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⋇ *Dedications* ⋇

I dedicate my dissertation work to my family and many friends.

A special feeling of gratitude to my loving parents, especially my father for supporting me spiritually throughout writing this thesis and my life in general and whose words of encouragement and push for tenacity ring in my ears.

My sisters, have never left my side and are very special.

My brothers, for helping me all time when I need.

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I dedicate this work to all my friends. For being there with me all the doctorate years.

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Mouna Yahiaoui

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INTRODUCTION

Differential equations and dynamical systems appear naturally in the description of many phenomena for which local processes are known. For instance, most physical laws, such as conservation of mass, energy, and momentum are local laws. The central problem is then to obtain global information on these phenomena. These elementary processes are typically nonlinear and, assuming continuity of the states of the system (the dependent variables) in time and space (the independent variables), their evolution is governed by nonlinear differential equations. For example, in classical physics, the gravitational forces between masses is nonlinear as are the electromagnetic interactions. In hydrodynamics, the nonlinearity of the Navier-Stokes equation comes from inertial effects. Also, autocatalytic chemical reactions are described by nonlinear differential equations through the massaction law. These nonlinear effects give rise to complex structures whose complete description can be extremely difficult. Once the local equations are formulated in a particular context, the next problem is to solve these equations. Already, in this simple statement, there is an ambiguity. For the physicist, the applied mathematician or the chemist, to solve an equation means to obtain global information on the solution and if possible, derive a closed-form solution for which the state of the dependent variables may be predicted for all given independent variables. In this sense, an equation can be solved if it can be locally represented by known functions. The mathematician, however, is often interested in a more fundamental problem related to the existence and uniqueness of the solutions, a prerequisite of any subsequent analytical approach.

The basic idea underlying these works is that the solution can always be represented by the combination of known functions or by perturbation expansions. The notion of integrability was then introduced to describe the property of equations for which all local and global information can be obtained either explicitly from the solutions or implicitly from the constants of the motion.

Integrability and dynamical systems have become such important theories that they have acquired over the years different meanings for different people.

The first attempt to solve differential equations either explicitly or by series expansions goes back to Euler, Newton, and Leibniz. The theory of integration for the equations of motion was subsequently expanded by the work of the analysts and mechanicians associated with the names of Lagrange, Poisson, Hamilton, and Liouville in the late 18th and 19th centuries.

Clearly, Hilbert formulated his 16th problem by dividing it into two parts. The first part, which studies the mutual disposition of the maximal number (in the sense of Harnack) of separate branches of an algebraic curve, and also the corresponding investigation for nonsingular real algebraic varieties; and the second part, which poses the question of the maximal number and relative position of the limit cycles of the polynomial system

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P\Big(x(t), y(t)\Big), \\ \dot{y} = \dfrac{dy}{dt} = Q\Big(x(t), y(t)\Big), \end{cases}$$

where P and Q are polynomials of degree n. For this problem, Lloyd [1988] stated that the striking aspect is that the hypothesis is algebraic, while the conclusion topological. Nonlinear dynamics, which plays an important role in the study of almost all disciplines of science and engineering, including mathematics, mechanics, aeronautics, electrical circuits, control systems, population problems, economics, financial systems, stock markets, ecological systems, etc. The phenomenon of limit cycle was first discovered and studied by Poincaré [71] who presented the breakthrough qualitative theory of differential equations. In order to determine the existence of limit cycles for a given differential equation and the properties of limit cycles, Poincaré introduced the well-known method of Poincaré Map, which is still the most basic tool for studying the stability of periodic orbits. The driving force behind the study of limit cycle theory was the invention of triode vacuum tube which was able to produce stable self-excited oscillations of constant amplitude. It was noted that such a kind of oscillation phenomenon could not be described by linear differential equations. At the end of the 1920s Van der Pol [76] developed a differential equation to describe the oscillations of constant amplitude of a triode vacuum tube. Later a more general equation called Liénard equation [58] was developed, for which Van der Pol's equation is a special case.

There exist three main open problems in the qualitative theory of real planar differential systems [9, 15, 13, 19, 36, 61, 64], the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The determination of the number of limit cycles is the most important topics that related to the second part of the unsolved Hilbert 16th problem [50]. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals [72].

This thesis is structured in three chapters, the first chapter is devoted to reminders of some preliminary notions on planar differential systems used subsequently in the second and third chapters. These last two chapters are mainly devoted to our results.

The second chapter is devided in two parts:

In the first part, we will determine the first integral, the non-existence of limit cycles and we give the curves which are formed by the orbits of a class of Kolmogorov systems of the form:

$$\begin{cases} x' = x\left(P\left(x,y\right) + R\left(x,y\right)\ln\left|\dfrac{A\left(x,y\right)}{B\left(x,y\right)}\right|\right), \\ y' = y\left(Q\left(x,y\right) + R\left(x,y\right)\ln\left|\dfrac{A\left(x,y\right)}{B\left(x,y\right)}\right|\right), \end{cases}$$

where $A,\ B,\ P,\ Q$, and R are homogeneous polynomials of degree a, a, n, n, and m respectively. The results obtained for this class was a subject of our publication [21].

In the second part, we will determine the first integral and we give the formula of the curves which are formed by orbits of a class of Kolmogorov systems of the form:

$$\begin{cases} x' = x(1 + ax^2 + bxy + cy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \\ y' = y(1 + nx^2 + mxy + sy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \end{cases}$$

where $a,\ b,\ c,\ n,\ m$ and s are real constants, the results obtained of this class was a subject of our publication [77].

The third chapter is devided in two parts:

In the first part, we will determine the first integral of a class of planar differential system of the form:

$$\begin{cases} x' = P(x,y) + x\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \\ y' = Q(x,y) + y\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \end{cases}$$

where $P,\ Q,\ M,\ N,\ R$, and S are homogeneous polynomials of degree $a,\ a,\ b,\ b,\ c$, and c respectively, and $\lambda,\ \beta \in \mathbb{R}$, the results obtained for this class was a subject of our publication [20].

In the second part, we will determine the first integral and limit cycles of a class of planar differential systems of the form:

$$\begin{cases} x' = x + P_5(x,y) + xR_8(x,y), \\ y' = y + Q_5(x,y) + yR_8(x,y), \end{cases}$$

where $P_5,\ Q_5,\ R_8$ are homogeneous polynomials, the results obtained for this class was a subject of our publication [78].

CHAPTER 1

SOME PRELIMINARY NOTIONS

Introduction

The objective of this chapter is to give some general notions used throughout this work and recalling essential notions with classical results. More precisely, the first section is about some general notions. The second section concerns some basic notions of dynamical systems, namely, we recall some basic notions for the qualitative study of dynamical systems. We start by defining dynamical systems, autonomous differential systems, solutions and periodic orbits, equilibrium points, phase portrait, limit cycles, the classification of the equilibrium points and the limit cycles in the plane $\mathbb{R}^2$ and the integrability of differential systems. We will also introduce a reminder of the fundamental theorems and criteria on the existence and non-existence of periodic solutions.

We present the most basic results on limit cycles. In particular, any topological configuration of a finite number of limit cycles is realizable by a suitable polynomial differential system. We present some results on the stability.

1.1 Dynamical systems

A dynamical system is a system that changes over time according to a set of fixed rules that determine how one state of the system moves to another state. It gives a functional description of the solution of a physical problem or the mathematical model describing the physical problem.

Definition 1.1.1. *A dynamical system consists of a phase (state) space E, $I = \mathbb{R}_+$ and a fuction $\phi : E \times I \to E$, where the time $t \in I$. For arbitrary states the following must hold:*

1) $\phi(x, 0) = x$ *(identity),*

2) $\phi\big(\phi(x, t), s)\big) = \phi(x, t + s)$ *(additivity).*

In other words, a dynamical system may be understood as a mathematical prescription for evolving the state of a system in time.
For each $x \in E$, the set $\{\phi(x, t) \mid t \in I\}$ is called the orbit (or trajectory) of the system through the point x.

1.1.1 Autonomous differential systems

An autonomous differential system is a system of ordinary differential equations which does not explicitly depend on the independent variable, when the variable is the time t, they are also called time-invariant systems.

Definition 1.1.2. *A system in the plane of the form:*

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P\Big(x(t), y(t)\Big), \\ \dot{y} = \dfrac{dy}{dt} = Q\Big(x(t), y(t)\Big), \end{cases} \tag{1.1}$$

is called autonomous planar differential system, where P and Q are functions depend solely on x, y.

Assume that P and Q are functions of class C^1 (so the conditions of Cauchy-Lipchitz are satisfied at any ordinary point in the system (1.1)). If P and Q are polynomials, then the number $d = \max\big(deg(P), deg(Q)\big)$ is called the degree of system (1.1). On the curve $P(x, y) = 0$, we say vertical isocline, the vector field is parallel to the y axis and on the curve $Q(x, y) = 0$, we say horizontal isocline, the vector field is parallel to the x axis.

1.1.2 Kolmogorov system

Definition 1.1.3. *The autonomous differential system on the plane given by*

$$\begin{cases} x' = \dfrac{dx}{dt} = xP(x,y), \\ y' = \dfrac{dy}{dt} = yQ(x,y), \end{cases} \tag{1.2}$$

where x and y represent the population density of two species at time t. P and Q are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.

Many mathematical models in biology science and population dynamics, frequently involve the systems of ordinary differential equations having the form Kolmogorov models are widely used in ecology to describe the interaction between two populations.

1.1.3 Vector fields, orbit, phase portrait

Definition 1.1.4. *[30] We say that X is a polynomial vector field of degree d on $\mathbb{R}^2$ if it can be written in the form*

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \ \text{ or } X = (P,Q),$$

where P and Q are polynomials in $\mathbb{R}[x,y]$ such that the maximum degree of P and Q is d.

Definition 1.1.5. *[30] We call that if a solution $x = f(t), y = g(t)$ of system (1.1) is a non-constant periodic function of t, then $\gamma = \{(x,y) : x = f(t), y = g(t)\}$ is called a periodic orbit of system (1.1).*

Corollary 1.1.1. *The orbits of a vector field X form a partition of the phase plane Ω.*

Definition 1.1.6. *A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.*
The phase portrait of a vector field X is a subset of the orbits in the phase plane.

1.1.4 Linear, Bernoulli and Riccati equations

Linear equation

Definition 1.1.7. *A differential equation of type*

$$y^{'} + a(x)y = f(x),$$

where f and a are continuous functions of x, is called a linear non-homogeneous differential equation of first order.

We consider two methods of solving linear differential equations of first order:

1. Using an integrating factor.

2. Method of variation of a constant.

- **Using an integrating factor**: If a linear differential equation is written in the standard form:

  $$y^{'} + a(x)y = f(x),$$

  the integrating factor is defined by the formula

  $$u(x) = \exp\left(\int a(x)dx\right).$$

  Multiplying the left side of the equation by the integrating factor $u(x)$ converts the left side into the derivative of the product $y(x)u(x)$.

  The general solution of the differential equation is expressed as follows:

  $$y(x) = \frac{\int u(x)f(x)dx + c}{u(x)},$$

  where c is an arbitrary constant.

- **Method of variation of a constant**: This method is similar to the previous approach. First it's necessary to find the general solution of the homogeneous equation:

  $$y^{'} + a(x)y = 0,$$

  The general solution of the homogeneous equation contains a constant of integration c. We replace the constant c with a certain (still unknown) function $c(x)$. By substituting this solution into the nonhomogeneous differential equation, we can determine the function $c(x)$.

  The described algorithm is called the method of variation of a constant. Of course, both methods lead to the same solution.

Bernoulli equation

Definition 1.1.8. *Bernoulli equation is one of the well known nonlinear differential equations of the first order. It is written as*

$$y' + a(x)y = b(x)y^m,$$

where $a(x)$ and $b(x)$ are continuous functions.

Solving method:

- If $m = 0$, the equation becomes a linear differential equation. In case of $m = 1$, the equation becomes separable.
- In general case, when $m \notin \{0, 1\}$, Bernoulli equation can be converted to a linear differential equation using the change of variable

  $$z = y^{1-m}.$$

  The new differential equation for the function $z(x)$ has the form:

  $$z^{'} + (1-m)a(x)z = (1-m)b(x).$$

Easy to solve it.

Riccati equation

Definition 1.1.9. *Riccati equation is one of the most intersting nonlinear differential equations of the first order. It is written in the form*

$$y' = a(x)y + b(x)y^2 + c(x),$$

where $a(x)$, $b(x)$, $c(x)$ are continuous functions of x.

The Riccati equation is used in different areas of mathematics (for example, in algebraic geometry and the theory of conformal mapping), and physics. It also appears in many applied problems.

Solving method: The differential equation given above is called the general Riccati equation. It can be solved with help of the following cases:

If a particular solution y_1 of a Riccati equation is known, the general solution of the equation is given by $y = y_1 + u$. Indeed, substituting the solution $y = y_1 + u$ into Riccati equation, we have

$$\begin{aligned}(y_1+u)^{'} &= a(x)(y_1+u) + b(x)(y_1+u)^2 + c(x),\\ \underline{y_1^{'}} + u^{'} &= \underline{a(x)y_1} + a(x)u + \underline{b(x)y_1^2} + 2b(x)y_1u + b(x)u^2 + \underline{c(x)}.\end{aligned}$$

The underlined terms in the left and in the right side can be canceled because y_1 is a particular solution satisfying the equation. As a result we obtain the differential equation for the function u

$$u^{'} = b(x)u^2 + [2b(x)y_1 + a(x)]u,$$

which is a Bernoulli equation. Substitution of $z = \dfrac{1}{u}$ converts the given Bernoulli equation into a linear differential equation that allows integration.

1.1.5 Solutions and periodic solutions

Definition 1.1.10. *We call that* $(x(t), y(t))$, $t \in I$ *where* $I \subset \mathbb{R}$, *is a solution of the system* (1.1) *if the vector field* $X = (P, Q)$ *is tangent to the trajectory representing this solution in the phase plane.i.e :*

$$\forall t \in I : P(x(t), y(t))\, x' + Q(x(t), y(t))\, y' = 0$$

Definition 1.1.11. *A periodic solution of system* (1.1) *is defined by* $\big(x(t), y(t)\big)$ *if there exists a real* $T > 0$ *such as* $\forall t \in \mathbb{R}$*:*

$$\begin{cases} x(t+T) = x(t) \\ y(t+T) = y(t). \end{cases}$$

The smallest number $T > 0$ *is called the period of this solution.*

The interest in periodic solutions of non-linear systems goes back to the beginning of this century. Already Poincaré (1912) investigated periodic solutions of non-linear dynamical systems. He studied fixed points of area-preserving one-to-one transformations of simply connected areas on the plane. However, he could not prove his well known last geometric problem" himself. Birkhoff solved this problem some years later and extended it to an arbitrary dimension of the state space. Based on this theorem, Birkhoff, Lewis (1933) have proved the existence of an infinite number of periodic solutions of a conservative system in the neighborhood of a known periodic solution, which could also be a fixed point of the "general stable type".

1.1.6 Flow

Definition 1.1.12. *[70] Let E be an open subset of $\mathbb{R}^2$ and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the initial value of problem* (1.1) *defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by*

$$\phi_t(x_0) = \phi(t, x_0).$$

is called the flow of the differential system (1.1) *or the flow defined by the differential system* (1.1)*, ϕ_t is also referred to as the flow of the vector field $f(x)$.*

1.2 Equilibria

To know the aspect of the trajectories of system (1.1), at least locally, we must look for its equilibrium points.

Definition 1.2.1. *[70] A trajectory that reduces to a point, or a constant solution*

$$\begin{cases} x(t) = x_0, \\ y(t) = y_0 \end{cases}$$

is called an equilibrium solution. The equilibrium solutions or equilibria are found by solving the nonlinear equations

$$\begin{cases} P(x_0, y_0) = 0, \\ Q(x_0, y_0) = 0. \end{cases}$$

Each such (x_0, y_0) in Ω is a trajectory whose graphic in the phase plane is a single point, called an equilibrium point. In applied literature, it may be called a critical point, stationary point or rest point.

1.2.1 Stability of an equilibrium point

Let (x_0, y_0) be an equilibrium point of the system (1.1). Notes by

$$Y(t) = \Big(P\big(x(t), y(t)\big), Q\big(x(t), y(t)\big)\Big) \textit{ and } Y_0 = \big(P(x_0, y_0), Q(x_0, y_0)\big).$$

Definition 1.2.2. *We say that (x_0, y_0) is stable if and only if*

$$\forall \epsilon > 0, \exists \eta > 0, \|(x, y) - (x_0, y_0)\| < \eta \implies \|Y(t) - Y_0\| < \epsilon, \quad \forall t > 0.$$

The point (x_0, y_0) is asymptotically stable if and only if (x_0, y_0) is stable and

$$\lim_{t \to \infty} \| Y(t) - Y_0 \| = 0.$$

1.2.2 Linear homogeneous system

Let a second order linear homogeneous system with constant coefficients be given in cartesian coordinates as

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y, \\ \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y, \end{cases}$$

this system of equations is autonomous since the right hand sides of the equations do not explicitly contain the independent variable t.

