Biopolymer Applications Journal (BAJ) MEDJANE et al. Vol 2, N°1, 2023, pp.30-35

e-ISSN : 2800-1729

Analytical and numerical investigation of displacements and stresses in thick-walled FGM cylinder with exponentially-varying properties

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Received: 09 January 2022; Accepted: 29 January 2022; Published: 29 January 2023

Abstract

In this paper, analytical solutions are presented for computing mechanical displacements and stresses in a thick-walled cylindrical made of functionally graded materials (FGMs) under mechanical loading. The pressure vessel is subject to axisymmetric internal pressure. The analytical model of the pressurized vessel is constructed, where the radial continuous varying of elastic modulus along the thickness was assumed. It has been assumed that the elastic modulus is varying through thickness of the FGM material according to an exponential distribution along the thickness. Navier's equation, which is a second-order ordinary differential equation, was derived from the mechanical equilibrium equation. The distributions of the displacement and stresses were determined by the exact solution to Navier's equation. The effect of the inhomogeneity parameter and internal pressure on radial displacement, radial, tangential and axial stresses in a functionally graded cylinder is analyzed. Results obtained clarify the influence of the mechanical field, non-homogeneity parameter on the elastic response of the functionally graded cylindrical vessel. This is depicted graphically by using radial displacement and stresses in a pressurized functionally graded cylinder. Thus, these parameters have remarkable effects on the distributions of radial displacement and radial and circumferential stresses. The inhomogeneity constant, which includes continuously varying volume fraction of the constituents, is a useful parameter from a design point of view in that it can be tailored for specific applications to control the stress distribution.

Keywords: *Stresses, elasticity, FGM, Thick cylinders, mechanics, exponential distribution.*

I. Introduction

Functionally graded materials (FGMs) are the advanced form of composite materials. The main advantage of using FGM over composite material is that it provides the freedom of continuous and directional gradation of specific properties on a macroscopic level, which eliminates the effect of interlaminar stresses and leads to increase in the overall performance of component [1]. FGM structures are most commonly used in civil structures, mechanical structures like, helicopter rotor blades, robot arms, turbine blades and space erectable booms, etc. [2-3]

Thick-walled cylindrical vessels and hollow cylinders are commonly used components in different structural applications and device systems in many industries involving aerospace and submarine structures, civil engineering, mechanical engineering, pipes, sensors and actuators, etc. [4-7]. Until recently, these cylindrical objects are made of isotropic homogeneous materials which can only be optimized for their applications by material selection and very limited dimensional design. In recent decades, functionally graded materials have become widespread due to possibility they offer for optimization of the design in terms of the spatial distribution of material properties.

Within the framework of theory of elasticity, the analysis of such hollow structures subjected to internal and/or external pressures and temperature may be of great interest regarding the importance of design in solid mechanics.

FG Materials are microscopically non homogeneous but at macro level, the mechanical properties vary continuously from one surface to another by smoothly varying the volume fractions of the material constituents [8,9]. Obviously FGM could be used in a variety of applications which have made them very attractive. In addition to their good thermal properties, they are corrosion and erosion resistant and have high fracture resistance [10, 11]. FGM are fabricated by continuously changing the volume fraction of two basic materials, usually ceramic and metal, in one or more directions. The FGM that are thus formed exhibit isotropic yet non-homogenous thermal and mechanical properties.

These kinds of materials are treated as non-homogenous with material contents that vary continuously along one spatial direction.

