s}-\sum_{n=0}^{N}a_n\rho^{n+s}-\frac{\left(E+mc^2\right)}{\beta\hbar c}\sum_{n=0}^{N}b_n\rho^{n+s}$$
$$+\frac{\gamma}{\hbar c}\sum_{n=-1}^{N-1}b_{n+1}\rho^{n+s}=0$$

$$\sum_{n=-3}^{N-3}\frac{\gamma^2\beta^2}{m^2c^4}\,(n+3+s)\,b_{n+3}\rho^{n+s}-\frac{\gamma\beta}{mc^2}\sum_{n=-2}^{N-2}(2n+2s+5-\kappa+\frac{\gamma\beta}{mc^2})b_{n+2}\rho^{n+s}$$
$$+\sum_{n=-1}^{N-1}(n+s+1-\kappa+\frac{2\gamma\beta}{mc^2})b_{n+1}\rho^{n+s}-\sum_{n=0}^{N}b_n\rho^{n+s}+\frac{\left(E-mc^2\right)}{\beta\hbar c}\sum_{n=0}^{N}a_n\rho^{n+s}$$
$$+\frac{\gamma}{\hbar c}\sum_{n=-1}^{N-1}a_{n+1}\rho^{n+s}=0.$$

From these equations, the recurrence relationship between coefficients can be extracted as

$$\frac{\gamma^2\beta^2}{m^2c^4}sa_0\rho^{-3+s}+\frac{\gamma^2\beta^2}{m^2c^4}\,(1+s)\,a_1\rho^{-2+s}+\frac{\gamma^2\beta^2}{m^2c^4}\,(2+s)\,a_2\rho^{-1+s}-\frac{\gamma\beta}{mc^2}(1+2s+\kappa+\frac{\gamma\beta}{mc^2})a_0\rho^{-2+s}$$
$$-\frac{\gamma\beta}{mc^2}(3+2s+\kappa+\frac{\gamma\beta}{mc^2})a_1\rho^{-1+s}+(1+s+\kappa+\frac{2\gamma\beta}{mc^2})a_0\rho^{-1+s}+\frac{\gamma}{\hbar c}b_0\rho^{-1+s}=0$$

$$\frac{\gamma^2\beta^2}{m^2c^4}sb_0\rho^{-3+s}+\frac{\gamma^2\beta^2}{m^2c^4}\,(1+s)\,b_1\rho^{-2+s}+\frac{\gamma^2\beta^2}{m^2c^4}\,(2+s)\,b_2\rho^{-1+s}-\frac{\gamma\beta}{mc^2}(1+2s-\kappa+\frac{\gamma\beta}{mc^2})b_0\rho^{-2+s}$$
$$-\frac{\gamma\beta}{mc^2}(3+2s-\kappa+\frac{\gamma\beta}{mc^2})b_1\rho^{-1+s}+(1+s-\kappa+\frac{2\gamma\beta}{mc^2})b_0\rho^{-1+s}+\frac{\gamma}{\hbar c}a_0\rho^{-1+s}=0$$

we should assert this all of the coefficients $a_0, b_0, a_1, b_1, a_2, b_2 \neq 0$ then we clearly find that

$$\frac{\gamma^2\beta^2}{m^2c^4}sa_0\rho^{-3+s}=0$$
$$\frac{\gamma^2\beta^2}{m^2c^4}sb_0\rho^{-3+s}=0 \tag{A.39}$$

that imply $s=0$. Moreover, we can obtain

$$\left(\frac{\gamma^2\beta^2}{m^2c^4}a_1-\frac{\gamma\beta}{mc^2}(1+\kappa+\frac{\gamma\beta}{mc^2})a_0\right)\rho^{-2}=0$$
$$\left(\frac{\gamma^2\beta^2}{m^2c^4}b_1-\frac{\gamma\beta}{mc^2}(1-\kappa+\frac{\gamma\beta}{mc^2})b_0\right)\rho^{-2}=0 \tag{A.40}$$

which gives [18, 20]

$$\begin{aligned} a_1 &= \frac{(1+\kappa+\frac{\gamma\beta}{mc^2})}{\gamma\beta}mc^2a_0 \\ b_1 &= \frac{(1-\kappa+\frac{\gamma\beta}{mc^2})}{\gamma\beta}mc^2b_0 \end{aligned} \tag{A.41}$$

also we have

$$\left(\frac{2\gamma^2\beta^2}{m^2c^4}a_2-\frac{\gamma\beta}{mc^2}(3+\kappa+\frac{\gamma\beta}{mc^2})a_1+(1+\kappa+\frac{2\gamma\beta}{mc^2})a_0+\frac{\gamma}{\hbar c}b_0\right)\rho^{-1}=0$$
$$\left(\frac{2\gamma^2\beta^2}{m^2c^4}b_2-\frac{\gamma\beta}{mc^2}(3-\kappa+\frac{\gamma\beta}{mc^2})b_1+(1-\kappa+\frac{2\gamma\beta}{mc^2})b_0+\frac{\gamma}{\hbar c}a_0\right)\rho^{-1}=0 \tag{A.42}$$