In matrix form, the system of equations can be written as

$$X' = AX, \quad \textit{where } X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The equilibrium positions can be found by solving the stationary equation

$$AX = 0.$$

This equation has the unique solution $X = 0$ if the matrix A is nonsingular, i.e. provided that $detA \neq 0$. In the case of a singular matrix, the system has an infinite number of equilibrium points.

Classification of equilibrium points is determined by the eigenvalues λ_1, λ_2 of the matrix A. The numbers λ_1, λ_2 can be found by solving the auxiliary equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

There are four different types of equilibrium points:

Node: Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 > 0$.

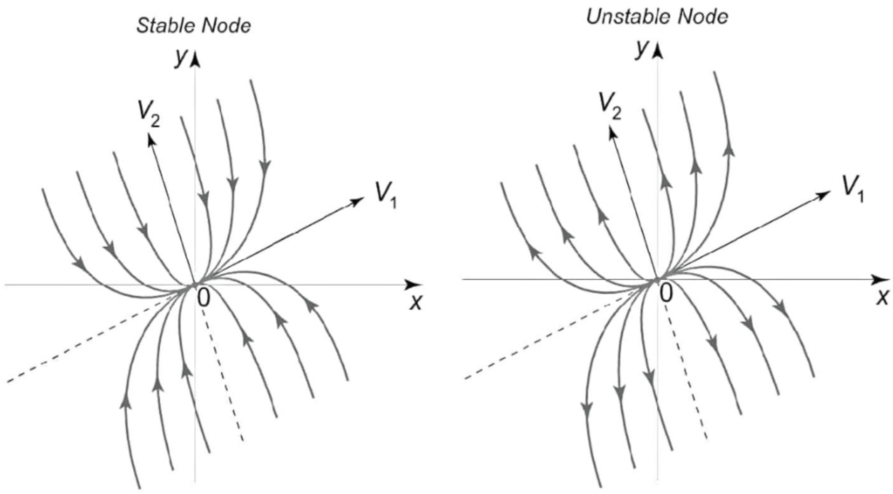

Figure 1.1: Node point.

Saddle: Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 < 0$.

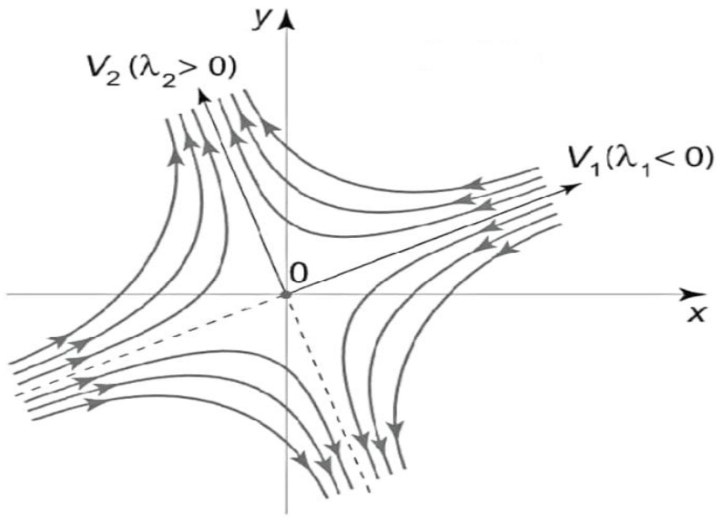

Figure 1.2: Saddle point.

Focus: Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $Re\lambda_1 = Re\lambda_2 \neq 0$.

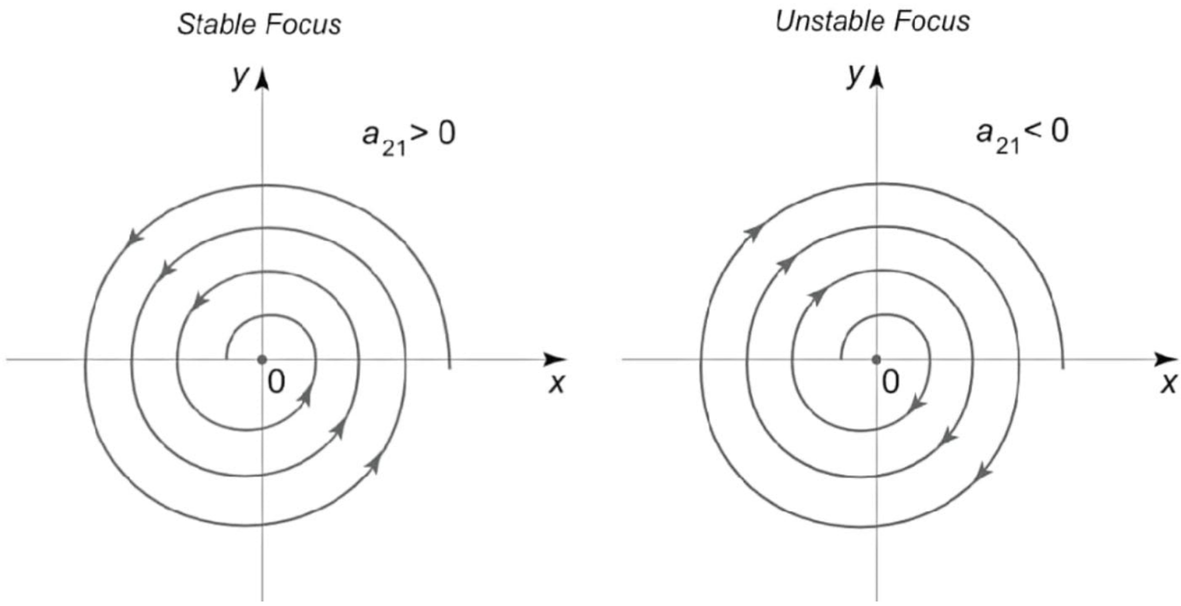

Figure 1.3: Focus point.

Center: Let λ_1 and λ_2 are purely imaginary numbers and $Re\lambda_1 = Re\lambda_2 = 0$.

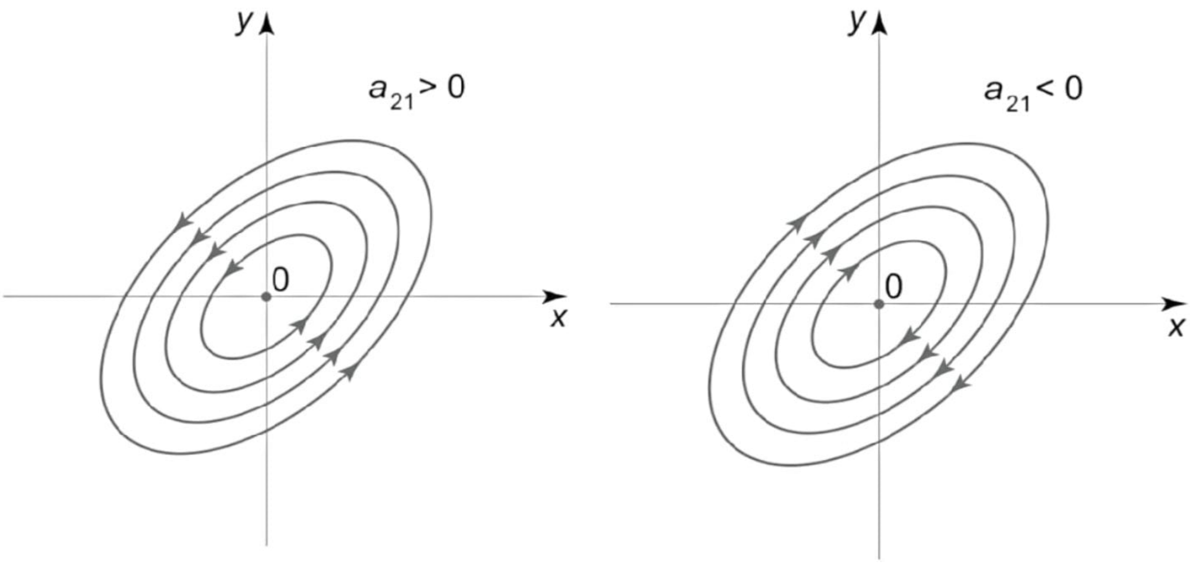

Figure 1.4: Center point.

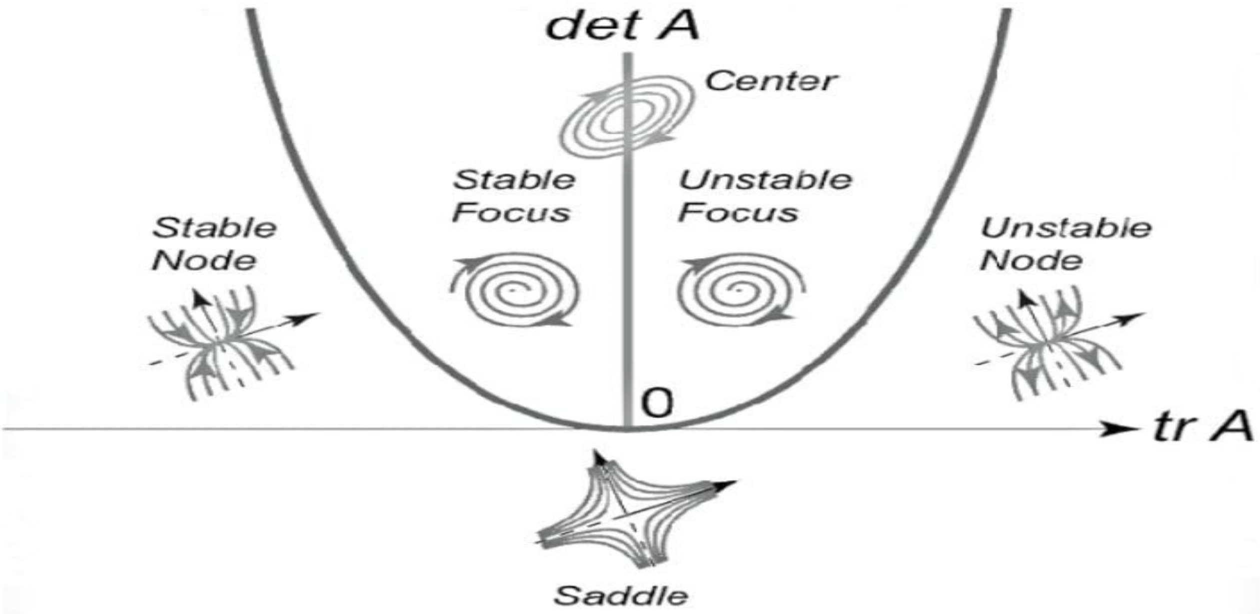

Figure 1.5: Equilibrium points.

1.2.3 Invariant algebraic curves

Definition 1.2.3. *A real polynomial U is called algebraic solution of the polynomial differential system* (1.1) *if*

$$\frac{\partial U(x,y)}{\partial x}P(x,y)+\frac{\partial U(x,y)}{\partial y}Q(x,y)=K(x,y)U(x,y),$$

for some polynomial K, called the cofactor of U. The corresponding cofactor of U is always polynomial whether U is algebraic or nonalgebraic.

Remark 1.2.1. *The curve $U(x,y)=0$ is an invariant under the flow of differential system* (1.1) *and the set $\{(x,y)\in\mathbb{R}^2, U(x,y)=0\}$ is formed by orbits of system* (1.1).

Definition 1.2.4. *[30] If the algebraic curve $U=0$ is invariant by a vector field X of degree d, then K is a polynomial of degree at most $d-1$. We say that $U(x,y)=0$, $\forall(x,y)\in\mathbb{R}^2$, is an **algebraic** curve of X, it is a polynomial of variables x and y, otherwise it is said **non-algebraic**.*

Remark 1.2.2. *One of the main applications of invariant algebraic curves is in constructing first integrals and integrating factors of Darboux type: that is, functions which are expressible as products of invariant algebraic curves and exponential factors.*

1.3 Integrating factor

1.3.1 Exact equation

Definition 1.3.1. *A differential equation of type $P(x,y)dx+Q(x,y)dy=0$ is called an exact differential equation if there exists a function of two variables $u(x,y)$ with continuous partial derivatives such that*

$$du(x,y)=P(x,y)dx+Q(x,y)dy.$$

The general solution of an exact equation is given by $u(x,y)=c$, where c is an arbitrary constant.

Let functions P and Q have continuous partial derivatives in a certain domain Ω. The differential equation $P(x,y)dx+Q(x,y)dy=0$ is an exact equation if and only if

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}.$$

1.3.2 Integrating factor

Definition 1.3.2. *[36] Consider a differential equation of type* $P(x,y)dx+Q(x,y)dy=0$*, where* P *and* Q *are functions of two variables* x *and* y *continuous in a certain region* $\Omega \subseteq \mathbb{R}^2$*, if*

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y},$$

the equation is not exact. However, we can try to find so-called integrating factor, which is an analytic function $R : \Omega \to \mathbb{R}$ *that is notidentically zero on* Ω *such that the equation becomes exact after multiplication by this factor. If so, then the relationship*

$$\frac{\partial\big(RQ(x,y)\big)}{\partial x} = \frac{\partial\big(RP(x,y)\big)}{\partial y}, \tag{1.3}$$

is true.

Remark 1.3.1. *The condition 1.3 can be written in the form:*

$$Q\frac{\partial R}{\partial x} + R\frac{\partial Q}{\partial x} = P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y}.$$

The last expression is the partial differential equation of first order that defines the integrating factor R.

Remark 1.3.2. *Unfortunately, there is no general method to find the integrating factor. However, we can mention some particular cases for which the partial differential equation can be solved and as a result we can construct the integrating factor.*

Proposition 1.3.1. *[37] Let* Ω *be an open subset of* $\mathbb{R}^2$ *i.e* $\Omega \subseteq \mathbb{R}^2$ *and let* $R : \Omega \to \mathbb{R}$ *be an analytic function which is not identically zero on* Ω*. The function R is an integrating factor of the system* (1.1) *on* Ω *if one of the following equivalent conditions hold*

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y},$$

$$div(RP, RQ) = 0,$$

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R\,div(P,Q),$$

where $div(P,Q)$ is the divergence of vector field (P,Q) given by

$$div(X) = div(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The first itegral H associated to the integrating factor R is given by

$$H(x,y)=\int R(x,y)P(x,y)dy+h(x),$$

where h is chosen such that $\dfrac{\partial H}{\partial x}=-RQ$. Then

$$\begin{cases}\dot{x}=RP=\dfrac{\partial H}{\partial y},\\ \dot{y}=RQ=-\dfrac{\partial H}{\partial x}.\end{cases}$$

1.4 First integral

Definition 1.4.1. *System* (1.1) *is integrable on an open set* Ω *of* $\mathbb{R}^2$ *if there exists a non constant* C^1*-function* $H:\Omega\to\mathbb{R}$*, called a first integral of the system on* Ω*, which is constant on the trajectories of the system* (1.1) *contained in* Ω*, i.e. if*

$$\frac{dH\left(x,y\right)}{dt}=\frac{\partial H\left(x,y\right)}{\partial x}P\left(x,y\right)+\frac{\partial H\left(x,y\right)}{\partial y}Q\left(x,y\right)\equiv 0 \text{ in the points of } \Omega.$$

It is well known that for differential systems defined on the plane $\mathbb{R}^2$ *the existence of a first integral determines their phase portrait.*

Example 1.4.1. *The polynomial system*

$$\begin{cases}\dot{x}=-y-b(x^2+y^2),\\ \dot{y}=x,\end{cases}$$

where $b\in\mathbb{R}$*, has the first integral* $H(x,y)=e^{2by}(x^2+y^2)$*,*
and has the integrating factor $R(x,y)=e^{2by}$*. From it we can obtain the first integral, and the other way around, see [37].*

Proposition 1.4.1. *[37]*

1. *If system* (1.1) *has two integrating factors* R_1 *and* R_2 *defined in* Ω*, then the functions* R_1/R_2*, which is defined in* $\Omega-\{R_2=0\}$*, and* $R_1R_2/(R_2^1+R_2^2)$*, which is defined in* $\Omega-\big(\{R_1=0\}\cap\{R_2=0\}\big)$*, are first integrals of* (1.1)*.*
2. *If system* (1.1) *has an integrating factor* R *and a first integral* H*, both defined in* Ω*, then the function* RH *is another integrating factor defined in* Ω*.*

1.5 Limit cycles

1.5.1 Periodic orbits

Definition 1.5.1. *A periodic orbit of* (1.1) *is any closed solution curve of* (1.1) *which is not an equilibrium point of* (1.1)*.*
The system (1.1) *correspond to periodic solutions of* (1.1) *since* $\phi(., x_0)$ *defines a closed solution curve of* (1.1) *if and only if for all* $t \in \mathbb{R}$*,*

$$\phi(t + T, x_0) = \phi(t, x_0)$$

for some $T > 0$*.*
The minimal T *for which this equality holds is called the period of the periodic orbit* $\phi(., x_0)$*.*

1.5.2 Limit cycles

Definition 1.5.2. *[74] A limit cycle of a vector field* X *in dimension* 2 *is a periodic orbit* Γ *which is isolated on one side, i.e., not approached by periodic orbits, all belonging to one side of* Γ*. (If* X *is analytic a limit cycle is necessarily isolated on both sides). The limit cycle is* ***algebraic*** *if it is contained in an algebraic curve in the plane, otherwise it is said* ***non-algebraic****.*

Remark 1.5.1. *Limit cycles appears only in the non-linear differential systems.*

The interest of the limit cycle, as an isolated periodic orbit, appears often in several branches of science and technology. The fact when the system admits a limit cycle implies the existence of an isolated periodic solution. The general problem of finding the number of limit cycles for dynamical systems is a complicated problem that has a linking with the 16th problem of Hilbert that it's not yet solved. The intensive study of the existence of limit cycles for dynamical systems is well justified because the existence and properties of limit cycles for a dynamical system gives the important information and introduce the interesting properties of solutions of the dynamical system studied.