Analytical solutions for the elastic behaviors of cylinders were provided by Tutuncu and Ozturk [12], Sburlati [09], Nejad and Fatehi [13], among others. Nejad and Fatehi [13] proposed an exact solution of an elastoplastic rotating thick-walled cylindrical pressure containers built of functionally graded materials. Chen and Lin [14] used the transmission matrix approach to statistically evaluate the elastic behavior of thick-walled, single or multi-layered, and arbitrarily graded cylinders/spheres. Nie et al. [15] presented a method for tailoring materials that are graded by a radially varying volume fraction rule to achieve through-the-thickness either a constant circumferential stress or a constant in-plane shear stress. This method is applicable to linear elastic hollow cylinders and spheres. Nie and Batra [16] employ the Airy stress function to derive analytical solutions for plane strain static deformations of a functionally graded (FG) hollow circular cylinder with Young's modulus E and Poisson's ratio ν taken to be functions of the radius r. Besides assuming that the material properties vary according to a power-law function of r, some investigators have presumed that the elastic modulii is exponential functions of r. Based on the assumption that Poisson's ratio ν is constant and Young's modulus E is an exponential function of *r* [17–19], have analyzed stresses and displacements in FG cylindrical pressure vessels. Existing analytical solution for elastic field in pipes are available when Young's modulus is expressed by the power law expression [20-22]. Ghannad and Gharooni [23] employed the first-order shear deformation theory to analyze the elastic characteristics of pressured thick FGM cylinders with exponential variation. A numerical elastic study was performed by Chen and Lin [24] for a thick cylinder or sphere constructed of exponentially graded materials. Dai et al. [25] reviewed most of the researches in years (2005–2015) on FGM cylindrical structures.

Existing analytical solution for elastic field in pipes are available when Young's modulus is expressed by the power law expression. For the exponential law, the most existing solutions in the literature are numerical. In order to fill this gap, an exponential function form for the spatial variation of the stiffness proposed was used in this work. We study the distribution of stresses in a thick wall of a cylinder in FGM under internal pressure. The equilibrium equation is transformed into a nonlinear differential equation that we have solved analytically.

However, designing of a cylinders made of FGM requires understanding and quantifying the elastic field in these elements. In an attempt to achieve this goal, this study concentrates on the analysis of elastic field in a cylinder FGM subjected to internally and/or externally pressurized.

In this paper, using the elasticity solution, the elastic behavior of the functionally graded thick-walled tube subjected to axisymmetric mechanical load is investigated in this work. In Section 2 the basic equations of the FGM long tube and the analysis of mechanical behaviors of the tube are described. Section 3 gives the results and discusses the obtained results.

II. Problem formulation

We consider a hollow FGM cylinder. The cylinder has inner and outer radii, respectively, R_{in} and R_{out}. It is subjected to mechanical loading. The materials are assumed linearly elastic and isotropic and the graded materials Young's modulus depend only on the radial direction while Poisson's ratio is assumed constant. The schematization of the thick tube is shown in Fig. 1.

For convenience, the axisymmetric load conditions require considering a cylindrical coordinate system (r, θ, z).

To ascertain the effect of the inhomogeneity, expression of Young's modulus is considered across the thickness:

$$E(r) = E_i e^{\left(\frac{\alpha(r-R_i)}{R_o-R_i}\right)} \quad (1)$$

Where $E(r)$ is Young's modulus. E_i is the values of Young's modulus at r = R_{in}. α is the parameter of the inhomogeneity material. Homogenous case is obtained when $\alpha = 0$.

Let $\boldsymbol{u}$ be the displacement field, where $u_i\,(i = r,\theta,z)$ are the components of displacement, respectively. The length of the cylinder is sufficiently large so that we confine our attention to the plain strain problem.

In the case of plane elasticity, the displacement, stress and strain fields may be written:

$$\boldsymbol{u} = \boldsymbol{u}(r), \quad \sigma_{ij} = \sigma_{ij}(r), \quad \varepsilon_{ij} = \varepsilon_{ij}(r) \quad (2a,b,c)$$

The cylinder's material is graded through the r-direction, thus the material properties are functions of r. Let u_r be the displacement component in the radial direction and since the elastic field is axisymmetric and independent of z, the kinematic relations (strain-displacement) are given:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \varepsilon_{\theta\theta} = \frac{u_r}{r}, \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{r\theta} = \varepsilon_{\theta z} = 0 \quad (3a,b,c)$$

The cylinder is made of elastic inhomogeneous material which is described by Hooke's law which is given by :

$$\sigma_{ij} = 2\mu(r)\varepsilon_{ij} + \lambda(r)\varepsilon_{ll}\delta_{ij} \quad (4)$$