On the other side we additionally find

$$\begin{aligned}&-\frac{\gamma\beta}{mc^2}(2N+1+\kappa+\frac{\gamma\beta}{mc^2})a_N\rho^{N-2}+(N+1+\kappa+\frac{2\gamma\beta}{mc^2})a_N\rho^{N-1}+(N+\kappa+\frac{2\gamma\beta}{mc^2})a_{N-1}\rho^{N-2}\\ &-a_N\rho^N-a_{N-1}\rho^{N-1}-a_{N-2}\rho^{N-2}-\frac{\left(E+mc^2\right)}{\beta\hbar c}\left(b_N\rho^N+b_{N-1}\rho^{N-1}+b_{N-2}\rho^{N-2}\right)\\ &+\frac{\gamma}{\hbar c}\left(b_N\rho^N+b_{N-1}\rho^{N-1}\right)=0\end{aligned}$$

$$\begin{aligned}&-\frac{\gamma\beta}{mc^2}(2N+1-\kappa+\frac{\gamma\beta}{mc^2})b_N\rho^{N-2}+(N+1-\kappa+\frac{2\gamma\beta}{mc^2})b_N\rho^{N-1}+(N-\kappa+\frac{2\gamma\beta}{mc^2})b_{N-1}\rho^{N-2}\\ &-b_N\rho^N-b_{N-1}\rho^{N-1}-b_{N-2}\rho^{N-2}+\frac{\left(E-mc^2\right)}{\beta\hbar c}\left(a_N\rho^N+a_{N-1}\rho^{N-1}+a_{N-2}\rho^{N-2}\right)\\ &+\frac{\gamma}{\hbar c}\left(a_N\rho^N+a_{N-1}\rho^{N-1}\right)=0\end{aligned}$$

we can demonstrate from that the following [18, 20]

$$\left(-a_N-\frac{\left(E+mc^2\right)}{\beta\hbar c}b_N\right)\rho^N=0$$
$$\left(\frac{\left(E-mc^2\right)}{\beta\hbar c}a_N-b_N\right)\rho^N=0 \tag{A.43}$$

that it mean

$$\frac{\left(E^2 - m^2c^4\right)}{\left(\beta\hbar c\right)^2} + 1 = 0 \tag{A.44}$$

so we get [18, 20]

$$\beta^2 = \frac{\left(m^2c^4 - E^2\right)}{\left(\hbar c\right)^2} \tag{A.45}$$

More than that, we work to show the recurrence equations which is given as follows

$$\left((N + 1 + \kappa + \frac{2\gamma\beta}{mc^2})a_N - a_{N-1} - \frac{\left(E + mc^2\right)}{\beta\hbar c}b_{N-1} + \frac{\gamma}{\hbar c}b_N\right)\rho^{N-1} = 0 \tag{A.46}$$

$$\left((N + 1 - \kappa + \frac{2\gamma\beta}{mc^2})b_N - b_{N-1} + \frac{\left(E - mc^2\right)}{\beta\hbar c}a_{N-1} + \frac{\gamma}{\hbar c}a_N\right)\rho^{N-1} = 0 \tag{A.47}$$

and then the subtraction of $\frac{\left(mc^2 - E\right)}{\beta\hbar c}$×(A.46) from (A.47) leads to

$$\left[\frac{\left(mc^2 - E\right)}{\beta\hbar c}(N + 1 + \kappa + \frac{2\gamma\beta}{mc^2}) - \frac{\gamma}{\hbar c}\right]a_N = \left[N + 1 - \kappa + \frac{2\gamma\beta}{mc^2} - \frac{\gamma}{\left(\hbar c\right)^2}\frac{\left(mc^2 - E\right)}{\beta}\right]b_N \tag{A.48}$$