Example 1.5.1. *[51] Consider the nonlinear system*

$$\begin{cases} x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + yx^2), \\ y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + xy^2). \end{cases}$$

We change to polar coordinates, the equations becomes much simpler. We compute

$$\begin{cases} r'\cos\theta - r\theta'\sin\theta = \frac{1}{2}(r - r^3)\cos\theta - \sin\theta, \\ r'\sin\theta + r\theta'\cos\theta = \frac{1}{2}(r - r^3)\sin\theta + \cos\theta, \end{cases}$$

from which we conclude, after equating the coefficients of $cos\theta$ *and* $sin\theta$*,*

$$\begin{cases} r' = \frac{1}{2}r(1 - r^2), \\ \theta' = 1. \end{cases}$$

We can now solve this system explicitly, since the equations are decoupled. Rather than do this, we will proceed in a more geometric fashion. From the equation $\theta' = 1$*, we conclude that all nonzero solutions spiral around the origin in the counterclockwise direction. From the first equation, we see that solutions do not spiral toward* ∞*. Indeed, we have* $r' = 0$ *when* $r = 1$*, so all solutions that start on the unit circle stay there forever and move periodically around the circle. Since* $r' > 0$ *when* $0 < r < 1$*, we conclude that nonzero solutions inside the circle spiral away from the origin and toward the unit circle. Since* $r' < 0$ *when* $r > 1$*, solutions outside the circle spiral toward it.*

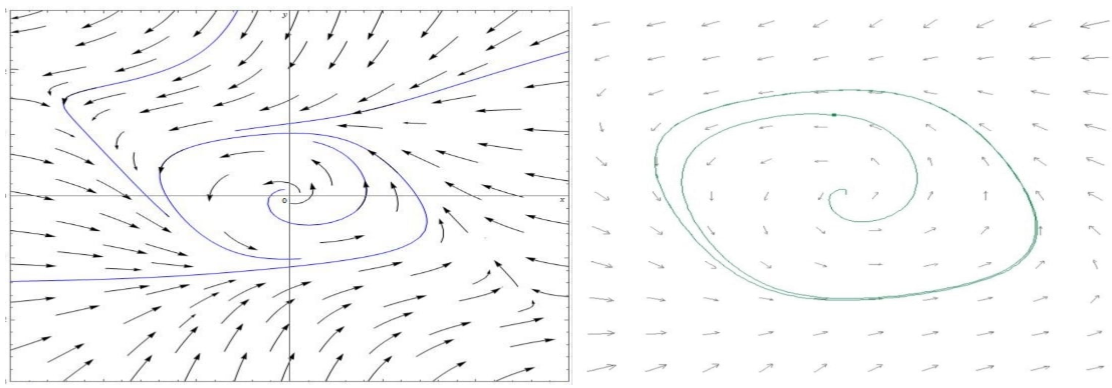

Figure 1.6: Phase plane of example 1.6.1

1.5.3 Classification of limit cycles

There are three types of limit cycles: stable limit cycle, unstable limit cycle and semi-stable limit cycle, see *[70]*.

Stable limit cycles

A stable limit cycle has a physic interpretatioun such as limit oscillation of the system studied. It is a periodic solution which the other solutions tend to it.

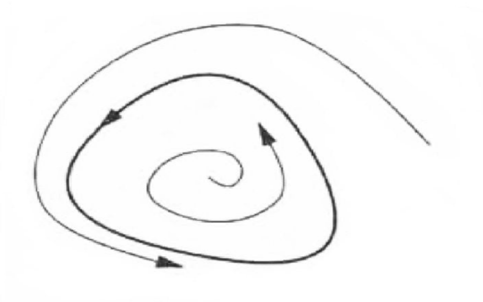

Figure 1.7: Stable limit cycle.

Unstable limit cycles

An unstable limit cycle does not appear physicly such as an oscillation. It constitutes a separation on each side which the trajectories move away towards other singular points or towards infinity.

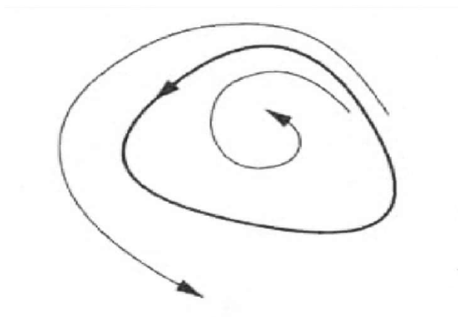

Figure 1.8: Unstable limit cycle.

Semi-stable limit cycles

A semi-stable limit cycle is a closed trajectory such as the trajectories tend on one side but move away on the other side.

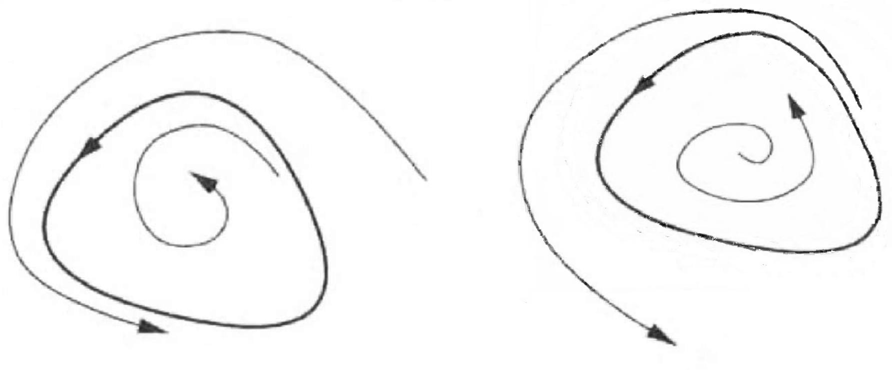

Figure 1.9: Semi-stable limit cycle.

Theorem 1.5.1. *Let $\gamma(t)$ a periodic orbit of the system (1.1) of period T. γ is a stable limit cycle if*

$$\int_0^T div(\gamma(t))dt < 0,$$

where $div(\gamma(t))$ is the divergence of the system, defined by

$$div(\gamma(t)) = \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big)\gamma(t),$$

γ is a unstable limit cycle if

$$\int_0^T div(\gamma(t))dt > 0.$$

In the case that the amount $\int_0^T div(\gamma(t))dt$ equal zero, an advanced study is necessary to determine if the orbit γ is a stable limit cycle, or unstable limit cycle or semi-stable limit or it is only a periodic orbit belonging to continuous band of closed orbits.

1.6 The inverse integrating factor

Another important tool in the study of planar differential systems is the inverse integrating factor.

Definition 1.6.1. *Consider the planar differential system*

$$\begin{cases} x' = P(x,y), \\ y' = Q(x,y), \end{cases}$$

where P and Q are C^2-functions in the variables x and y. Let X be its associated vector field and let

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.4}$$

Let Ω be the domain of definition of system (1.1)*, and let W be an open subset of Ω. A non-zero C^1-function $V : W \to \mathbb{R}$ is an inverse integrating factor of system* (1.1) *on W if it is a solution of the linear partial differential equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V, \tag{1.5}$$

also written $XV = V divX$. As we deduce from this equation, the gradient $(\partial V/\partial x, \partial V/\partial y)$ of the set of curves $V^{-1}(0) = \Sigma$ is orthogonal to the vector field X. So X is tangent to $\{V = 0\}$, and then this curve is formed by trajectories of X. Moreover, $V^{-1}(0)$ is an invariant algebraic curve of (1.1) *with cofactor $divX$.*

Proposition 1.6.1. *[37] Let V be an inverse integrating factor of system* (1.1) *defined in the open subset $W \subseteq \mathbb{R}^2$. Then,*

1. *The function $\frac{1}{V}$, defined in $W - \{V = 0\}$, is an integrating factor of system* (1.1)*. Moreover, the function*

$$H(x,y) = -\int \frac{P(x,y)}{V(x,y)}dy + \int \left(\frac{Q(x,y)}{V(x,y)} + \frac{\partial}{\partial x}\int \frac{P(x,y)}{V(x,y)}dy\right)dx, \tag{1.6}$$

   is a first integral of (1.1)*.*

2. *If system* (1.1) *has a first integral H, then the function*

$$V_H(x,y) = \frac{P(x,y)}{-\dfrac{\partial H(x,y)}{\partial y}} = \frac{Q(x,y)}{\dfrac{\partial H(x,y)}{\partial x}},$$

   is an inverse integrating factor of (1.7)*. Moreover, the system*

$$\begin{cases} x' = \dfrac{P(x,y)}{V_H} = -\dfrac{\partial H(x,y)}{\partial y}, \\[2ex] y' = \dfrac{Q(x,y)}{V_H} = \dfrac{\partial H(x,y)}{\partial x}, \end{cases} \tag{1.7}$$

   is Hamiltonian in $W - \{V = 0\}$.

Proof. *The first part of the proposition follows from the computation*

$$X\frac{1}{V} = P\left(\frac{1}{V}\right)_x + Q\left(\frac{1}{V}\right)_y = -\frac{XV}{V^2} = -\frac{1}{V}divX.$$

The expression of H can be obtained as in previous section.
To prove the second part, we note that $\frac{1}{V_H}$ is an integrating factor of (1.1), *so system* (1.7) *is Hamiltonian in $W - \{V = 0\}$.*

Remark 1.6.1. *[37] Proposition* (1.4.1) *can be applied also to inverse integrating factors.*

The following lemma (see [37]) gives a linear property of the inverse integrating factors.

Lemma 1.6.2. *[37] Let $V_1, ..., V_p$ be inverse integrating factors of system* (1.1)*, $a_1, ..., a_p \in \mathbb{R}$. Then, the function $V = \sum_{i=1}^{p} a_i V_i$ is an inverse integrating factor of system* (1.1).

The function V is an inverse integrating factor of the system (1.1) on open set $\Omega \subseteq \mathbb{R}^2$ if $V \in C^l(\Omega), V \neq 0$ on Ω and

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = RV,\ (x, y) \in \Omega. \tag{1.8}$$

It is easy to verify that the function $R = \frac{1}{V}$ defines an integrating in $\Omega - \{V = 0\}$ of the system (1.1).

Corollary 1.6.3. *[51] Let H be a first integral of a planar system. If H is not constant on any open set, then there are no limit cycles.*

Proof. *Suppose there is a limit cycle γ, let $c \in \mathbb{R}$ be the constant value of H on γ. If $X(t)$ is a solution that spirals toward γ, then $H(X(t)) = c$ by continuity of H. In corollary* (1.7.3) *we found an open set whose solutions spiral toward γ, thus H is constant on an open set.*

The inverse integrating factor is among the tools that are used in the study of the existence and non-existence of limit cycles, we even can determine their formulas using the inverse integrating factor. This method is introduite by Giacomini, Llibre and Viano in 1996 and is based on the following Criteria:

Criterion 1. *[43] Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$, $(x(t), y(t))$ a periodic solution of (P, Q) of period T, $R : U \to \mathbb{R}$ a C^1-map such that $\int_0^T R(x(t), y(t))dt \neq 0$, and $V = V(x, y)$ is a C^1-solution of linear partial differential equation (1.8) then the closed trajectory*

$$\gamma = \{(x(t), y(t)) \in U : t \in [0, T]\}$$

is contained in

$$\Sigma = \{(x, y) \in U : V(x, y) = 0\},$$

and γ is not contained in a period annulus of (P, Q). Moreover, if the vector field (P, Q) is analytic, then γ is a limit cycle.

Let W be an open subset of $\mathbb{R}^2$. A C^1-vector field (P, Q) defined in W is a C^1-map which associates to each point $(x, y) \in W$ a vector $\big(P(x, y), Q(x, y)\big)$ in $\mathbb{R}^2$ based at (x, y).
The system

$$\begin{cases} x' = P(x, y), \\ y' = Q(x, y), \end{cases} \tag{1.9}$$

is called the differential system associated to the vector field (P, Q). If the solution $x = x(t)$, $y = y(t)$ of system (1.9) is a nonconstant periodic function of t, it is called periodic. Then the locus of this solution in W is a closed trajectory of system (1.9). If a closed trajectory of (1.9) is isolated in the set of all closed trajectories of (1.9), it is called a limit cycle. A period annulus for system (1.9) is a closed annulus fulfilled of closed trajectories of system (1.9).

Proof. *We define $R(t) = R\big(x(t), y(t)\big)$ and $V(t) = R\big(x(t), u(t)\big)$.*
We consider the differential equation $\dot{z} = R(t)z$. Its general solution is

$$z(t) = z(0) \exp \int_0^t R(s)ds.$$

Since $V(x, y)$ is a solution of (1.9) and $(x(t), y(t))$ is a solution of (1.1), it follows that $z = V(t)$ is a solution of $\dot{z} = R(t)z$. So

$$V(t) = V(0) \exp \left(\int_0^t R(s)ds \right),$$

since $(x(t), y(t))$ is a periodic solution of period T, $V(T) = V(0)$.
Hence, since $\int_0^t R(t)dt \neq 0$, we get that $V(0) = 0$, and consequently $V(t) = 0$.

Now we suppose that γ is contained in a period annulus, then there is a closed annulus neighbourhood A of γ fulfilled of closed trajectories.
We note that perhaps γ is in the boundary of A. Since $\int_\gamma Rdt \neq 0$, if A is sufficiently narrow, then $\int_{\gamma'} Rdt \neq 0$ for any closed trajectory γ' in A. So $A \subset \Sigma$, in contradiction with the fact that Σ is a locally 1-dimensional manifold, except perhaps in finitely many points.
Hence γ is not contained in a period annulus.
It is well known in the theory of planar analytic vector fields that a periodic orbit is either a limit cycle, or it is contained in a period annulus, see [53]. Since the second possibility cannot occur, γ is a limit cycle.

Criterion 2. *[37] Let $(P.Q)$ a C^l-class vector field defined in the open subset U of $\mathbb{R}^2$. Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation:*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})V.$$

If γ is a limit cycle of (1.1)*, then γ is contained in:*

$$\Sigma = \{(x, y) \in U : V(x, y) = 0\}.$$

Definition 1.6.2 (Hyperbolic limit cycle)**.** *If $\int_0^T div(\gamma(t))dt$ is different of* 0*, we say that the limit cycle is hyperbolic.*

Example 1.6.1. *[41] The Liénard system,*

$$\begin{cases} x' = y - x(x^2-2)(x^2-1)(x^2-\frac{1}{4}), \\ y' = -x, \end{cases}$$

has exactly three limit cycles. Furthermore they are concentric and hyperbolic, see [41].

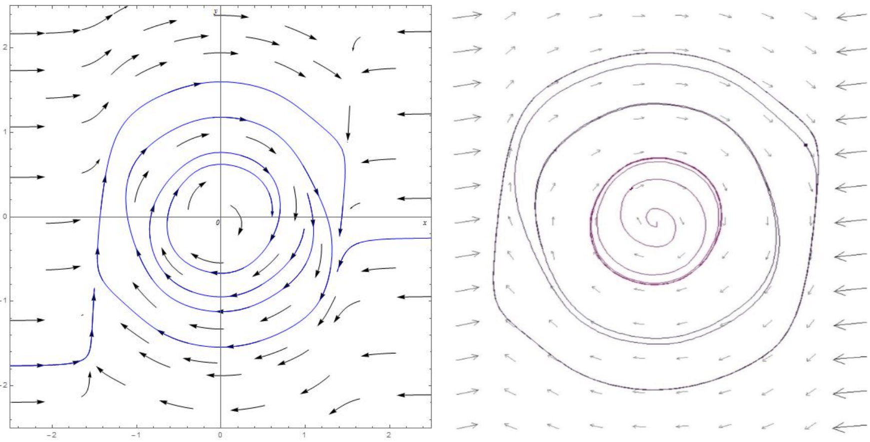

Figure 1.10: Phase plane of example 1.7.1

Corollary 1.6.4. *[51] A compact set K that is positively or negatively invariant contains either a limit cycle or an equilibrium point.*

Corollary 1.6.5. *[51] Let γ be a closed orbit and let U be the open region in the interior of γ. Then U contains either an equilibrium point or a limit cycle.*

Theorem 1.6.6. *[26] If a quadratic system has an algebraic limit cycle of degree 2, then after an affine change of variables, the limit cycle becomes the circle*

$$\Gamma = x^2 + y^2 - 1 = 0,$$

Moreover, Γ is the unique limit cycle of the quadratic system which can be written in the form

$$\begin{cases} x' = -y(ax + by + c) - (x^2 + y^2 - 1), \\ y' = x(ax + by + c), \end{cases}$$

with $a \neq 0$, $c^2 + 4(b + 1) > 0$ and $c^2 > a^2 + b^2$.