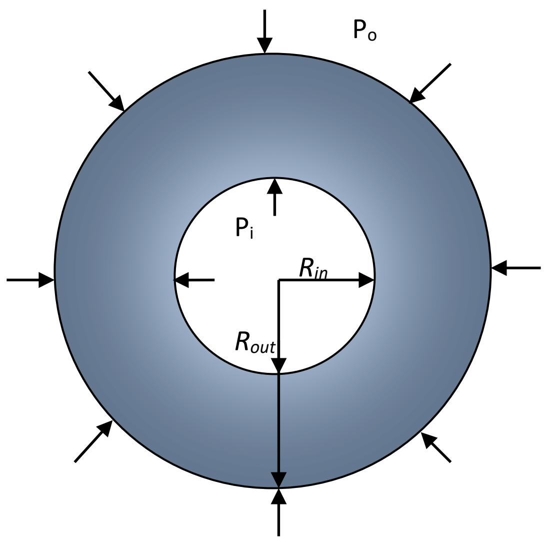

Figure 1. Thick circular cylinder with wall in FGM.

where: $\mu(r)=\frac{E(r)}{2(1+\nu)}$ and $\lambda(r)=\frac{\nu E(r)}{(1+\nu)(1-2\nu)}$ are Lamé's constants, $E(r)$ is Young's modulus which depends on spatial coordinates and $\nu$the Poisson's ratio.

The stress-strain relations in elastic steady state problem are given:

$$\begin{cases}\sigma_{rr}=\left(2\mu(r)+\lambda(r)\right)\cdot\varepsilon_{rr}+\lambda(r)\cdot\varepsilon_{\theta\theta}\\ \sigma_{\theta\theta}=\left(2\mu(r)+\lambda(r)\right)\cdot\varepsilon_{\theta\theta}+\lambda(r)\cdot\varepsilon_{rr}\\ \sigma_{zz}=\lambda(r)\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}\right)\end{cases} \tag{5}$$

where σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are the radial, circumferential and axial components of the Cauchy stress tensor.

III. Analytical solution

The equilibrium equation for a cylindrical vessel in an axisymmetric problem, disregarding the body forces and the inertia terms, is written in cylindrical coordinates as follows:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \tag{6}$$

To obtain the equilibrium equation in terms of the displacement component for the cylinder made of FG Material, the functional relationship of the material properties must be known.

For convenience, we assume:

$$\lambda(r)=\psi(\nu)\cdot E(r),2\mu(r)+\lambda(r)=\zeta(\nu)E(r)$$
$$\psi(\nu)=\frac{\nu}{(1+\nu)(1-2\nu)},\zeta(\nu)=\frac{(1-\nu)}{(1+\nu)(1-2\nu)}, \tag{7}$$

Substituting the kinematic relations in Eq. (3) into Hooke's law in Eq. (4) and then into the equations of equilibrium Eq. (6) gives the following differential equation governing the radial displacement field:

$$r^2\frac{d^2u}{dr^2}+(Ar+1)r\frac{du}{dr}-\left(1-\frac{\psi(\nu)}{\zeta(\nu)}Ar\right)u=0 \tag{8}$$

where: $A=\frac{dE(r)}{E(r)dr}$

Substituting Eqs. (1) in Eq. (8) then we obtain the following form of the differential equation:

$$r^2\frac{d^2u}{dr^2}+\left(\frac{\alpha}{R_{out}-R_{in}}r+1\right)r\frac{du}{dr}-\left(1-\frac{\psi(\nu)}{\zeta(\nu)}\frac{\alpha}{R_{out}-R_{in}}r\right)u=0 \tag{9}$$

The solution for the differential equation (9) can be expressed:

$$u(r)=Ar^{-\frac{1}{2}}e^{\left(\frac{\alpha r}{2(R_{in}-R_{out})}\right)}M\left(a_1,b_1,\frac{\alpha r}{(R_{in}-R_{out})}\right)+Br^{-\frac{1}{2}}e^{\left(\frac{\alpha r}{2(R_{in}-R_{out})}\right)}W\left(a_1,b_1,\frac{\alpha r}{(R_{in}-R_{out})}\right) \tag{10}$$

With: $a_1=\frac{3\nu-1}{2\nu-2}$, $b_1=1$

The basic properties of Kummer's function can be found in various mathematical references (Abramowitz and Stegun [26]) and they are available in many computer programs.