we do the same thing with terms of ρ^{N-2} and we obtain

$$\begin{aligned} &-\frac{\gamma\beta}{mc^2}(2N + 1 + \kappa + \frac{\gamma\beta}{mc^2})\frac{\left(mc^2 - E\right)}{\beta\hbar c}a_N + \frac{\left(mc^2 - E\right)}{\beta\hbar c}(N + \kappa + \frac{2\gamma\beta}{mc^2})a_{N-1} \\ &+ \frac{\gamma}{\left(\hbar c\right)^2}\frac{\left(mc^2 - E\right)}{\beta}b_{N-1} = -\frac{\gamma\beta}{mc^2}(2N + 1 - \kappa + \frac{\gamma\beta}{mc^2})b_N + (N - \kappa + \frac{2\gamma\beta}{mc^2})b_{N-1} + \frac{\gamma}{\hbar c}a_{N-1} \end{aligned} \tag{A.49}$$

The final and the important relations are

$$\begin{aligned} &\frac{\gamma^2\beta^2}{m^2c^4}\left(n + 3\right)a_{n+3} - \frac{\gamma\beta}{mc^2}(2n + 5 + \kappa + \frac{\gamma\beta}{mc^2})a_{n+2} + (n + 2 + \kappa + \frac{2\gamma\beta}{mc^2})a_{n+1} \\ &- a_n - \frac{\left(E + mc^2\right)}{\beta\hbar c}b_n + \frac{\gamma}{\hbar c}b_{n+1} = 0 \end{aligned} \tag{A.50}$$

$$\begin{aligned} &\frac{\gamma^2\beta^2}{m^2c^4}\left(n + 3\right)b_{n+3} - \frac{\gamma\beta}{mc^2}(2n + 5 - \kappa + \frac{\gamma\beta}{mc^2})b_{n+2} + (n + 1 - \kappa + \frac{2\gamma\beta}{mc^2})b_{n+1} \\ &- b_n + \frac{\left(E - mc^2\right)}{\beta\hbar c}a_n + \frac{\gamma}{\hbar c}a_{n+1} = 0 \end{aligned} \tag{A.51}$$

The rest is the quest for the energy formula, so we let $n \to N$ in the equations (A.50) and (A.51) and we know that our series functions are defined from 0 to N. Therefore, all the coefficients have order bigger than N are zero and thus the two equations become

$$-a_N - \frac{\left(E+mc^2\right)}{\beta\hbar c} b_N = 0 \tag{A.52}$$

$$-b_N + \frac{\left(E-mc^2\right)}{\beta\hbar c} a_N = 0 \tag{A.53}$$

and then

$$\frac{a_N}{b_N} = -\frac{\left(E+mc^2\right)}{\beta\hbar c} \tag{A.54}$$

$$\frac{a_N}{b_N} = \frac{\beta\hbar c}{\left(E-mc^2\right)}. \tag{A.55}$$

Leading to

$$2\frac{a_N}{b_N} = -\frac{\left(E+mc^2\right)}{\beta\hbar c} + \frac{\beta\hbar c}{\left(E-mc^2\right)}$$

and the use of (A.45) gives

$$\frac{a_N}{b_N} = \frac{\beta\hbar c}{\left(E-mc^2\right)}. \tag{A.56}$$

By replacing in (A.48) we get

$$\frac{\beta\hbar c}{\left(E-mc^2\right)}\left[\frac{\left(mc^2-E\right)}{\beta\hbar c}(N+1+\kappa+\frac{2\gamma\beta}{mc^2}) - \frac{\gamma}{\hbar c}\right] = \left[N+1-\kappa+\frac{2\gamma\beta}{mc^2} - \frac{\gamma}{\left(\hbar c\right)^2}\frac{\left(mc^2-E\right)}{\beta}\right]$$

then

$$2(N+1) + \frac{4\gamma\beta}{mc^2} - \frac{\gamma}{\left(\hbar c\right)^2}\frac{\left(mc^2-E\right)}{\beta} - \frac{\gamma\beta}{\left(mc^2-E\right)} = 0$$

once more

$$\frac{2\gamma}{mc^2}\beta^2 + (N+1)\beta - \frac{\gamma}{\left(\hbar c\right)^2}mc^2 = 0 \tag{A.57}$$

This is a second order equation in β where its solutions are

$$\beta = \sqrt{\frac{\left(m^2c^4-E^2\right)}{\left(\hbar c\right)^2}} = \frac{mc^2}{4\gamma}\left[-(N+1) \pm \sqrt{(N+1)^2 + \frac{8\gamma^2}{(\hbar c)^2}}\right]. \tag{A.58}$$

From this we take out the energy formula

$$E_N = mc^2\sqrt{1-\left(\frac{\hbar c}{4\gamma}\right)^2\left[-(N+1)\pm\sqrt{(N+1)^2+\frac{8\gamma^2}{(\hbar c)^2}}\right]^2} \tag{A.59}$$

Bibliography

[1] G. Kane, Modern Elementary Particle Physics, Cambridge University Press (2017).