General case: The number of limit cycles of a polynomial differential equation is the subject of the second part of Hilbert's sixteenth problem. The Poincaré-Bendixson theorem and that of Bendixson-Dulac predict the existence, respectively the non-existence, of limit cycles for nonlinear differential equations in two dimensions.

1.7 Criteria for existence, non existence of limit cycles

1.7.1 Green's theorem

Definition 1.7.1. *Let Γ be a planar simple closed curve. Suppose that P and Q are two continuously differentiable functions defined on the interior of Γ; called D, then*

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy = \oint_\Gamma Pdy - Qdx.$$

1.7.2 Bendixon criterion

The search for periodic solutions (limit cycles) is conditioned by the Bendixon's criterion that there can be no closed orbit content in a simply related region of the plan, if the divergence of the field vectors keep a constant sign.
In this paragraph, we give results that allow to demonstrate the non-existence of periodic solutions for a two-order autonomous differential system.

Theorem 1.7.1 (Bendixon criterion)**.** *Let P and Q two functions owned at $C^1(\Omega, \mathbb{R})$, where Ω is a simply connected region in $\mathbb{R}^2$. Considering the autonomous system* (1.1). *If $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is not identically zero and does not change sign in Ω, then* (1.1) *has no closed orbit lying entirely in Ω.*

Proof. *Suppose that $\Gamma : X = X(t)$, $0 \leq t \leq T$ is a closed orbit of* (1.1) *lying entirely in Ω. If D denotes the interior of Γ, it follows from Green's theorem that*

$$\begin{aligned} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy &= \oint_\Gamma (Pdx - Qdy) \\ &= \oint_\Gamma (P\dot{y} - Q\dot{x})dt \\ &= \oint_\Gamma (PQ - QP) = 0, \end{aligned}$$

and if $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is not identically zero and does not change sign in D, then it follows from the continuity of $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ in D that the above double integral is either positive or negative. In either case this leads to a contradiction. Therefore, there is no closed orbit of (1.1) *lying entirely in Ω.*

A more general result of this type, which is also proved using Green's theorem, given by the following theorem:

1.7.3 Dulac criterion

Theorem 1.7.2 (Dulac criterion). *[70] Consider the system: $x' = f(x)$, $x = (x_1, x_2) \in \mathbb{R}^2$,*
$f = (f_1, f_2)$. Let Ω is a simply connected region in $\mathbb{R}^2$ and $\mu \in C^1(\Omega)$, if

$$div\big(\mu(x)f(x)\big) = \frac{\partial\big(\mu(x)f_1(x)\big)}{\partial x_1} + \frac{\partial\big(\mu(x)f_2(x)\big)}{\partial x_2},$$

does not vanish on any open subset of Ω, then the system does not admit a periodic orbit in Ω.

Proof. *Let ϕ be a periodic solution in Ω which surrounds a region $A \subset \Omega$, $div\big(\mu(x)f(x)\big)$ not identically zero and does not change sign in A then*

$$\iint_A div(\mu f)dA \neq 0,$$

using Green's theorem:

$$\iint_A div(\mu f)dA = \oint_\phi \mu f \overrightarrow{n} dl,$$

$\overrightarrow{n}$ is the normal to outward
$f\overrightarrow{n} = 0$, because ϕ is a periodic orbit then the vector field is tangent to ϕ, the normal is perpendicular to f and:

$$\oint_\phi \mu f \overrightarrow{n} dl = 0,$$

on the other hand

$$\iint_A div(\mu f)dA \neq 0,$$

and

$$\iint_A div(\mu f)dA \neq 0 = \oint_\phi \mu f \overrightarrow{n} dl = 0.$$

This leads to a contradiction. Therefore, there is no periodic orbit lying entirely in Ω

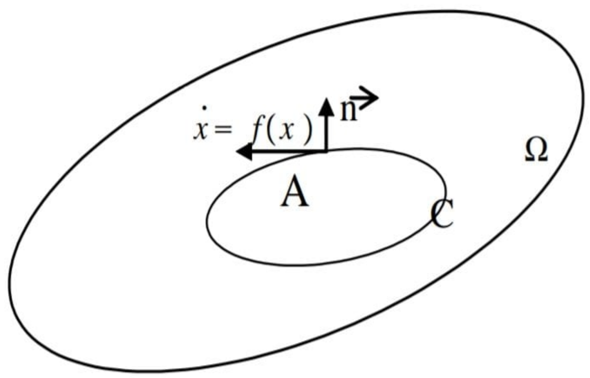

Figure 1.11: Dulac's creterion of non existence.

Theorem 1.7.3 (Poincaré-Bendixson)**.** *Suppose that* Ω *is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then* Ω *is a closed orbit.*

Theorem 1.7.4 (Existence)**.** *Let* C *and* C' *two closed curves, the second surrounds the first.*
If in each point of C *the speed vector* (P, Q) *of trajectory that passes through towards outside and if each point of* C'*, it is led towards the interior, so there is at least a limit cycle between* C *and* C'*.*

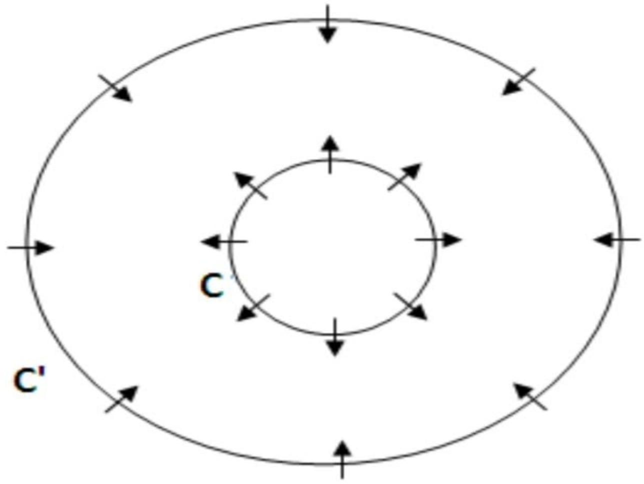

Figure 1.12: Existence of a limit cycle between C and C'.

Theorem 1.7.5. *[29] Let $f = 0$ be a non-singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D = 0$ lies outside all bounded components of $f = 0$. Choose the constants α and β so that*

$$\alpha D_x + \beta D_y \neq 0,$$

then the polynomial vector field of degree m,

$$\begin{cases} \dot{x} = \alpha f - D f_y, \\ \dot{y} = \beta f + D f_x, \end{cases}$$

has all the bounded components of $f = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.

Theorem 1.7.6 (Poincaré-Bendixon). *[3] Let P and Q two functions belonging to $C^1(\Omega, \mathbb{R})$, where Ω is a closed bounded of $\mathbb{R}^2$. Suppose that:*

- *The system* (1.1) *does not admit a fixed point in Ω.*
- *The solution $\Gamma = \{(x, y) = (\Phi(t), \Psi(t)),\ t \geq t_0\}$ remains inside of Ω.*

So one of the two following propositions is satisfied:

1. *Γ is a limit cycle,*

2. Γ rolls up in spirals over a limit cycle.

In both cases, the system (1.1) *admits a periodic solution.*

1.7.4 The Poincaré Map

The most basic tool for studying the stability of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881.
The idea of Poincaré's map is quite simple: If γ is a periodic orbit of the system:

$$x^{'} = f(x) \tag{1.10}$$

through the point x_0 and Σ is a hyperplane perpendicular to Γ at x_0, then for any point $x_0 \in \Sigma$ sufficiently near to x_0, the solution of (1.10) through x at $t = 0$, $\phi_t(x)$ will cross Σ again at a point $P(x)$ near x_0, the function $x \to P(x)$ is called the Poincaré map.
The Poincaré map can also be defined when $x_0 \in \Gamma$, which is not tangent to Γ at x_0.
In this case, the surface Σ is said to intersect the curve Γ transversally at x_0.
The next theorem establishes the existence and continuity of the Poincaré map $P(x)$ and of its first derivative $DP(x)$.

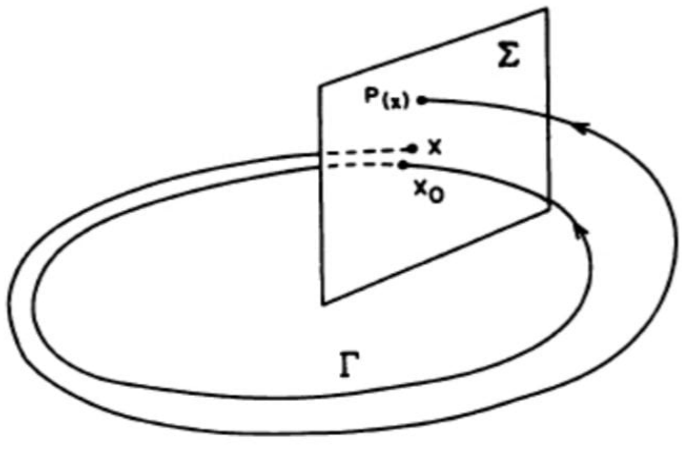

Figure 1.13: The Poincaré Map.

Theorem 1.7.7. *[70] Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$, suppose that, $\phi_t(x_0)$ is a periodic solution of (1.10) of period T and that the cycle:*

$$\Gamma = \{x \in \mathbb{R}^n / x = \phi_t(x_0),\ 0 \leq t \leq T\},$$

is contained in E. Let Σ the hyperplane orthogonal to Γ at x_0, i.e, let:

$$\Sigma = \{x \in \mathbb{R}^n / (x - x_0).f(x_0) = 0\},$$

then there is a $\delta > 0$ and a unique function $\tau(x)$, defined and continuously differentiable for $x \in N_\delta(x_0)$, such that: $\tau(x_0) = T$ and

$$\phi_{\tau(x)}(x) \in \Sigma,$$

for all $x \in N_\delta(x_0)$.

Proof. *The proof of this theorem is an immediate application of the implicit function theorem. For a given point $x_0 \in \Gamma \subset E$, define the function*

$$F(t, x) = [\phi_t(x) - x_0].f(x_0)$$

It then follows that $F \in C^1(\mathbb{R} \times E)$ and it follows from the periodicity of $\phi_t(x_0)$ that $F(T, x_0) = 0$.
Furthermore, since $\phi(t, x_0) = \phi_t(x_0)$ is a solution of (1.1) which satisfies $\phi(T, x_0) = x_0$, it follows that

$$\frac{\partial F(T, x_0)}{\partial t} = \frac{\partial \phi(T, x_0)}{\partial t}.f(x_0) = f(x_0).f(x_0) = |f(x_0)|^2 \neq 0,$$

since $x_0 \in \Gamma$ is not an equilibrium point of (1.1). Thus, it follows from the implicit function theorem, that there exists a $\delta > 0$ and a unique function $\tau(x)$ defined and continuously differentiable for all $x \in N_\delta(x_0)$ such that $\tau(x_0) = T$ and such that

$$F(\tau(x), x) = 0$$

for all $x \in N_\delta(x_0)$. Thus, for all $x \in N_\delta(x_0)$,

$$[\phi(\tau(x), x) - x_0].f(x_0) = 0,\ i.e.,\ \phi_{\tau(x)}(x) \in \Sigma.$$

Definition 1.7.2. *[70] Let Γ, Σ, δ and $\tau(x)$ be defined as in the previous theorem, then for $x \in N_\delta(x_0) \cap \Sigma$, the function $P(x) = \phi_{\tau(x)}(x)$ is called the Poincaré map for Γ at x_0.*

Theorem 1.7.8. *[70] Let E be an open subset of $\mathbb{R}^2$ and suppose that $f \in C^1(E)$. Let $\gamma(t)$ be a periodic solution of* (1.10) *of period T. Than the derivative of the Poincaré map $P(s)$ along a straight line Σ normal to $\Gamma = \{x \in \mathbb{R}^2 / x = \gamma(t) - \gamma(0), 0 \leq t \leq T\}$ at $x = 0$ is given by:*

$$P'(0) = exp \int_0^T \bigtriangledown . f(\gamma(t))dt.$$

Corollary 1.7.9. *[70] Under the hypotheses of the previous theorem, the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \bigtriangledown . f(\gamma(t))dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \bigtriangledown . f(\gamma(t))dt > 0.$$

It may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is zero.

In other words *[67].* When plotting the solutions to some nonlinear problems, the phase space can become overcrowded and the underlying structure may become obscured. To overcome these difficulties, a basic tool was proposed by Henri Poincaré at the end of the nineteenth century. As a simple introduction to the theory of Poincaré (or first return) maps consider two-dimensional autonomous systems of the form (1.1) that there is a curve or straight line segment, say, Σ, that is crossed transversely (no trajectories are tangential to Σ). Then Σ is called a Poincaré section. Consider a point r_0 lying on Σ. As shown in next figure, follow the flow of the trajectory until it next meets Σ at a point r_1. This point is known as the first return of the discrete Poincaré map $P : \Sigma \to \Sigma$, defined by

$$r_{n+1} = P(r_n),$$

where r_n maps to r_{n+1} and all points lie on Σ. Finding the function P is equivalent to solving the differential equations(1.1).Unfortunately, this is very seldom possible, and one must rely on numerical solvers to make any progress.

Theorem 1.7.10. *[67] Define the characteristic multiplier M to be*

$$M = \left.\frac{dP}{dr}\right|_{r_0},$$

where r_0 *is a fixed point of the Poincaré map* P *corresponding to a limit cycle, say,* Γ*. Then if*

1. $|M| < 1$, Γ *is a hyperbolic stable limit cycle,*
2. $|M| > 1$, Γ *is a hyperbolic unstable limit cycle,*
3. $|M| = 1$ *and* $\dfrac{d^2P}{dr^2} \neq 0$, *then the limit cycle is stable on one side and unstable on the other, in this case* Γ *is called a semistable limit cycle.*

CHAPTER 2

SOME CLASSES OF TWO-DIMENSIONAL KOLMOGOROV SYSTEMS

2.1 Introduction

This chapter is composed of two parts: in the first part, we determine the conditions for the non-existence of periodic orbits as well as the non-existence of limit cycles of Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + R(x,y)\ln\left|\dfrac{A(x,y)}{B(x,y)}\right|\right), \\ y' = y\left(Q(x,y) + R(x,y)\ln\left|\dfrac{A(x,y)}{B(x,y)}\right|\right), \end{cases}$$

where $A,\ B,\ P,\ Q$ and R are homogeneous polynomials of degree a, a, n, n and m respectively, the results obtained for this class was a subject of our publication [21].

In the second part, we introduce an explicit expression of invariant algebraic curves of the multi-parameter planar Kolmogorov system of the form

$$\begin{cases} x' = x(1 + ax^2 + bxy + cy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \\ y' = y(1 + nx^2 + mxy + sy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \end{cases}$$

where a, b, c, n, m and s are real constants. Then we proved that this system is integrable and we introduced an explicit expression for a first integral. The results obtained for this class was a subject of our publication [77].

2.2 On the class of two dimentional Kolmogorov systems

We are intersted in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + R(x,y)\ln\left|\dfrac{A(x,y)}{B(x,y)}\right|\right), \\ y' = y\left(Q(x,y) + R(x,y)\ln\left|\dfrac{A(x,y)}{B(x,y)}\right|\right), \end{cases} \tag{2.1}$$

where $A(x,y)B(x,y) \neq 0$ and $A,\ B,\ P,\ Q$ and R are homogeneous polynomials of degree a, a, n, n and m respectively. Our first result on the existence of algebraic curve of system (2.1) is the following:

Theorem 2.2.1. *Consider the Kolmogorov system* (2.1),
The curve $U(x,y) = xyQ(x,y) - xyP(x,y) = 0$ is an invariant algebraic curve of the differential system (2.1).