In the case of a cylinder subjected to internal and external pressures respectively, P_i and P_o, the constants A and B can be obtained from the following boundary conditions:

$$\begin{cases}\sigma_{rr}\big|_{r=R_{in}}=-P_i\\ \sigma_{rr}\big|_{r=R_{0ut}}=-P_o\end{cases} \tag{11}$$

IV. Results and discussion

The main objective of this work is to obtain tractable solutions to allow for further parametric studies. Stress and displacement solutions in the analytical solution form are presented in FGM thick-walled cylinders with exponentially-varying elastic modulus in the radial direction. Although the case of FGM cylinders with variation of elastic properties obeying a simple power law is extensively studied, the results for exponentially varying properties are scarcely available in the literature. A positive inhomogeneity constant refers to increasing stiffness in the radial direction.

We consider a thick cylinder whose elasticity modulus varies in radial direction and has the following characteristics: R_{in} =0.1 m, R_{out} =0.2m. The elasticity modulus, is taken Ei = 200GPa.The Poisson's ratio has a constant value ν = 0.3.

The applied internal and external pressures are P_i = 500MPa and P_o = 0 MPa, respectively. The constant of the inhomogeneity parameter can be adjusted in various ways to reproduce various conditions. However, the study was conducted for different values of inhomogeneity parameter α = -2, -1 0, 1 and 2.

We compared the results of our analyses with the results obtained from finite element method (FEM). A finite element model of the pressure vessel subjected to internal pressure was constructed using FE code. The geometry is axisymmetric. Consequently, half of cylindrical specimen was considered. Thus, we reduce the number of nodes and the computational time. In order to consider the radial continuous varying of the elastic modulus along the thickness of hollow cylinder with special function, the exponential expression of Young's modulus (Eq. (1).) was implemented into the FE model.

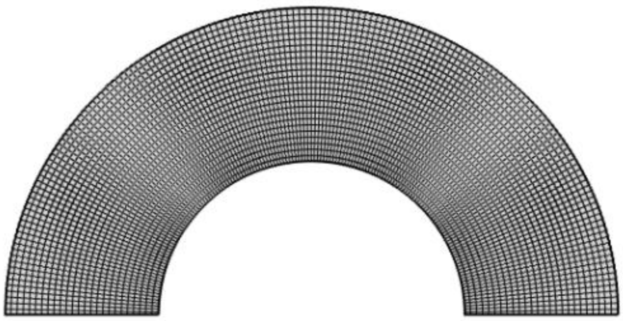

Figure 2. Numerical model geometry.

Internal and external pressures are applied to the nodes of inner and outer layers, respectively. The boundary condition and mesh distribution of the FE model employed in this study is presented in Fig. 2.

For different values of inhomogeneity parameter α = -2, -1 0, 1 and 2, dimensionless modulus of elasticity across the radial direction is plotted in Fig. 3. According to this figure, at the same radial position r in R_{in}< r < R_{out}, dimensionless modulus of elasticity increases as the parameter of inhomogeneity α increases.

For different values of α, the distribution of the radial displacement resulted from the analytical solution across the thickness of a cylinder is depicted in Fig. 4. It was obviously observed in Fig. 4 that the radial displacements have its maximum values in internal surface (r = R_{in}). The radial displacement values decrease gradually from inner surface through the outer surface. We observe an increase of the displacement with the decrease of α. The increase in radial displacement with the decrease of α is due to the fact that the global stiffness of the cylinder decreases.