[2] M. P. Hobson, G.P. Efstathiou, and A.N. Lasenby, General Relativity, Cambridge University press (2006).

[3] T. Padmanabhan, Gravitation, Cambridge University press (2010).

[4] P. G. Bergman, Introduction to the theory of relativity, Dover publication (1976).

[5] S. M. Carrol, Lecture notes on general relativity (1996). gr-qc/9712019v1.

[6] B. P. Abbott et al, Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 (2016).

[7] H. Weyl, Mathematische Zeitschrift , 2, 3, 384–411, in German (1918).

[8] E. Schrodinger, space-time structure, Cambridge University press (1950).

[9] Th. Kaluza, Zum Unitätsproblem in der Physik, Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.). 1921, 966–972 (1921).

[10] O. Klein, Z. Phys. 37, 903 (1926).

[11] B. Finzi: "Relativit'a Generale e Teorie Unitarie", in Cinquant'anni di Relativit'a (in Italian), ed. M. Pantaleo. Giunti. Firenze (1955).

[12] F. Cardone, R. Mignani and A. Petrucci, The Principle of Solidarity: Geometrical Description of Interactions, arXiv:1109.2010v1.

[13] G. Amelino-Camelia, DOUBLY-SPECIAL RELATIVITY: FIRST RESULTS AND KEY OPEN PROBLEMS, International Journal of Modern Physics D. 11, 1643-1669 (2002).

[14] J. Magueijo, Generalized Lorentz invariance with an invariant energy scale, Phys Rev D. 67, 044017 (2003).

[15] F. Cardone and R. Mignani, Energy and geometry: an introduction to deformed special relativity (Vol. 22), World Scientific (2004).

[16] S. De, S. Ghosh and S. Chakrabarty, Schrodinger equation of a particle in a uniformly accelerated frame and the possibility of a new kind of quanta, Mod. Phys. Lett. A. 30,1550182 (2015).

[17] S. Bhattacharyya, S. Gangopadhyay and A. Saha, Quantum mechanics of a particle in an accelerated frame and the equivalence principle, EPL. 120, 30005 (2018).

[18] C.C. Barros Jr, Quantum mechanics in curved space-time, Eur. Phys. J. C. 42, 119–126 (2005). arXiv:physics/0409064.

[19] C.C. Barros Jr, Quantum mechanics in curved space-time, (2005). arXiv:physics/0509011.

[20] C.C. Barros Jr, Quark confinement and curved spaces, Eur. Phys. J. C. 45, 421–425 (2006). arXiv:hep-ph/0504179v1.

[21] M. Özer, Electrostatic time dilation and redshift, Physics Letters B. 135212(2020).

[22] A. Bouda and A. Belabbas. Int J Theor Phys. 49, 2630- 2639 (2010). arXiv:gr-qc/1012.2245.

[23] A. Djehiche, A. Belabbas and A. Bouda, On a Possible Quantum Manifestation of the Perihelion Advance in Hydrogen like Atom, Canadian Journal of Physics, (ja)(2019).

[24] R. Descartes, Principles of Philosophy (1983), R. P. Miller and V. R. Miller (trans.), Dordrecht: D. Reidel.

[25] N. Huggett and C. Hoefer, Absolute and relational theories of space and motion, In E. N. Zalta (Ed.), The stanford encyclopedia of philosophy (Fall 2006 Edition) hhttp://plato.stanford.edu/archives/fall2006/entries/ spacetime-theoriesi.

[26] A. Einstein and F. A. Davis, The principle of relativity, Courier Corporation(2013).

[27] P. Ehrenfest, Gleichförmige Rotation starrer Körper und Relativitätstheorie, Physikalische Zeitschrift. 10, 918(1909).

[28] B. Ydri, Lectures on General Relativity, Cosmology and Quantum Black Holes, IOP Publishing Ltd (2017).

[29] T. Sunada, From Euclid to Riemann and Beyond, arXiv:1904.01845v1 [math.HO] 3 Apr 2019.

[30] A. Einstein, Kosmologische Betrachtungen zur allgemeinen Relativitäts-theorie. In : Das Relativitätsprinzip. Vieweg+ Teubner Verlag, Wiesbaden, p. 130-139(1922).