Proof. *We prove that $U(x,y) = xyQ(x,y) - xyP(x,y) = 0$ is an invariant algebraic curve of the differential system* (2.1).
Indeed, we have

$$\begin{aligned} \frac{\partial U(x,y)}{\partial x}x' + \frac{\partial U(x,y)}{\partial y}y' &= \frac{\partial U(x,y)}{\partial x}\left(P(x,y) + R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right) \\ &\quad + \frac{\partial U(x,y)}{\partial y}\left(Q(x,y) + R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right) \\ &= \frac{\partial U(x,y)}{\partial x}xR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| + \frac{\partial U(x,y)}{\partial y}yR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\ &\quad + \frac{\partial U(x,y)}{\partial x}xP(x,y) + \frac{\partial U(x,y)}{\partial y}yQ(x,y). \end{aligned}$$

Then, taking into account that if P and Q are homogeneous polynomials of degree n, we have

$$x\frac{\partial P(x,y)}{\partial x}+y\frac{\partial P(x,y)}{\partial y}=nP(x,y) \ \ \text{and} \ \ x\frac{\partial Q(x,y)}{\partial x}+y\frac{\partial Q(x,y)}{\partial y}=nQ(x,y).$$

Then, we have

$$\begin{aligned}
&\frac{\partial U(x,y)}{\partial x}xR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|+\frac{\partial U(x,y)}{\partial y}yR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\
=\ & \frac{\partial\big(xyQ(x,y)-xyP(x,y)\big)}{\partial x}xR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\
&+\frac{\partial\big(xyQ(x,y)-xyP(x,y)\big)}{\partial y}yR(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\
=\ & xy\Big(2Q(x,y)-2P(x,y)+(xQ_x+yQ_y)-(xP_x+yP_y)\Big)R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\
=\ & (n+2)xy\Big(Q(x,y)-P(x,y)\Big)R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \\
=\ & (n+2)U(x,y)R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|.
\end{aligned}$$

On the other hand, substituting

$$y\frac{\partial P(x,y)}{\partial y}=nP(x,y)-x\frac{\partial P(x,y)}{\partial x} \ \ \text{and} \ \ y\frac{\partial Q(x,y)}{\partial y}=nQ(x,y)-x\frac{\partial Q(x,y)}{\partial x},$$

in what follows, we get

$$\begin{aligned}
\frac{\partial U(x,y)}{\partial x}xP(x,y)+\frac{\partial U(x,y)}{\partial y}yQ(x,y) \ =\ & \frac{\partial\big(xyQ(x,y)-xyP(x,y)\big)}{\partial x}xP(x,y) \\
&+\frac{\partial\big(xyQ(x,y)-xyP(x,y)\big)}{\partial y}yQ(x,y) \\
=\ & \big(yQ(x,y)-yP(x,y)+xyQ_x-xyP_x\big)xP(x,y) \\
&+\big(xQ(x,y)-xP(x,y)+xyQ_y-xyP_y\big)yQ(x,y) \\
=\ & xy\Big(\big(Q(x,y)+P(x,y)\big)\big(Q(x,y)-P(x,y)\big)+ \\
& n\big(Q(x,y)-P(x,y)\big)Q(x,y)-x\big(Q(x,y) \\
& -P(x,y)\big)Q_x+x\big(Q(x,y)-P(x,y)\big)P_x\Big) \\
=\ & \big((n+1)Q(x,y)+P(x,y)-xQ_x+xP_x\big)U(x,y).
\end{aligned}$$

In short, we have

$$\frac{\partial U(x,y)}{\partial x}x\left(P(x,y)+R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right)+\frac{\partial U(x,y)}{\partial y}y\left(Q(x,y)+R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|\right)$$

$$= \left((n+1)Q(x,y) + P(x,y) - xQ_x + xP_x + (n+2)R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right| \right) U(x,y).$$

Therefore,

$$U(x,y) = xyQ(x,y) - xyP(x,y) = 0$$

is an invariant algebraic curve of the polynomial differential systems (2.1) *with the cofactor*

$$K(x,y) = (n+1)Q(x,y) + P(x,y) - x\frac{\partial}{\partial x}Q(x,y) + x\frac{\partial}{\partial x}P(x,y) + (n+2)R(x,y)\ln\left|\frac{A(x,y)}{B(x,y)}\right|.$$

Our second result on the existence of first integral of system (2.1) is the following

Theorem 2.2.2. *Consider a Kolmogorov system* (2.1)*, then the following statements hold.*

1. *If $n \neq m$, then system* (2.1) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n) \int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega \right) - \\ &\quad (n-m) \int_0^{\arctan\frac{y}{x}} \exp\left((m-n) \int_0^{w} M(\omega)\, d\omega \right) N(w)\, dw, \end{aligned}$$

where $M(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$,
$f_1(\theta) = P(\cos\theta, \sin\theta)(\cos\theta)^2 + Q(\cos\theta, \sin\theta)(\sin\theta)^2$,
$f_2(\theta) = R(\cos\theta, \sin\theta)\ln\left|\dfrac{A(\cos\theta,\sin\theta)}{B(\cos\theta,\sin\theta)}\right|$, *and*
$f_3(\theta) = (\cos\theta\sin\theta)(Q(\cos\theta,\sin\theta) - P(\cos\theta,\sin\theta))$,
and the curves which are formed by the trajectories of the differential system (2.1)*, in cartesian coordinates are written as*

$$x^2+y^2 = \begin{pmatrix} h\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) + \\ (n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\, d\omega\right) N(w)dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$. Moreover, system (2.1) *have no limit cycle.*

2. *If $n - m = 1$ then system* (2.1) *has the first integral*

$$\begin{aligned} H(x,y) &= (x^2+y^2)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega \right) \\ &\quad - \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\, d\omega \right) N(w)\, dw, \end{aligned}$$

where $M(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, *and the curves which are formed by the trajectories of the differential system* (2.1), *in cartesian coordinates are written as*

$$x^2 + y^2 = \left(\begin{array}{c} h \exp\left(-\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega \right) + \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega \right) \\ \int_0^{\arctan \frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\, d\omega \right) N(w)\, dw \end{array} \right)^2,$$

where $h \in \mathbb{R}$. *Moreover, the system* (2.1) *has no limit cycle.*

3. *If* $n = m$ *then system* (2.1) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} \left(M(\omega) + N(\omega)\right) d\omega \right),$$

and the curves which are formed by the trajectories of differential system (2.1), *in cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h \exp\left(\int_0^{\arctan \frac{y}{x}} \left(M(\omega) + N(\omega)\right) d\omega \right) = 0,$$

where $h \in \mathbb{R}$. *Moreover, the system* (2.1) *has no limit cycle.*

4. *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then system* (2.1) *has the first integral* $H = \dfrac{y}{x}$, *and the curves which are formed by the trajectories of the differential system* (2.1), *in cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$. *Moreover, system* (2.1) *has no limit cycle.*

Proof. *In order to prove our results we write the polynomial differential system* (2.1) *in polar coordinates* (r, θ), *defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$, *then system* (2.1) *becomes*

$$\begin{cases} r' = f_1(\theta)\, r^{n+1} + f_2(\theta)\, r^{m+1}, \\ \theta' = f_3(\theta)\, r^n, \end{cases} \tag{2.2}$$

where the trigonometric functions $f_1(\theta), f_2(\theta), f_3(\theta)$ *are given, and*

$$r' = \frac{dr}{dt}, \quad \theta' = \frac{d\theta}{dt}$$

Proof of statement (1) ***of Theorem2.2.2***

Assume that: $n \neq m$.

Taking as independent variable the coordinate θ, *the differential system* (2.2) *writes*

$$\frac{dr}{d\theta} = M(\theta)\, r + N(\theta)\, r^{1+m-n}, \tag{2.3}$$

where $M(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$ *and* $N(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, *which is a Bernoulli differential equation. By introducing*

the standard change of variables $\rho = r^{n-m}$ *we obtain the linear differential equation*

$$\frac{d\rho}{d\theta} = (n-m)\left(M(\theta)\rho + N(\theta)\right). \tag{2.4}$$

The general solution of the differential linear equation (2.4) *is*

$$\begin{aligned} \rho(\theta) &= \exp\left((n-m)\int_0^{\theta} M(\omega)\,d\omega\right)\times \\ &\left(\mu + (n-m)\int_0^{\theta}\exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw\right), \end{aligned}$$

where $\mu \in \mathbb{R}$, *which has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}}\exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &+ (m-n)\int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw. \end{aligned}$$

Let Γ *be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let* $h_\Gamma = H(\Gamma)$.

The curves $H = h$ *with* $h \in \mathbb{R}$, *which are formed by trajectories of the differential system* (2.1), *in cartesian coordinates are written as*

$$x^2+y^2 = \begin{pmatrix} h\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$. *Therefore the periodic orbit* Γ *is contained in the curve*

$$x^2+y^2 = \begin{pmatrix} h_\Gamma\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}}.$$

But this curve cannot contain the periodic orbit Γ *and consequently no limit cycle contained in the realistic quadrant* $(x>0, y>0)$, *because this curve in realistic quadrant has at most a unique point on every straight line* $y = \eta x$ *for all* $\eta \in]0, +\infty[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ *for all* $\eta \in]0, +\infty[$, *the abscissa is given by*

$$x = \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} h_\Gamma \exp\left((n-m)\int_0^{\arctan\eta} M(\omega)\,d\omega\right) + \\ (n-m)\exp\left((n-m)\int_0^{\arctan\eta} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\eta} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}},$$

at most a unique value of x *on every half straight* OX^+, *consequently at most a unique point in realistic quadrant* $(x > 0, y > 0)$. *So this curve cannot contain the periodic orbit.* Hence statement (1) of theorem (2.2.2) is proved.

Proof of statement (2) ***of Theorem 2.2.2***

Assume that: $n - m = 1$.

Taking as independent variable the coordinate θ, *the differential system* (2.2) *writes*

$$\frac{dr}{d\theta} = M(\theta)\, r + N(\theta), \tag{2.5}$$

where $M(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$ *and* $N(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, *which is a linear equation. The general solution of linear equation* (2.5) *is*

$$\rho(\theta) = \exp\left(\int_0^{\theta} M(\omega)\,d\omega\right)\left(\mu + \int_0^{\theta} \exp\left(-\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw\right),$$

where $\mu \in \mathbb{R}$, *which has the first integral*

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) - \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw.$$

Let Γ *be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let* $h_\Gamma = H(\Gamma)$. *The curves* $H = h$ *with* $h \in \mathbb{R}$, *which are formed by trajectories of the differential system* (2.1), *in cartesian coordinates are written as*

$$x^2 + y^2 = \begin{pmatrix} h\exp\left(\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \exp\left(\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^2,$$

where $h \in \mathbb{R}$. *Therefore the periodic orbit* Γ *is contained in the curve*

$$x^2 + y^2 = \begin{pmatrix} h_\Gamma\exp\left(\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + \exp\left(\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^2.$$

But this curve cannot contain the periodic orbit Γ *and consequently no limit cycle contained in the realistic quadrant* $(x > 0, y > 0)$*, because this curve in realistic quadrant has at most a unique point on every straight line* $y = \eta x$ *for all* $\eta \in]0, +\infty[$ *.*
To be convinced by this fact, we compute the abscissa points of intersection of this curve with straight line $y = \eta x$ *for all* $\eta \in]0, +\infty[$ *, the abscissa is given by*

$$x = \frac{1}{\sqrt{1+\eta^2}} \left(\frac{h_\Gamma \exp\left(\int_0^{\arctan \eta} M(\omega)\, d\omega\right) + \exp\left(\int_0^{\arctan \eta} M(\omega)\, d\omega\right)}{\int_0^{\arctan \eta} \exp\left(-\int_0^{w} M(\omega)\, d\omega\right) N(w)\, dw} \right)^2,$$

at most a unique value of x *on every half straight* OX^+*, consequently at most a unique point in realistic quadrant* $(x > 0, y > 0)$*. So this curve cannot contain the periodic orbit.*
Hence statement (2) of theorem (2.2.2) is proved.

Proof of statement (3) ***of theorem 2.2.2***

Assume that: $n = m$*.*
Taking as independent variable the coordinate θ*, this differential system* (2.2) *writes*

$$\frac{dr}{d\theta} = (M(\theta) + N(\theta))\, r. \tag{2.6}$$

The general solution of equation (2.6) *is*

$$r(\theta) = \mu \exp\left(\int_0^{\theta} (M(\omega) + N(\omega))\, d\omega\right),$$

where $\mu \in \mathbb{R}$*, which has the first integral*

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} (M(\omega) + N(\omega))\, d\omega\right).$$

Let Γ *be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant* $(x > 0, y > 0)$*, and let* $h_\Gamma = H(\Gamma)$*. The curves* $H = h$ *with* $h \in \mathbb{R}$*, which are formed by trajectories of the differential system* (2.1)*, in cartesian coordinates are written as*

$$\left(x^2 + y^2\right)^{\frac{1}{2}} - h \exp\left(\int_0^{\arctan \frac{y}{x}} (M(\omega) + N(\omega))\, d\omega\right) = 0,$$

where $h \in \mathbb{R}$*. Therefore the periodic orbit* Γ *is contained in the curve*

$$\left(x^2 + y^2\right)^{\frac{1}{2}} = h_\Gamma \exp\left(\int_0^{\arctan \frac{y}{x}} (M(\omega) + N(\omega))\, d\omega\right).$$

But this curve cannot contain the periodic orbit Γ, *and consequently no limit cycle contained in the realistic quadrant* $(x > 0, y > 0)$, *because this curve in realistic quadrant has at most a unique point on every straight line* $y = \eta x$ *for all* $\eta \in \left]0, +\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ *for all* $\eta \in \left]0, +\infty\right[$, *the abscissa is given by*

$$x = \frac{h_\Gamma}{\sqrt{(1+\eta^2)}} \exp\left(\int_0^{\arctan \eta} \left(M(\omega) + N(\omega)\right) d\omega\right),$$

at most a unique value of x *on every half straight* OX^+, *consequently at most a unique point in realistic quadrant* $(x > 0, y > 0)$. *So this curve cannot contain a periodic orbit.* Hence statement (3) of Theorem 2.2.2 is proved.

Proof of statement* (4) *of Theorem 2.2.2

Assume that $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then from system* (2.2) *it follows that* $\theta' = 0$. *So the straight lines through the origin of coordinates of the differential system* (2.1) *are invariant by the flow of this system. Hence,* $\frac{y}{x}$ *is a first integral of the system, then curves which are formed by the trajectories of the differential system* (2.1), *in cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$, *since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.* This completes the proof of statement (4) of Theorem 2.2.2.

2.2.1 Example of Kolmogorov system

The following example is given to illustrate our results.

Example

If we take $A(x,y) = 5x^2 + 4y^2$, $B(x,y) = x^2 + y^2$,
$P(x,y) = x^4 + x^3y + 2x^2y^2 + xy^3 + y^4$, $Q(x,y) = x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$,
and $R(x,y) = 3x^2 - xy + 3y^2$, *then system* (2.6) *writes*

$$\begin{cases} x' = x\left((x^4 + x^3y + 2x^2y^2 + xy^3 + y^4) + (3x^2 - xy + 3y^2)\ln\left|\dfrac{5x^2+4y^2}{x^2+y^2}\right|\right), \\ y' = y\left((x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4) + (3x^2 - xy + 3y^2)\ln\left|\dfrac{5x^2+4y^2}{x^2+y^2}\right|\right), \end{cases} \tag{2.7}$$

the Kolmogorov system (2.7) *in polar coordinates* (r, θ) *becomes*

$$\begin{cases} r' = \left(1 + \dfrac{3}{4}\sin 2\theta - \dfrac{1}{8}\sin 4\theta\right) r^5 + (3 - \cos\theta \sin\theta) \ln\left(\dfrac{9}{2} + \dfrac{1}{2}\cos 2\theta\right) r^3, \\ \theta' = \left(\cos^2\theta \sin^2\theta\right) r^4, \end{cases}$$

where

$$f_1(\theta) = 1 + \tfrac{3}{4}\sin 2\theta - \tfrac{1}{8}\sin 4\theta,$$

$$f_2(\theta) = (3 - \cos\theta\sin\theta)\ln\left(\tfrac{9}{2} + \tfrac{1}{2}\cos 2\theta\right) \quad \textit{and}$$

$$f_3(\theta) = \cos^2\theta \sin^2\theta.$$

In the realistic quadrant $(x > 0, y > 0)$ *it is the case* (1) *of the Theorem* 2.2.2*, then the Kolmogorov system* (2.7) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2 + y^2\right)\exp\left(-2\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right) \\ &\quad -2\int_0^{\arctan\frac{y}{x}} \exp\left(-2\int_0^{w} M(\omega)\, d\omega\right) B(w)\, dw, \end{aligned}$$

where

$$M(\omega) = \frac{1 + \dfrac{3}{4}\sin 2\omega - \dfrac{1}{8}\sin 4\omega}{\cos^2\omega \sin^2\omega},$$

$$N(w) = \frac{\left(3 - \cos w \sin w\right)\ln\left(\dfrac{9}{2} + \dfrac{1}{2}\cos 2w\right)}{\cos^2 w \sin^2 w}.$$

The curves $H = h$ *with* $h \in \mathbb{R}$*, which are formed by trajectories of the differential system* (2.7)*, in cartesian coordinates are written as*

$$\begin{aligned} x^2 + y^2 &= 2\exp\left(2\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right)\int_0^{\arctan\frac{y}{x}} \exp\left(-2\int^{w} N(\omega)\, d\omega\right) N(w)\, dw \\ &\quad + h\exp\left(2\int_0^{\arctan\frac{y}{x}} M(\omega)\, d\omega\right), \end{aligned}$$

where $h \in \mathbb{R}$*. The system* (2.7) *has no periodic orbits, and consequently no limit cycle.*