[31] E. Hubble, A relation between distance and radial velocity among extra-galactic nebulae, Proceedings of the National Academy of Sciences. 15, 168-173(1929).

[32] R. J. Lambourne, Relativity, gravitation and cosmology, Cambridge University Press(2010).

[33] H. Mo, F. Van den Bosch and S.White, Galaxy formation and evolution, Cambridge University Press(2010).

[34] M. Mobberley, Supernovae: And How to Observe Them, Springer Science and Business Media(2007).

[35] P. J. E. Peebles and B. Ratra, The cosmological constant and dark energy, Reviews of modern physics. 75, 559(2003).

[36] T. Padmanabhan, Cosmological constant-the weight of the vacuum, Physics Reports. 380, 235-320(2003).

[37] J. F. Hawley and K. A. Holcomb, Foundations of modern cosmology, Oxford University Press(2005).

[38] L. Järv, M. Rünkla, M. Saal and O. Vilson, Nonmetricity formulation of general relativity and its scalar-tensor extension, Phys Rev D. 97, 124025 (2018).

[39] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, Physics Reports. 258, 1-171 (1995).

[40] T. Ortin, Gravity and Strings, Cambridge University Press (2015).

[41] M. Blagojevic, Gravitation and gauge symmetries, CRC Press (2001).

[42] N. Rosen, Weyl's geometry and physics. Foundations of Physics. 12, 213-248(1982).

[43] G. G. Nyambuya, A Perdurable Defence to Weyl's Unified Theory. Journal of Modern Physics. 5, 1244 (2014).

[44] J. Aalbers, Conformal Symmetry in Classical Gravity, Master thesis (2013). http://dspace. library. uu. nl/handle/1874/280136.

[45] E. Scholz, WEYL GEOMETRY IN LATE 20TH CENTURY PHYSICS, arXiv:1111.3220v1 [math.HO] 14 Nov 2011.

[46] C. Romero, Is Weyl unified theory wrong or incomplete?, arXiv:1508.03766v2 [gr-qc] 25 Aug 2015.

[47] W. O. Straub, On the Failure of Weyl's 1918 Theory, (2013). http://vixra.org/abs/1401.0168.

[48] A. S. Eddington, The Mathematical Theory of Relativity, 2nd ed. Cambridge Univ. Press(1930).

[49] J. E. M. Aguilar and C. Romero, Inducing the cosmological constant from five-dimensional Weyl space, Foundations of Physics. 39, 1205(2009).

[50] J. M. Overduin and, P. S. Wesson, Kaluza-Klein Gravity, arXiv:gr-qc/9805018v1 7 May 1998.

[51] F. Ravndal, Oskar Klein and the fifth dimension, arXiv:1309.4113v1 [physics.hist-ph] 16 Sep 2013.

[52] P. S. Wesson, Space-time-matter: modern Kaluza-Klein theory, World Scientific (1999).

[53] A. Einstein, Ideas and Opinions, trans. Sonja Bergmann, Crown Publisher (1954).

[54] O. Roldan and C.C. Jr. Barros, Geometrical Aspects of non-gravitational interactions, arXiv:gr-qc/160303429, (2016).

[55] A. Belabbas, Les Interactions Fondamentales et la Structure de l'Espace-temps, arXiv preprint arXiv:1401.0800, (2014).

[56] J.-F. Pascual-S´anchez, The harmonic gauge condition in the gravitomagnetic equations, arXiv preprint gr-qc/0010075, (2000).

[57] B. Mashhoon, Gravitoelectromagnetism: a brief review, arXiv preprint gr-qc/0311030, (2003).

[58] S. M. Kopeikin, Gravitomagnetism and the Speed of Gravity, International Journal of Modern Physics D. 15, 305-320(2006).

[59] M. Bahram, F. Gronwald, and H. I. Lichtenegger, Gravitomagnetism and the clock effect. Gyros, Clocks, Interferometers...: Testing Relativistic Graviy in Space. Springer, Berlin, Heidelberg. 83-108(2001).

[60] U. G. Alberto, and A. Gallerati, Superconductor in a weak static gravitational field, Eur. Phys. J. C. 77, 549(2017).

[61] H. Thirring, Phys. Z. 19, 33 (1918); 22, 29 (1921); J. Lense and H. Thirring, Phys. Z. 19, 156 (1918); B. Mashhoon, F. W. Hehl and D. S. Theiss, Gen. Rel. Grav. 16, 711 (1984).

[62] B. Mashhoon and F. W. Hehl, Nonlocal Gravitomagnetism, Universe. 5, 195(2019).

[63] P. Huei, On Calculation of Magnetic-Type Gravitation and Experiment, Gen. Rel. Grav., 15, 725-735(1983).

[64] A. I. Arbab, Gravitomagnetism: a novel explanation of the precession of planets and binary pulsars, Astrophys Space Sci. 330, 61–68(2010).

[65] B. Mashhoon, F. Gronwald and H. I. Lichtenegger, Gravitomagnetism and the clock effect. In Gyros, Clocks, Interferometers...: Testing Relativistic Graviy in Space, Springer, Berlin, Heidelberg, 83-108(2001).

[66] J. Larmor, On the theory of the magnetic influence on spectra; and on the radiation from moving ions, Phil. Mag. 44, 503-512(1897).

[67] L. Hui-qing, Plane gravitational waves under a non-zero cosmological constant, Chinese astronomy and astrophysics. 12, 186-190(1988).

[68] S. Carlip, S. Deser, A. Waldron, and D. K. Wise, Cosmological topologically massive gravitons and photons, Classical and Quantum Gravity. 26, 075008(2009).

[69] A. I. Arbab, Quantized Maxwell's equations, Optik. 136, 64-70(2017).

[70] A, Blasi and N. Maggiore, Massive gravity and Fierz–Pauli theory, Eur. Phys. J. C. 77, 614(2017).

[71] K. Aoki1 and S. Mukohyama, Massive gravitons as dark matter and gravitational waves, Phys Rev D. 94, 024001(2016).

[72] M. Fierz and W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. Roy. Soc. Lond A. 173, 211(1939).

[73] C. De Rham and G. Gabadadze, Generalization of the Fierz-Pauli action, Phys Rev D. 82, 044020(2010).

[74] T. Diemer and M. J. Hadley, Charge and the topology of spacetime, Classical and Quantum Gravity. 16, 3567(1999).

[75] T. Kim and H. Kim, Geometrical interpretation of electromagnetism in a 5-dimensional manifold, Classical and Quantum Gravity. 34, 155006(2017).

[76] M. Özer, On the equivalence principle and a unified description of gravitation and electromagnetism. arXiv preprint gr-qc/9910062, (1999).

[77] A. Bakopoulos and P. Kanti, From GEM to electromagnetism, Gen Relativ Gravit. 46, 1742(2014).

[78] A. L. Proca, Sur la théorie ondulatoire des électrons positifs et négatifs, Journal de Physique et le Radium. 7, 347-353(1936).

[79] E. C. G. Stueckelberg, On the Existence of Heavy Electrons, Phys Rev. 52, 41(1937).

[80] H. Ruegg and M. Ruiz-Altaba, The stueckelberg field, International Journal of Modern Physics A. 19, 3265-3347(2004).

[81] M. Suzuki, Slightly massive photon, Phys Rev D. 38, 1544(1988).

[82] S. Kouwn, P. Oh and C. G. Park, Massive photon and dark energy, Physical Review D. 93, 083012(2016).

[83] J. Yepez, Einstein's vierbein field theory of curved space, arXiv preprint arXiv:1106.2037 (2011).

[84] M. Lanzagorta, Quantum information in gravitational fields, Morgan & Claypool (2014).

[85] V. I. Dokuchaev and Y. N. Eroshenko, Stationary solutions of the Dirac equation in the gravitational field of a charged black hole, Journal of Experimental and Theoretical Physics. 117, 72-77(2013).

[86] M. V. Gorbatenko and V. P. Neznamov, Stationary bound states of dirac particles in collapsar fields, Physics of Particles and Nuclei, 45, 220-222(2014).

[87] M. A. Vronsky, et al, Stationary bound states of dirac particles in the schwarzschild gravitational field, arXiv preprint arXiv:1301.7595 (2013).

[88] M. V. Gorbatenko and V. P. Neznamov, Stationary bound states of spin-half particles in the Kerr and Kerr-Newman gravitational fields, arXiv preprint arXiv:1303.1127 (2013).

[89] H. Reissner, Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie, Annalen der Physik. 355, 106-120(1916).

[90] G. Nordstrom, On the energy of gravitation field in Einstein's theory, Proc. Ned. Ac. Wet. 20, 1238-1245(1918).

[91] M. A. Melvin, Pure magnetic and electric geons. Phys. Letters, 8(1964).

[92] M. Özer, On the equivalence principle and a unified description of gravitation and electromagnetism. arXiv:gr-qc/9910062v5 8 Jan 2020.