2.3 Invariant algebraic curves and first integral of a class of Kolmogorov systems

In the second part we introduce an explicit expression of invariant algebraic curves of the multi-parameter planar Kolmogorov system of the form

$$\begin{cases} x' = x(1 + ax^2 + bxy + cy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \\ y' = y(1 + nx^2 + mxy + sy^2 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4), \end{cases} \tag{2.8}$$

where $a,\ b,\ c,\ n,\ m$ and s are real constants.
Our first result of the existence of algebraic curve of the system (2.8) about the algebraic curves is the following

Theorem 2.3.1. *Consider a multi-parameter planar Kolmogorov system* (2.8)*, then the curve*

$$U(x,y) = xy(nx^2 + mxy + sy^2) - xy(ax^2 + bxy + cy^2)$$

is an invariant algebraic curve of system (2.8)*.*

Proof. *We prove that*

$$U(x,y) = xy(nx^2 + mxy + sy^2) - xy(ax^2 + bxy + cy^2),$$

is an invariant algebraic curve of the differential system (2.8)*.*
Denoting by $P(x,y) = ax^2 + bxy + cy^2,\ Q(x,y) = nx^2 + mxy + sy^2$ *and*
$R(x,y) = 1 - (a+1)x^4 - bx^3y - (c+n+2)x^2y^2 - mxy^3 - (s+1)y^4$
Indeed, we have

$$\begin{aligned} &\frac{\partial U(x,y)}{\partial x}x(1 + ax^2 + bxy + cy^2 - (a+1)x^4 - bx^3y(c+n+2)x^2y^2 - mxy^3 - (s+1)y^4) + \\ &\frac{\partial U(x,y)}{\partial y}y(1 + nx^2 + mxy + sy^2 - (a+1)x^4 - bx^3y(c+n+2)x^2y^2 - mxy^3 - (s+1)y^4) \\ &= \left(\frac{\partial U(x,y)}{\partial x}x + \frac{\partial U(x,y)}{\partial y}y\right)R(x,y) + \frac{\partial U(x,y)}{\partial x}xP(x,y) + \frac{\partial U(x,y)}{\partial y}yQ(x,y) \end{aligned}$$

And,

$$\left(\frac{\partial U(x,y)}{\partial x}x + \frac{\partial U(x,y)}{\partial y}y\right)R(x,y) = 4xy\big(Q(x,y) - P(x,y)\big)R(x,y) = 4R(x,y)U(x,y)$$

On the other hand,

$$\frac{\partial U(x,y)}{\partial x}xP(x,y)+\frac{\partial U(x,y)}{\partial y}yQ(x,y) = xy\Big(-P^2(x,y)+Q^2(x,y)+x(2nx+my)P(x,y)$$

$$-x(2ax+by)P(x,y)+y(mx+2sy)Q(x,y)-y(bx+2cy)Q(x,y)\Big).$$

Taking into account that

$y(bx+2cy)=2P(x,y)-x(2ax+by)$ *and* $y(mx+2sy)=2Q(x,y)-x(2nx+my)$.

Then, we have

$$\frac{\partial U(x,y)}{\partial x}xP(x,y)+\frac{\partial U(x,y)}{\partial y}yQ(x,y)=$$

$$\Big(3Q(x,y)+P(x,y)+(b-m)xy+2(a-n)x^2\Big)U(x,y).$$

In short, we have

$$\Big(\frac{\partial U(x,y)}{\partial x}x+\frac{\partial U(x,y)}{\partial y}y\Big)R(x,y)+\frac{\partial U(x,y)}{\partial x}xP(x,y)+\frac{\partial U(x,y)}{\partial y}yQ(x,y)$$

$$=\Big(3Q(x,y)+P(x,y)+(b-m)xy+2(a-n)x^2+4R(x,y)\Big)U(x,y).$$

Therefore,

$U(x,y)=xy(nx^2+mxy+sy^2)-xy(ax^2+bxy+cy^2)=0$ *is an invariant algebraic curve of the polynomial differential system* (2.8) *with the cofactor*

$$K(x,y)=(b-m)xy+2(a-n)x^2+3Q(x,y)+P(x,y)+4R(x,y).$$

Hence, Theorem 2.3.1 is proved.

Our second results of the existence of first integral of the system (2.8) are the following

Theorem 2.3.2. *Consider a multi-parameter planar Kolmogorov system* (2.8), *then the following statements hold.*

1. *If* $f_3(\theta)\neq 0$, *then system* (2.8) *has the first integral*

$$H(x,y)=\frac{\exp\big(\int_0^{\arctan\frac{y}{x}}D(w)dw\big)+(x^2+y^2-1)\int_0^{\arctan\frac{y}{x}}\exp(-\int_0^s D(w)dw)C(s)ds}{x^2+y^2-1}.$$

   where

$$A(\theta)=\frac{2}{f_3(\theta)},\ B(\theta)=\frac{2f_1(\theta)}{f_3(\theta)},\ C(\theta)=\frac{2f_2(\theta)}{f_3(\theta)},\ D(w)=B(w)+2C(w).$$

$f_1(\theta) = acos^4\theta + ssin^4\theta + bcos^3\theta sin\theta + mcos\theta sin^3\theta + (n+c)cos^2\theta sin^2\theta,$

$f_2(\theta) = -1 - acos^4\theta - ssin^4\theta - bcos^3\theta sin\theta - mcos\theta sin^3\theta - (n+c)cos^2\theta sin^2\theta,$ *and*

$f_3(\theta) = (n-a)cos^3\theta sin\theta + (s-c)cos\theta sin^3\theta + (m-b)cos^2\theta sin^2\theta.$

Moreover the phase portrait of the differential system (2.8)*, in cartesian coordinates is given by*

$$x^2 + y^2 = \frac{h + \exp\left(\int_0^{\arctan\frac{y}{x}} D(w)dw\right) - \int_0^{\arctan\frac{y}{x}} \exp(-\int_0^s D(w)dw)C(s)ds}{h - \int_0^{\arctan\frac{y}{x}} \exp(-\int_0^s D(w)dw)C(s)ds},$$

where $h \in \mathbb{R}$.

2. *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*, then system* (2.8) *has the first integral*

$$H(x,y) = \frac{y}{x}.$$

Moreover the phase portrait of the differential system (2.8)*, in cartesian coordinates is given by*

$$y - hx = 0,$$

where $h \in \mathbb{R}$.

Proof. *In order to prove our results we write the polynomial differential system* (2.8) *in polar coordinates* (r,θ)*, defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$*, then the system* (2.8) *becomes*

$$\begin{cases} r' = \dfrac{dr}{dt} = r + f_1(\theta)r^3 + f_2(\theta)r^5, \\ \theta' = \dfrac{d\theta}{dt} = f_3(\theta)r^2, \end{cases} \tag{2.9}$$

where the trigonometric functions $f_1(\theta), f_2(\theta)$ *and* $f_3(\theta)$ *are given.*

Proof of statement (1) ***of Theorem*** (2.3.2)**.**

Assume that $f_3(\theta) \neq 0$, $\forall \theta \in \mathbb{R}$.

Taking as independent variable the coordinate θ*, the differential system* (2.9) *writes*

$$\frac{dr}{d\theta} = \frac{1}{f_3(\theta)}\frac{1}{r} + \frac{f_1(\theta)}{f_3(\theta)}r + \frac{f_2(\theta)}{f_3(\theta)}r^3. \tag{2.10}$$

Via the change of variables $\rho = r^2$ *, this equation is transformed into Riccati equation*

$$\frac{d\rho}{d\theta} = A(\theta) + B(\theta)\rho + C(\theta)\rho^2, \tag{2.11}$$

Fortunately, the equation (2.11) *is integrable, since it possesses the particular solution* $\rho = 1$*, by introducing the standard change of variables* $\rho = z + 1$*, we obtain the Bernoulli equation*

$$\frac{dz}{d\theta} = D(\theta)z + C(\theta)z^2, \tag{2.12}$$

By introducing the standard change of variables $u = \dfrac{1}{z}$ *we obtain the linear equation*

$$\frac{du}{d\theta} = -D(\theta)u - C(\theta). \tag{2.13}$$

The general solution of linear equation (2.13) *is*

$$u(\theta) = \exp\left(-\int_0^\theta D(w)dw\right)\left(\lambda - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds\right),$$

where $\lambda \in R$.

Then the general solution of linear equation (2.12) *is*

$$z(\theta) = \frac{\exp\left(\int_0^\theta D(w)dw\right)}{\lambda - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds},$$

where $\lambda \in R$.

The general solution of linear equation (2.11) *is*

$$\rho(\theta) = \frac{\lambda + \exp\left(\int_0^\theta D(w)dw\right) - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds}{\lambda - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds},$$

where $\lambda \in R$.

Then the general solution of linear equation (2.10) *is*

$$r^2(\theta) = \frac{\lambda + \exp\left(\int_0^\theta D(w)dw\right) - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds}{\lambda - \int_0^\theta \exp\left(-\int_0^s D(w)dw\right)C(s)ds},$$

where $\lambda \in R$.

By passing to cartesian coordinates, we deduce the first integral is

$$H(x,y) = \frac{\exp\left(\int_0^{\arctan\frac{y}{x}} D(w)dw\right) + (x^2+y^2-1)\int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^s D(w)dw\right)C(s)ds}{x^2+y^2-1},$$

The curves $H = h$ *with* $h \in R$*, which are formed by trajectories of the differential system* (2.8)*, in cartesian coordinates are written as*

$$x^2 + y^2 = \frac{h + \exp\left(\int_0^{\arctan\frac{y}{x}} D(w)dw\right) - \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^s D(w)dw\right)C(s)ds}{h - \int_0^{\arctan\frac{y}{x}} \exp\left(-\int_0^s D(w)dw\right)C(s)ds},$$

where $h \in R$*.*

Hence statement (1) of Theorem 2.3.2 is proved.

***Proof of statement* (2) *of Theorem* 2.3.2.**

Assume now that $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*. Then from system* (2.9) *it follows that* $\theta' = 0$*. So the straight lines through the origin of coordinates of the differential system* (2.8) *are invariant by the flow of this system. Hence,* $\frac{y}{x}$ *is a first integral of the system, then curves which are formed by the trajectories of the differential system* (2.8)*, in cartesian coordinates are written as* $y - hx = 0$*, where* $h \in \mathbb{R}$*.*

This completes the proof of statement (2) of Theorem 2.3.2.

CHAPTER 3

SOME CLASSES OF PLANAR DIFFERENTIAL SYSTEMS

3.1 Introduction

This chapter is contained of two parts: In the first part we are interested in studying the existence of a first integral and the curves which are formed by the trajectories of a class of a two-dimensional planar differential system of the form:

$$\begin{cases} x' = P\left(x,y\right) + x\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \\ y' = Q(x,y) + y\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \end{cases}$$

where $P,\ Q,\ M,\ N,\ R$ and S are homogeneous polynomials of degree a, a, b, b, c, c respectively, and $\lambda, \beta \in \mathbb{R}$, the results obtained for this class was a subject of our publication [20].

In the second part, we are interested to give an explicit expression of invariant algebraic curves of multi-parameter polynomial differential systems planar of degree nine, then we proved that these systems are integrable and we introduce an explicit expression of a first integral. Moreover, we determine sufficient conditions for these systems to possess two limit cycles: one of them is algebraic and the other one is non-algebraic, we consider the differential system

$$\begin{cases} x' = \dfrac{dx}{dt} = x + P_5(x,y) + xR_8(x,y), \\ y' = \dfrac{dy}{dt} = y + Q_5(x,y) + yR_8(x,y), \end{cases}$$

where P_5, Q_5 and R_8 are polynomials of variables x, y.

3.2 Explicit expression for a first integral for a class of two-dimensional differential systems

First, we are studying the existence of a first integral of the two-dimensional differential system of the form

$$\begin{cases} x' = P(x,y) + x\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \\ y' = Q(x,y) + y\left(\lambda x \exp\left(\dfrac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\dfrac{R(x,y)}{S(x,y)}\right)\right), \end{cases} \tag{3.1}$$

where P, Q, M, N, R and S are homogeneous polynomials of degree a, a, b, b, c and c respectively and λ, $\beta \in \mathbb{R}$. Concrete examples exhibiting the applicability of our result are introduced.

Our main results on the existence of first integral of the system (3.1) are the following

Theorem 3.2.1. *Consider a differential system* (3.1)*, then the following statements hold.*

1. *If $a \neq 2$ then system* (3.1) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{a-2}{2}} \exp\left((2-a)\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \\ &\quad - (a-2)\int^{\arctan\frac{y}{x}} \exp\left((2-a)\int^{w} A(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where,

$$A(\theta) = \frac{f_2(\theta)}{f_3(\theta)}, \quad B(\theta) = \frac{f_1(\theta)}{f_3(\theta)},$$

$$f_1(\theta) = \lambda(\cos\theta)\exp\left(\frac{M(\cos\theta,\sin\theta)}{N(\cos\theta,\sin\theta)}\right) + \beta(\sin\theta)\exp\left(\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)}\right),$$
$$f_2(\theta) = P(\cos\theta,\sin\theta)\cos\theta + Q(\cos\theta,\sin\theta)\sin\theta \ \text{ and}$$
$$f_3(\theta) = (\cos\theta)Q(\cos\theta,\sin\theta) - (\sin\theta)P(\cos\theta,\sin\theta),$$

are trigonometric functions.

The curves which are formed by the trajectories of the differential system (3.1)*, in cartesian coordinates are written as*

$$x^2+y^2 = \begin{pmatrix} h\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ (a-2)\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((2-a)\int_0^{w} A(\omega)\,d\omega\right) B(w)\,dw \end{pmatrix}^{\frac{2}{a-2}},$$

where $h \in \mathbb{R}$.

2. *If* $a = 0$ *then system* (3.1) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right),$$

and the curves which are formed by the trajectories of the differential system (3.1)*, in cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

3. *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*, then system* (3.1) *has the first integral* $H = \dfrac{y}{x}$*, and the curves which are formed by the trajectories of the differential system* (3.1)*, in cartesian coordinates are written as* $y - hx = 0$*, where* $h \in \mathbb{R}$.

Proof. *In order to prove our results we write the polynomial differential system* (3.1) *in polar coordinates* (r,θ)*, defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$*, then system* (3.1) *becomes*

$$\begin{cases} r' = f_1(\theta)\,r^2 + f_2(\theta)\,r^a, \\ \theta' = f_3(\theta)\,r^{a-1}, \end{cases} \tag{3.2}$$

where the trigonometric functions $f_1(\theta), f_2(\theta)$ *and* $f_3(\theta)$ *are given in introduction.*

Proof of statement (1) ***of Theorem*** 3.2.1

Assume that $a \neq 2$.

Taking as independent variable the coordinate θ*, the differential system* (3.2) *writes*

$$\frac{dr}{d\theta} = A(\theta)\,r + B(\theta)\,r^{3-a}, \tag{3.3}$$

which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^{a-2}$ *we obtain the linear equation*

$$\frac{d\rho}{d\theta} = (a-2)\left(A(\theta)\rho + B(\theta)\right). \tag{3.4}$$

The general solution of linear equation (3.4) *is*

$$\begin{aligned} \rho(\theta) &= \exp\left((a-2)\int^{\theta} A(\omega)\,d\omega\right) \times \\ &\left(\mu + (a-2)\int^{\theta} \exp\left((2-a)\int^{w} A(\omega)\,d\omega\right) B(w)\,dw\right), \end{aligned}$$

where $\mu \in \mathbb{R}$, *which has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{a-2}{2}} \exp\left((2-a)\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \\ &+ (a-2)\int^{\arctan\frac{y}{x}} \exp\left((2-a)\int^{w} A(\omega)\,d\omega\right) B(w)\,dw. \end{aligned}$$

Let Γ *be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let* $h_\Gamma = H(\Gamma)$.

The curves $H = h$ *with* $h \in \mathbb{R}$, *which are formed by trajectories of the differential system* (3.1), *in cartesian coordinates are written as*

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left((a-2)\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ (a-2)\exp\left((a-2)\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \\ \int^{\arctan\frac{y}{x}} \exp\left((2-a)\int^{w} A(\omega)\,d\omega\right) B(w)\,dw \end{array}\right)^{\frac{2}{a-2}},$$

where $h \in \mathbb{R}$.

Hence statement (1) of theorem 3.2.1 is proved.

Proof of statement (2) ***of Theorem*** 3.2.1

Assume that $a = 2$.

Taking as independent variable the coordinate θ, *this differential system* (3.2) *writes*

$$\frac{dr}{d\theta} = \big(A(\theta) + B(\theta)\big)r. \tag{3.5}$$

The general solution of equation (3.5) *is*

$$r(\theta) = \mu\exp\left(\int^{\theta} \left(A(\omega) + B(\omega)\right)d\omega\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right) d\omega\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3.1)*, in cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

Hence statement (2) of Theorem 3.2.1 is proved.

Proof of statement (3) ***of theorem*** 3.2.1

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from system (3.2) *it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system* (3.1) *are invariant by the flow of this system. Hence, $\dfrac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system* (3.1)*, in cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories.*

This completes the proof of statement (3) of Theorem 3.2.1.

3.2.1 Examples of the system

The following examples are given to illustrate our result

Example 1

If we take $\lambda = 1, \beta = -2$, $P(x,y) = 2x - 3y$, $Q(x,y) = 3x + 2y$, $M(x,y) = x^2 + 2y^2$, $N(x,y) = x^2 + y^2$, $R(x,y) = x^4 + 3x^2y^2 + y^4$ and $S(x,y) = x^4 + 2x^2y^2 + y^4$ then system (3.1) *writes*

$$\begin{cases} x' = (2x-3y) + x\left(x\exp\left(\dfrac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\dfrac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \\ y' = (3x+2y) + y\left(x\exp\left(\dfrac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\dfrac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \end{cases} \tag{3.6}$$

The differential system (3.6) *in polar cordinates* (r, θ) *becomes*

$$\begin{cases} r' = \left((\cos\theta)\exp(1+\sin^2\theta) - 2(\sin\theta)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4\theta\right)\right) r^2 + 2r, \\ \theta' = 3, \end{cases} \tag{3.7}$$

where $f_1(\theta) = (\cos\theta)\exp(1+\sin^2\theta) - 2(\sin\theta)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4\theta\right)$, $f_2(\theta) = 2$ *and* $f_3(\theta) = 3$, *it is the case (1) of theorem* (3.2.1).

The differential system (3.6) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{-1}{2}} \exp\left(\frac{2}{3}\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{2}{3}\omega\right) B(w)\,dw,$$

where $B(w) = \dfrac{(\cos w)\exp(1+\sin^2 w) - 2(\sin w)\exp(\frac{9}{8} - \frac{1}{8}\cos 4w)}{3}$.

The curves $H = h$ *with* $h \in \mathbb{R}$, *which are formed by trajectories of the differential system* (3.6) *in cartesian coordinates are written as*

$$x^2+y^2 = \left[\left(h + \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{2}{3}\omega\right) B(w)\,dw\right)\exp\left(-\frac{2}{3}\arctan\frac{y}{x}\right)\right]^{-2},$$

where $h \in \mathbb{R}$.

Example 2

If we take $\lambda = 1, \beta = -2$,
$P(x,y) = 5x^2 + 2xy,\ Q(x,y) = -2xy + 5y^2,\ M(x,y) = x^2 + 2y^2$,
$N(x,y) = x^2 + y^2,\ R(x,y) = y$ *and* $S(x,y) = x$, *then system* (3.1) *writes*

$$\begin{cases} x' = (5x^2+2xy) + x\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{y}{x}\right)\right), \\ y' = (-2xy+5y^2) + y\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{y}{x}\right)\right). \end{cases} \tag{3.8}$$

The differential system (3.8) *in polar cordinates* (r, θ) *becomes*

$$\begin{cases} r' = \left[7\cos^3\theta + 3\sin^3\theta - 2(\sin\theta)(-1+\exp\tan\theta) + (\cos\theta)\Big(-2+\exp(1+\sin^2\theta)\Big)\right] r^2, \\ \theta' = (3\cos\theta\sin^2\theta - 7\cos^2\theta\sin\theta)r, \end{cases}$$

where

$$A(w)+B(w) = \frac{7\cos^3 w + 3\sin^3 w - 2(\sin w)(-1+\exp\tan w) + (\cos w)\Big(-2+\exp(1+\sin^2 w)\Big)}{3\cos w\sin^2 w - 7\cos^2 w\sin w},$$

it is the case (2) *of Theorem* 3.2.1.
The the differential system (3.8) *has the first integral*

$$H(x,y) = (x^2+y^2)^{\frac{1}{2}}\exp\left(\int_0^{\arctan\frac{y}{x}}\big(A(w)+B(w)\big)dw\right),$$

where $h \in \mathbb{R}$. *The curves* $H = h$ *with* $h \in \mathbb{R}$, *which are formed by trajectories of the differential system* (3.8) *in cartesian coordinates are written as*

$$(x^2+y^2)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}}\big(A(w)+B(w)\big)dw\right) = 0,$$

where $h \in \mathbb{R}$.

Example 3

If we take $\lambda = 1, \beta = -2$,
$P(x,y) = x^3+xy^2,\ Q(x,y) = y^3+yx^2,\ M(x,y) = x^2+2y^2,$
$N(x,y) = x^2+y^2,\ R(x,y) = x^4+3x^2y^2+y^4$ *and* $S(x,y) = x^4+2x^2y^2+y^4$,
then system (3.1) *writes*

$$\begin{cases} x' = (x^3+xy^2) + x\left(x\exp\left(\dfrac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\dfrac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \\ y' = (y^3+yx^2) + y\left(x\exp\left(\dfrac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\dfrac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \end{cases} \tag{3.9}$$

The differential system (3.9) *in polar cordinates* (r,θ) *becomes*

$$\begin{cases} r' = r^3 + \left((\cos\theta)\exp(1+\sin^2\theta) - 2(\sin\theta)\exp\left(\dfrac{9}{8} - \dfrac{1}{8}\cos 4\theta\right)\right) r^2, \\ \theta' = 0, \end{cases} \tag{3.10}$$

it is the case (3) *of theorem* (3.2.1), *then from system* (3.10) *it follows that* $\theta' = 0$. *Hence,* $\dfrac{y}{x}$ *is a first integral of the system, then curves which are formed by the trajectories of the differential system* (3.9), *in cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$, *since all straight lines through the origin are formed by trajectories.*

3.3 Polynomial differential systems with explicit expression for limit cycles

In the second part, we obtain by a more intuitive and understandable method a new result for an another class of differential systems of degree 9.

$$\begin{cases} x' = \dfrac{dx}{dt} = x + P_5(x,y) + xR_8(x,y), \\ y' = \dfrac{dy}{dt} = y + Q_5(x,y) + yR_8(x,y), \end{cases} \tag{3.11}$$

where

$$P_5(x,y) = -(a+2)\,x^5 + (4+4b)\,x^4y - (2a+4)\,x^3y^2 + (8+4b)\,x^2y^3 - (a+2)\,xy^4 + 4y^5,$$

$$Q_5(x,y) = -4x^5 - (a+2)\,x^4y + (4b-8)\,x^3y^2 - (2a+4)\,x^2y^3 + (4b-4)\,xy^4 - (a+2)\,y^5,$$

and

$$\begin{aligned} R_8(x,y) &= (a+1)\,x^8 - 4bx^7y + (4a+4)\,x^6y^2 - 12bx^5y^3 + (6a+6)\,x^4y^4 - 12bx^3y^5 \\ &\quad + (4a+4)\,x^2y^6 - 4bxy^7 + (a+1)\,y^8, \end{aligned}$$

where a, b are real constants.

First, we introduce an explicit expression of invariant algebraic curves of a multi-parameter polynomial differential planar system of degree nine (3.11), then we proved that this system is integrable and we introduce an explicit expression of a first integral. Moreover, we determine sufficient conditions for these systems to possess two limit cycles: one of them is algebraic and the other one is showen to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

Our main results on the existence of critical points and algebraic curve are the following:

Theorem 3.3.1. *Consider a multi-parameter polynomial differential planar system* (3.11)*, then the following statements hold.*

1. *The origin $O\,(0,0)$ is the unique critical point at finite distance.*
2. *The curve*

$$U\,(x,y) = x^4 + y^4 + 2x^2y^2 - 1$$

is an invariant algebraic curve of system (3.11) *with cofactor*

$$K(x,y) = (-4)\left(x^2+y^2\right)^2\left((-a-1)\left(x^2+y^2\right)^2 + 4bxy\left(x^2+y^2\right)+1\right).$$

Proof. ***Proof of statement*** (1) ***of theorem*** 3.3.1

The point $A(x_0,y_0)\in\mathbb{R}^2$ *is a critical point of system* (3.11), *if*

$$\begin{cases} x_0 + P_5(x_0,y_0) + x_0R_8(x_0,y_0) = 0, \\ y_0 + Q_5(x_0,y_0) + y_0R_8(x_0,y_0) = 0, \end{cases}$$

we have

$$y_0P_5(x_0,y_0) - x_0Q_5(x_0,y_0) = 4x_0^6 + 4y_0^6 + 12x_0^2y_0^4 + 12x_0^4y_0^2 = 0,$$

then $x_0 = 0, y_0 = 0$, *is the unique solution of this equation. Thus the origin is the unique critical point at finite distance of the system* (3.11).

This completes the proof of statement (1) of theorem (3.3.1).

Proof of statement (2) ***of theorem*** 3.3.1

An computation shows that $U(x,y) = x^4 + y^4 + 2x^2y^2 - 1$ *satisfy the linear partial differential equation in definition* (1.2.3) *of chapter 1, the associated cofactor being*

$$K(x,y) = (-4)\left(x^2+y^2\right)^2\left((-a-1)\left(x^2+y^2\right)^2 + 4bxy\left(x^2+y^2\right)+1\right),$$

then the curve $U(x,y) = 0$ *is an invariant algebraic curve of system* (3.11) *with cofactor* $K(x,y)$.

This completes the proof of statement (2) of theorem3.3.1.

Our second results on the existence of first integral of the system (3.11) are the following:

Theorem 3.3.2. *The system* (3.11) *has the first integral*

$$H(x,y) = \frac{\left(x^2+y^2\right)^2 + \left(1-\left(x^2+y^2\right)^2\right)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right) f(\arctan\frac{y}{x})}{\left(\left(x^2+y^2\right)^2-1\right)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where $f(\arctan\dfrac{y}{x}) = \int_0^{\arctan\frac{y}{x}} \exp(-as - b\cos 2s)ds$.

Proof. *In order to prove our result, we write the polynomial differential system* (3.11) *in polar coordinates* (r,θ), *defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$, *then the system* (3.11) *becomes*

$$\begin{cases} r' = \dfrac{dr}{dt} = r + (-2-a+2b\sin 2\theta)\,r^5 + (1+a-2b\sin 2\theta)\,r^9, \\ \theta' = \dfrac{d\theta}{dt} = -4r^4. \end{cases} \tag{3.12}$$

Since θ' *is negative for all* $t \in \mathbb{R}$*, the orbits* $\big(r(t), \theta(t)\big)$ *of system* (3.12) *have the opposite orientation with respect to those* $\big(x(t), y(t)\big)$ *of system* (3.11)*. Taking* θ *as an independent variable, we obtain the equation*

$$\frac{dr}{d\theta} = \frac{-1}{4r^3} + \left(\frac{1}{4}a + \frac{1}{2} - \frac{1}{2}b\sin 2\theta\right) r + \left(\frac{-1}{4} - \frac{a}{4} + \frac{1}{2}b\sin 2\theta\right) r^5. \qquad (3.13)$$

Via the change of variables $\rho = r^4$*, this equation* (3.13) *is transformed into the Riccati equation*

$$\frac{d\rho}{d\theta} = (-a - 1 + 2b\sin 2\theta)\rho^2 + (a + 2 - 2b\sin 2\theta)\rho - 1. \qquad (3.14)$$

This equation is integrable, since it possesses the particular solution $\rho = 1$*. By introducing the standard change of variables* $y = \rho - 1$ *we obtain the Bernoulli equation*

$$\frac{dy}{d\theta} = (-a + 2b\sin 2\theta)y + (-1 - a + 2b\sin 2\theta)y^2. \qquad (3.15)$$

By introducing the standard change of variables $z = \dfrac{1}{y}$ *we obtain the linear equation*

$$\frac{dz}{d\theta} = (a - 2b\sin 2\theta)\, z + (1 + a - 2b\sin 2\theta)\,. \qquad (3.16)$$

The general solution of linear equation (3.16) *is*

$$\begin{aligned} z(\theta) &= 1, \\ z(\theta) &= \frac{\lambda + \int_0^\theta (1 + a - 2b\sin 2w)\exp\left(\int_0^w (-a + 2b\sin 2s)ds\right)dw}{\exp\left(\int_0^\theta (-a + 2b\sin 2s)ds\right)}, \end{aligned}$$

where $\lambda \in \mathbb{R}$*.*

Then the general solution of equation (3.15) *is*

$$\begin{aligned} y\,(\theta) &= 1, \\ y\,(\theta) &= \frac{\exp(\int_0^\theta (-a + 2b\sin 2s)ds\Big)}{\lambda + \int_0^\theta (1 + a - 2b\sin 2w)\exp\left(\int_0^w (-a + 2b\sin 2s)ds\right)dw}, \end{aligned}$$

where $\lambda \in \mathbb{R}$*.*

Then the general solution of equation (3.14) *is*

$$\begin{aligned} \rho(\theta) &= 1, \\ \rho(\theta) &= \frac{\exp\big(a\theta + b\cos 2\theta\big)\big(h + f(\theta)\big)}{-1 + \exp\big(a\theta + b\cos 2\theta\big)\big(h + f(\theta)\big)}, \end{aligned}$$

where $h = (1+\lambda)\exp(-b) \in \mathbb{R}$.

Consequently, the general solution of (3.13) *is*

$$\begin{aligned} r(\theta,h) &= 1, \\ r(\theta,h) &= \left(\frac{\exp\big(a\theta + b\cos 2\theta\big)\big(h + f(\theta)\big)}{-1 + \exp\big(a\theta + b\cos 2\theta\big)\big(h + f(\theta)\big)}\right)^{\frac{1}{4}}, \end{aligned}$$

where $h \in \mathbb{R}$.

From these solution we obtain a first integral in the variables (x,y) *of the form*

$$H(x,y) = \frac{(x^2+y^2)^2 + \Big(1-(x^2+y^2)^2\Big)\exp\Big(a\arctan\frac{y}{x} + b\cos(2\arctan\frac{y}{x})\Big)f(\arctan\frac{y}{x})}{\Big((x^2+y^2)^2 - 1\Big)\exp\Big(a\arctan\frac{y}{x} + b\cos(2\arctan\frac{y}{x})\Big)}.$$

Hence, Theorem 3.3.2 *is proved.*

Our third results on the existence of limit cycles of the system (3.11) are the following:

Theorem 3.3.3. *Consider a multi-parameter polynomial differential planar system* (3.11)*, then:*

1. *The system* (3.11) *has an explicit limit cycle, given in cartesian coordinates by*

$$(\Gamma_1) : x^4 + y^4 + 2x^2y^2 - 1 = 0.$$

2. *If* $a > 0$ *and* $b \in \mathbb{R} - \{0\}$*, then system* (3.11) *has non-algebraic limit cycle* (Γ_2)*, explicitly given in polar coordinates* (r,θ)*, by the equation*

$$r(\theta, r_*) = \left(\frac{\exp(a\theta + b\cos 2\theta)\left(\dfrac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi) + f(\theta)\right)}{-1 + \exp(a\theta + b\cos 2\theta)\left(\dfrac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi) + f(\theta)\right)}\right)^{\frac{1}{4}},$$

where

$$f(\theta) = \int_0^\theta \exp(-as - b\cos 2s)ds \ .$$

Moreover, the non-algebraic limit cycle (Γ_2) *lies inside the algebraic limit cycle* (Γ_1).

Proof. ***Proof of statement*** (1) ***of theorem*** 3.3.3

The curves $H = h$ *with* $h \in \mathbb{R}$*, which are formed by trajectories of the differential system*

(3.11), *in cartesian coordinates are written as*

$$\begin{aligned}\left(x^2+y^2\right)^2 &= 1,\\ \left(x^2+y^2\right)^2 &= \left(\frac{\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(h+f(\arctan\frac{y}{x})\right)}{-1+\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(h+f(\arctan\frac{y}{x})\right)}\right),\end{aligned}$$

where $h\in\mathbb{R}$. *Notice that system* (3.11) *has a periodic orbit if and only if equation* (3.13) *has a strictly positive* 2π *periodic solution. This, moreover, is equivalent to the existence of a solution of* (3.13) *that fulfills* $r(0,r_*)=r(2\pi,r_*)$ *and* $r(\theta,r_*)>0$ *for any* θ *in* $[0,2\pi]$. *The solution* $r(\theta,r_0)$ *of the differential equation* (3.13) *such that* $r(0,r_0)=r_0$ *is*

$$r(\theta,r_0)=\left(\frac{\exp(a\theta+b\cos 2\theta)\left(\frac{r_0^4}{(r_0^4-1)\exp(b)}+f(\theta)\right)}{-1+\exp(a\theta+b\cos 2\theta)\left(\frac{r_0^4}{(r_0^4-1)\exp(b)}+f(\theta)\right)}\right)^{\frac{1}{4}},$$

where $r_0=r(0)$.

We have the particular solution $\rho(\theta)=1$ *of the differential equation* (3.13), *from this solution we obtain* $r^4(\theta,1)=1>0$, *for all* $\theta\in[0,\pi]$ *is particular solution of the differential equation* (3.13).