[93] K. Nordtvedt, Equivalence principle for massive bodies including rotational energy and radiation pressure, Phys Rev. 180, 1293(1969).

[94] K. Nordtvedt, Equivalence Principle for Massive Bodies. IV. Planetary Bodies and Modified Eötvös-Type Experiments, Phys Rev. 180, 1293(1969).

[95] P. Ogonowski, Time dilation as field, Journal of Modern Physics. 3, 200-207 (2012).

[96] A. I. Arbab, The Generalized Newton's Law of Gravitation Versus the General Theory of Relativity, Journal of Modern Physics. 3, 1231-1235(2012).

[97] M. Nowakowski, The Consistent Newtonian limit of Einstein's gravity with a cosmological constant, International Journal of Modern Physics D. 10, 649-661(2001).

[98] M. Nowakowski, J. C. Sanabria and A. Garcia, Gravitational equilibrium in the presence of a positive cosmological constant, Phys Rev D. 66, 023003(2002).

[99] E. Gessner, Cosmological constant and the flat rotation curves of galaxies, Astrophysics and space science. 194, 197-205(1992).

[100] E. Ozbay, K. Guven and K. Aydin, Metamaterials with negative permeability and negative refractive index: experiments and simulations, Journal of Optics A: Pure and Applied Optics. 9, S301(2007).

[101] I. I. Smolyaninov, Metamaterial model of tachyonic dark energy, Galaxies. 2, 72-80(2014).

[102] G. Grynberg, A. Aspect and C. Fabre, Introduction to quantum optics: from the semi-classical approach to quantized light. Cambridge university press(2010).

[103] I. Merches, D. Tatomir and R. E. Lupu, Basics of quantum electrodynamics. CRC Press(2012).

[104] S. Weinberg, The quantum theory of fields. Vol. 1: Foundations. Cambridge University Press (1995).

[105] W. Greiner and J. Reinhardt, Quantum electrodynamics, Springer Science & Business Media (2008).

[106] V. Mukhanov and S. Winitzki, Introduction to quantum effects in gravity, Cambridge university press (2007).

[107] Bjorken, J. D., & Drell, S. D. (1965). Relativistic quantum mechanics. McGraw-Hill.

[108] W. E. Lamb Jr and R. C. Retherford, Fine structure of the hydrogen atom by a microwave method, Phys Rev. 72(3), 241(1947).

[109] E. A. Uehling, Polarization effects in the positron theory, Physical Review. 48, 55(1935).

[110] H. A. Bethe, The electromagnetic shift of energy levels, Phys Rev. 72, 339(1947).

[111] V. F. Weisskopf, On the self-energy and the electromagnetic field of the electron, Phys Rev. 56, 72(1939).

[112] S. W. Hawking, Particle creation by black holes, Communications in mathematical physics. 43, 199-220(1975).

[113] R. Dick, Advanced Quantum Mechanics. Advanced Quantum Mechanics. Series: Graduate Texts in Physics, ISBN: 978-3-319-25674-0. Springer International Publishing (Cham) (2016), Edited by Rainer Dick.

[114] E. Elizalde, Ten physical applications of spectral zeta functions, Springer Berlin Heidelberg(2012).

[115] P. W. Milonni, The quantum vacuum: an introduction to quantum electrodynamics, Academic press(1994).

[116] S. E. Rugh and H. Zinkernagel, The quantum vacuum and the cosmological constant problem, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics. 33, 663-705(2002).

[117] S. Weinberg, The cosmological constant problem, Reviews of modern physics. 61, 1 (1989).

[118] N. Zettili, Quantum Mechanics: concepts and applica- tions, United Kingdom (2009).

[119] P. D. Mannheim, Comprehensive solution to the cosmological constant, zero-point energy, and quantum gravity problems, General Relativity and Gravitation. 43, 703-750 (2011).

[120] C. Wetterich, Graviton fluctuations erase the cosmological constant, Physics Letters B. 773, 6-19(2017).

[121] I. J. R. Aitchison, Nothing's plenty the vacuum in modern quantum field theory, Contemporary Physics. 26, 333-391(1985).

[122] T. Müller and F. Grave, Catalogue of spacetimes, arXiv preprint arXiv:0904.4184(2009).

[123] H. M. Wiseman, Quantum trajectories and quantum measurement theory, Quantum and Semiclassical Optics: Journal of the European Optical Society Part B. 8, 205(1996).