This is an algebraic limit cycle for the differential systems (3.11), *corresponding of course to an invariant algebraic curve* $U(x,y)=0$.

More precisely, in cartesian coordinates $r^2=x^2+y^2$ *and* $\theta=\arctan\left(\frac{y}{x}\right)$, *the curve* (Γ_1) *defined by this limit cycle is*

$$(\Gamma_1): x^4+y^4+2x^2y^2-1=0$$

Hence, statement (1) of theorem (3.3.3) is proved.

Proof of statement (2) ***of theorem*** 3.3.3

A periodic solution of system (3.11) *must satisfy the condition* $r(2\pi,r_0)=r(0,r_0)$, *which leads to a unique value* $r_0=r_*$, *given by*

$$r_*=\sqrt[4]{\frac{e^b f(2\pi)}{1-e^{-2\pi a}+e^b f(2\pi)}},$$

r_* *is the intersection of the periodic orbit with the* OX_+ *axis. After the substitution of these value* r_* *into* $r(\theta,r_0)$ *we obtain*

$$r(\theta,r_*)=\left(\frac{\exp(a\theta+b\cos 2\theta)\left(\frac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi)+f(\theta)\right)}{-1+\exp(a\theta+b\cos 2\theta)\left(\frac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi)+f(\theta)\right)}\right)^{\frac{1}{4}}.$$

In what follows it is proved that $r(\theta, r_*) > 0$. *Indeed, we have*

$$r(\theta, r_*) = \left(\frac{\exp(a\theta + b\cos 2\theta)\left(\frac{1}{1-e^{2\pi a}} f(2\pi) - \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw\right)}{-1 + \exp(a\theta + b\cos 2\theta)\left(\frac{1}{1-e^{2\pi a}} f(2\pi) - \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw\right)} \right)^{\frac{1}{4}},$$

According to $a > 0$ *and* $b \in \mathbb{R} - \{0\}$, *hence*

$$\frac{1}{1-e^{2\pi a}} < 0, \; f(2\pi) > 0, \; \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw \geq 0, \;\; \forall [0, \pi],$$

then we have

$$r(\theta, r_*) > 0, \; \forall \theta \in [0, \pi].$$

This is a limit cycle for the differential system (3.11). *It is not algebraic, due to the expression*

$$\exp\left(a \arctan \frac{y}{x} + b\cos\left(2\arctan \frac{y}{x}\right)\right)$$

More precisely, in cartesian coordinates $r^2 = x^2 + y^2$ *and* $\theta = \arctan\left(\frac{y}{x}\right)$, *the curve* (Γ_2) *defined by this limit cycle is* $(\Gamma_2) : F(x, y) = 0$ *where*

$$F(x,y) = (x^2+y^2)^2 - \frac{\exp\left(a\arctan\frac{y}{x} + b\cos(2\arctan\frac{y}{x})\right)\left(\frac{r_*^4}{(r_*^4-1)\exp b} + f(\arctan\frac{y}{x})\right)}{-1 + \exp\left(a\arctan\frac{y}{x} + b\cos(2\arctan\frac{y}{x})\right)\left(\frac{r_*^4}{(r_*^4-1)\exp b} + f(\arctan\frac{y}{x})\right)}.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial F *in the variables* x *and* y *satisfies that there is a positive integer* n *such that*

$$\frac{\partial^{(n)} F(x,y)}{(\partial x)^n} = 0,$$

and this is not the case, therefore the curve

$$(\Gamma_2) : F(x,y) = 0$$

is non-algebraic and the limit cycle will also be non-algebraic.
According to $a > 0$ *and* $b \in \mathbb{R} - \{0\}$, *hence*

$$\frac{1}{1-e^{2\pi a}} < 0, \; f(2\pi) > 0, \; \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw \geq 0, \;\; \forall [0, \pi],$$

we get

$$\exp\left(a\theta + b\cos 2\theta\right)\left(\frac{1}{1-e^{2\pi a}} f\left(2\pi\right) - \int_{\theta}^{2\pi} \exp\left(-aw - b\cos 2w\right) dw\right) < 0,\ \forall \left[0,\pi\right],$$

then we have $r\left(\theta, r_*\right) < 1,\ \forall\theta \in \left[0,\pi\right].$ *We conclude that system* (3.11) *has two limit cycles, the non-algebraic* (Γ_2) *lies inside the algebraic one* (Γ_1).

This completes the proof of theorem (3.3.3).

3.3.1 Examples of the system

The following examples are given to illustrate our result.

Example 1

If we take $a = 3$ *and* $b = 1$, *then system* (3.11) *writes*

$$\begin{cases} x' = x - 5x^5 + 8x^4y - 10x^3y^2 + 12x^2y^3 - 5xy^4 + 4y^5 + 4x^9 - 4x^8y + \\ 16x^7y^2 - 12x^6y^3 + 24x^5y^4 - 12x^4y^5 + 16x^3y^6 - 4x^2y^7 + 4xy^8, \\ y' = y - 4x^5 - 5x^4y - 4x^3y^2 - 10x^2y^3 - 5y^5 + 4x^8y - 4x^7y^2 + 16x^6y^3 - \\ 12x^5y^4 + 24x^4y^5 - 12x^3y^6 + 16x^2y^7 - 4xy^8 + 4y^9. \end{cases} \tag{3.17}$$

The system (3.17) *has the first integral*

$$H\left(x,y\right) = \frac{\left(x^2+y^2\right)^2 + \left(1-\left(x^2+y^2\right)^2\right)\exp\left(3\arctan\frac{y}{x} + \cos\left(2\arctan\frac{y}{x}\right)\right) f(\arctan\frac{y}{x})}{\left(\left(x^2+y^2\right)^2 - 1\right)\exp\left(3\arctan\frac{y}{x} + \cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where

$$f\big(\arctan\frac{y}{x}\big) = \int_0^{\arctan\frac{y}{x}} \exp(-3s - \cos 2s) ds.$$

The system (3.17) *has an algebraic limit cycle* (Γ_1) *whose expression is*

$$(\Gamma_1) : x^4 + y^4 + 2x^2y^2 - 1 = 0.$$

This system (3.17) *has a non-algebraic limit cycle* (Γ_2) *whose expression in polar coordinates* (r,θ) *is*

$$r\left(\theta, r_*\right) = \left(\frac{\exp\left(3\theta + \cos 2\theta\right)\left(\dfrac{e^{6\pi}}{1-e^{6\pi}} f(2\pi) + f(\theta)\right)}{-1 + \exp\left(3\theta + \cos 2\theta\right)\left(\dfrac{e^{6\pi}}{1-e^{6\pi}} f(2\pi) + f(\theta)\right)}\right)^{\frac{1}{4}},$$

where $\theta \in \mathbb{R}$, *with*

$$f(\theta) = \int_0^{\theta} \exp(-3s - \cos 2s) ds,$$

and the intersection of the limit cycle with the OX_+ *axis is the point having* r_*

$$r_* = \sqrt[4]{\frac{e\int_0^{2\pi} \exp(-3s - \cos 2s)ds}{1 - e^{-6\pi} + e\int_0^{2\pi} \exp(-3s - \cos 2s)ds}} = 0.764\,60$$

We conclude that system (3.17) *has two limit cycles. Since* $r_* = 0.764\,60 < 1$*, the non-algebraic lies inside the algebraic one.*

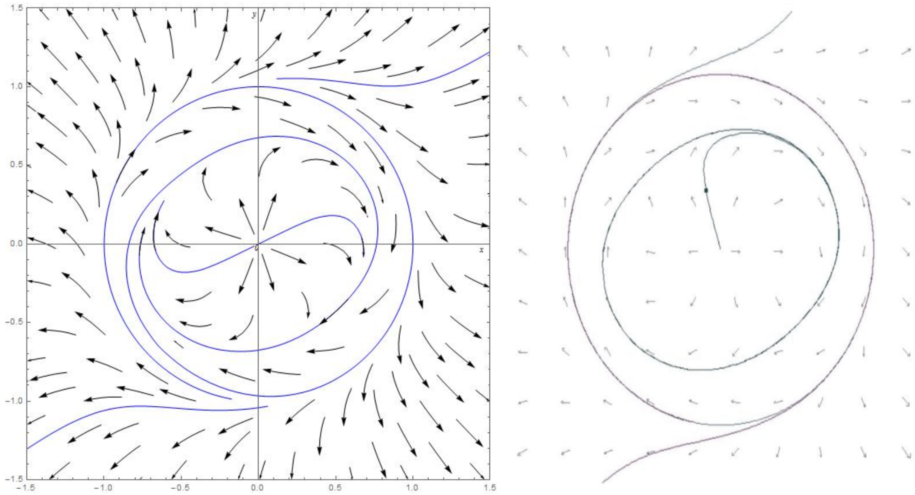

Figure 3.1: Phase plane of example 1

Example 2

If we take $a = 5$ *and* $b = -2$*, then system* (3.11) *writes*

$$\begin{cases} x' = x - 7x^5 - 4x^4y - 14x^3y^2 - 7xy^4 + 4y^5 + 6x^9 + 8x^8y + 24x^7y^2 + 24x^6y^3 + \\ 36x^5y^4 + 24x^4y^5 + 24x^3y^6 + 8x^2y^7 + 6xy^8, \\ y' = y - 4x^5 - 7x^4y - 16x^3y^2 - 14x^2y^3 - 12xy^4 - 7y^5 + 6x^8y + 8x^7y^2 + \\ 24x^6y^3 + 24x^5y^4 + 36x^4y^5 + 24x^3y^6 + 24x^2y^7 + 8xy^8 + 6y^9, \end{cases} \tag{3.18}$$

The system (3.18) *has the first integral*

$$H(x,y) = \frac{(x^2+y^2)^2 + \left(1 - (x^2+y^2)^2\right)\exp\left(5\arctan\frac{y}{x} - 2\cos\left(2\arctan\frac{y}{x}\right)\right) f(\arctan\frac{y}{x})}{\left((x^2+y^2)^2 - 1\right)\exp\left(5\arctan\frac{y}{x} - 2\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where

$$f\left(\arctan\frac{y}{x}\right) = \int_0^{\arctan\frac{y}{x}} \exp(-5s + 2\cos 2s)ds.$$

The system (3.18) *has an algebraic limit cycle* (Γ_1) *whose expression is*

$$(\Gamma_1) : x^4 + y^4 + 2x^2y^2 - 1 = 0.$$

This system (3.18) *has a non-algebraic limit cycle* (Γ_2) *whose expression in polar coordinates* (r, θ) *is*

$$r(\theta, r_*) = \left(\frac{\exp(5\theta - 2\cos 2\theta)\left(\frac{e^{10\pi}}{1-e^{10\pi}} f(2\pi) + f(\theta)\right)}{-1 + \exp(5\theta - 2\cos 2\theta)\left(\frac{e^{10\pi}}{1-e^{10\pi}} f(2\pi) + f(\theta)\right)} \right)^{\frac{1}{4}},$$

where $\theta \in \mathbb{R}$, *with*

$$f(\theta) = \int_0^{\theta} \exp(-5s + 2\cos 2s)ds,$$

and the intersection of the limit cycle with the OX_+ *axis is the point having* r_*

$$r_* = \sqrt[4]{\frac{e^{-2}\int_0^{2\pi} \exp(-5s + 2\cos 2s)ds}{1 - e^{-10\pi} + e^{-2}\int_0^{2\pi} \exp(-s + 2\cos 2s)ds}} = 0.584\,60$$

We conclude that system (3.18) *has two limit cycles. Since* $r_* = 0.584\,60 < 1$, *the non-algebraic limit cycle lies inside the algebraic limit cycle.*

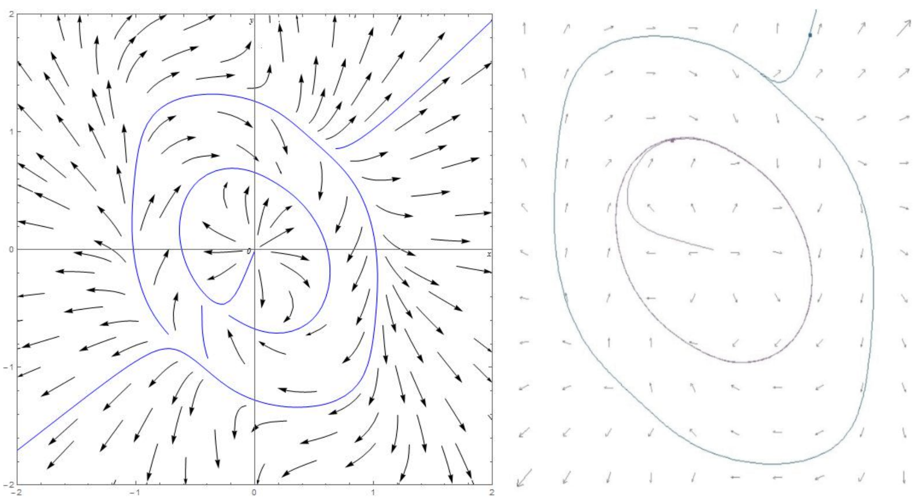

Figure 3.2: Phase plane of example 2

CONCLUSION

In this thesis we are interested in the qualitative study of the planar differential polynomial systems as well as that of the planar differential systems of the Kolmogorov type. It is important for a differential system to know if it admits or not a first integral, a periodic solution, moreover if this periodic solution is isolated, one speaks by definition of a limit cycle. On the other hand, the calculation of the first integral of a planar differential system completely determines the phase portrait of the system. For models resulting from practice, it is important to study these questions: first integral, periodic solution, limit cycle, phase portrait. The results obtained in this thesis revolve around these questions.
In the first chapter we presented some basic notions, concerning the qualitative theory of differential systems, in particular planar differential systems.
In the second chapter we have dealt with classes of planar differential systems of the Kolmogorov type. This chapter is divided into two parts, in the first part we have determined the exact expression of the first integral and the formula of the curves which are formed by the orbits of a class of planar differential systems. In the second part we have determined the exact expression of the first integral and the formula of the curves which are formed by the orbits of a class of planar differential systems of Kolmogorov type we used the Riccati equation.
In the third chapter we have treated a class of planar differential systems. This chapter is divided into two parts, in the first part we have determined the exact expression of the first integral and we have demonstrated the non-existence of limit cycles for a class of planar differential systems. In the second part of this chapter, we have determined the

exact expression of the first integral and the formula of the curves which are formed by the orbits of a class of planar polynomial differential systems. Then we determined the conditions so that these differential systems have two cycles limiting an algebraic cycle and not algebraic given explicitly. To our knowledge, it is rare to find, in the literature of differential systems, a differential system with a non-algebraic limit cycle given explicitly. For the perspectives, given the techniques that we have used to find a class of systems with a non-algebraic limit cycle, it is possible to hope to find a class of quadratic differential systems which admit a non-algebraic limit cycle and given it explicitly. Note that this issue is an open issue so far.

On the other hand, we have studied classes of Kolmogorov systems from the point of view of integrability and non-existence of limit cycles. There remains the problem of existence of limit cycle given explicitly for a class of system of Kolmogorov type. To our knowledge there is not an example of a Kolmogorov system with a non-algebraic limit cycle given explicitly.

Our investment in the future is in this direction and this thesis serves as a powerful tool in the search for the first integral and the existence of limit cycle.

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الملخص الهدف من هذه الأطروحة هي الدراسة النوعية لبعض الفئات من النظم التفاضلية كثيرات الحدود في المستوي. النتائج المتحصل عليها في هذه الدراسة متعلقة بتصرفات وسلوكيات الحلول في المستوي و وجود الحلول الدورية بصفة خاصة الدورات النهائية لبعض النظم التفاضلية ، بالاضافة الى ذلك تمكنا من تحديد الصيغة الرياضياتية

لكل التكاملات الأولية ودورات النهائية الجبرية أو الغير الجبرية

كلمات مفتاحية : نظام تفاضلي ، منحى صامد ، حل دوري ، حل جبري ، حل غيرجبري

……………………………………………………………………………………………….

Abstract The objective of this thesis is the qualitative study of some classes of planar polynomial differential systems. The results obtained in this study concerns the integrability, the phase portraits and the existence of limit cycles of some classes of differential systems. In addition, we give an explicitly expression of the first integrals and limits cycles algebraic or non-algebraic found for all the classes studied.

Keywords: Differential system, invariant curve, periodic solution, algebraic limit cycle, non-algebraic limits cycle.

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Résumé L’objectif de cette thèse est l'étude qualitative de quelques classes des systèmes différentiels planaires polynômiaux. Les résultats obtenus dans cette étude concernent l'intégrabilité, les portraits de phase et l'existence des cycles limites de quelques classes des systèmes différentielles. De plus on détermine explicitement l’expression des intégrales premières et des cycles limites algébriques ou non algébriques trouvés pour toutes les classes étudiées.

Mots clés: Système différentiel, courbe invariante, solution périodique, cycle limite algébrique, cycle limite non algébrique.