[124] P. F. Gonzalez-Diaz, Quantized cosmological constant. Modern Physics Letters A. 2, 551-554(1987).

[125] E. H. Wichmann and N. M. Kroll, Vacuum polarization in a strong Coulomb field, Phys Rev. 101, 843(1956).

[126] J. B. Jimenez and A. L. Maroto, Cosmological electromagnetic fields and dark energy, Journal of Cosmology and Astroparticle Physics. 2009, 016 (2009).

Abstract

In this thesis, we have developed a geometrical approach to the electromagnetism based on the C. C. Barros idea. In this approach, the electromagnetic field, analogous to gravity, affects the electromagnetic spacetime structure. This spacetime is also affected by the passive particle's properties, and that is why we called it the electromagnetic universe to distinguish it from gravity. Therefore, in electron-proton classical interaction, the electron should follow the geodesic world line in a spacetime curved by the proton electromagnetic field. Hence, the metric of the system was established. This is what brings new dynamics to the charged particles, and thus causes new phenomena such as the perihelion advance, and the time slowdown in the electromagnetic universe. In the quantum electron-proton interaction with the support of a novel constant proposition, the Lamb shift of the hydrogen atom was explained. This constant is called electromagnetic lambda term. It is related to the electromagnetic vacuum. Unlike the cosmological constant the contribution of that constant is considered as an induced electromagnetic field comes from vacuum.

ملخّص

فكرة باروس غير المسبوقة تقول بأنه يمكن ربط التّأثير الكهرومغناطيسي مباشرة بانحناء الزّمكان. انطلاقا من هذه الفكرة، قمنا في هذا العمل بوضع وصف هندسي لتأثير الكهرومغناطيسية بحيث يؤثّر الحقل الكهرومغناطيسي على بنية الزّمكان، لكن في هذه الحالة الزّمكان ليس هو نفسه زمكان الجاذبية لأن خصائصه مختلفة و لأنه في هذه الحالة يتأثّر بخصائص الجسيمات المتأثّرة بانحناء الزّمكان. و لهذا ومن أجل التفريق نطلق على زمكاننا الجديد اسم الكون الكهرومغناطيسي. من هذا المنطلق حاولنا ايجاد مترية تفاعل شحنتين على سبيل المثال الكترون مع بروتون وذلك انطلاقا من معادلات اينشتاين المعدّلة، التي تعبر في هذه الحالة عن مصدر الكهرومغناطيسية. الالكترون المتفاعل مع البروتون هو فقط يتبع خط الجيوديسي في الزمكان المنحني. و من ثم فإن الوصف الهندسي مع المترية التي تم تحديدها يضيفان وصفا ديناميكيا جديدا لحركة الجسيمات وبظواهر جديدة كتقدم الحضيض في الحركة الدائرية للشحنات و تباطئ الزمن في الكون الكهرومغناطيسي. بالعودة إلى ميكانيك الكم تحديدا الى ذرة الهيدروجين فإن وصفنا الهندسي و بالاستعانة باقتراح ثابت جديد مكنونا جميعا من إعطاء تفسير لإنزياح لامب. هذا الثابت بمثابة الثابت الكوني لأينشتاين لكن تأثيره مختلف حيث اعتبرعلى أنه متعلق بالفراغ الكهرومغناطيسي أو هو حقل كهرومغناطيسي إضافي يعبر عن الفراغ في الكون الكهرومغناطيسي.

Résumé

À la lumière de l'idée de C. C. Barros, une approche géométrique de l'électromagnétisme a été présentée de sorte que le champ électromagnétique, de façon analogue à la gravité, affecte la structure de l'espace-temps électromagnétique. Dans ce cas, la structure est également liée aux propriétés de la particule passive, et c'est pour cela nous l'appelons l'univers électromagnétique. Lors de l'interaction électron-proton, l'électron doit suivre une géodésique dans un espace-temps courbée par le champ électromagnétique du proton. Ainsi, la métrique du système est établie. C'est ce qui apporte une nouvelle dynamique aux particules chargées, et provoque ainsi de nouveaux phénomènes tels que décalage du périhélie, et la dilatation du temps dans l'univers électromagnétique. Lors dans le cas quantique de l'interaction électron-proton, avec l'introduction d'une nouvelle constante, le décalage de Lamb de l'atome d'hydrogène a été expliqué. Cette constante est appelée terme lambda électromagnétique. Elle est liée au vide d'électromagnétique. Contrairement à la constante cosmologique, la contribution de cette constante est considérée comme un champ électromagnétique générer à partir du vide.