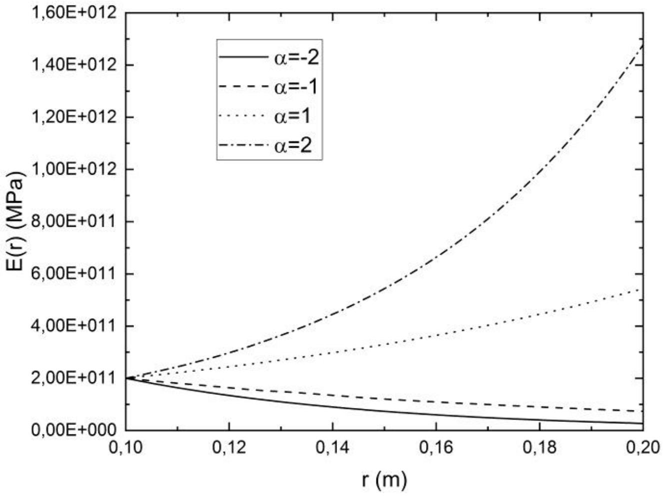

Figure 3. Radial distribution of elastic modulus for different values of inhomogeneity parameter.

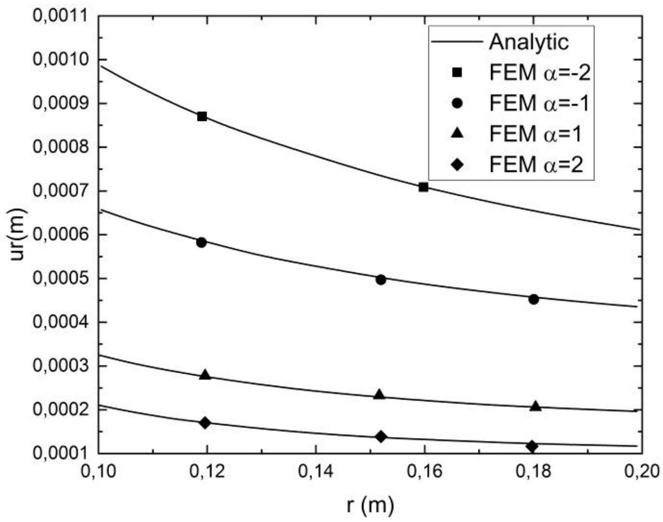

Figure 4. Radial distribution for radial displacement for different values of inhomogeneity parameter.

The radial stress was plotted in Fig. 5 for different values of inhomogeneity parameter α =−2, −1, 1 and 2. The variation in the radial stress of heterogeneous material is similar to that of homogenous material.

For all values of α considered, the magnitude of the radial stress has a monotonic behavior increasing in r. It was seen that, for each fixed r in R_{in}< r < R_{out} the radial stress magnitude decrease as α increases. It is due to the fact that the global stiffness of the cylinder increases. Increasing the inhomogeneity constant causes an increase in the stresses.

The circumferential stress was plotted along the radial direction and shown in Fig. 6. The hoop stress along the radius decreases monotonically respect to r for all different negative values of α considered (similar to thick cylinders made of isotropic materials), while the hoop stress increases respect to r for positive values of α. It was seen that, for each fixed value of r in r<~ 0.14 m, the hoop stress magnitude decrease as α increases. However, for each fixed value of r in r >~ 0.14 m, the hoop stress magnitude increase as α increases. Indeed, higher values of α mean increasing stiffness.

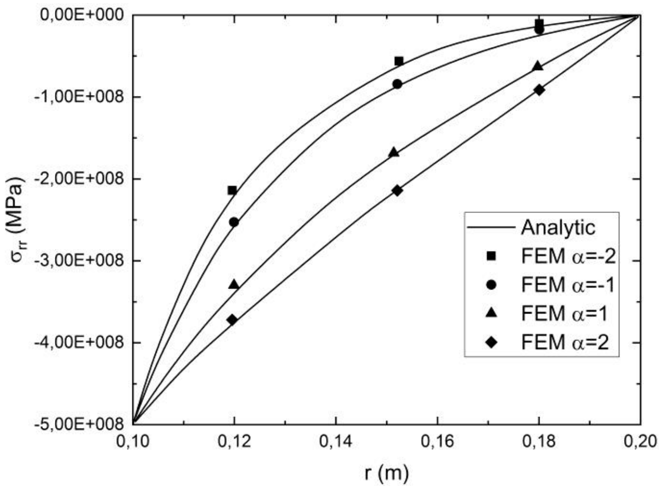

Figure 5. Radial distribution of radial stress for different values of inhomogeneity parameter α.

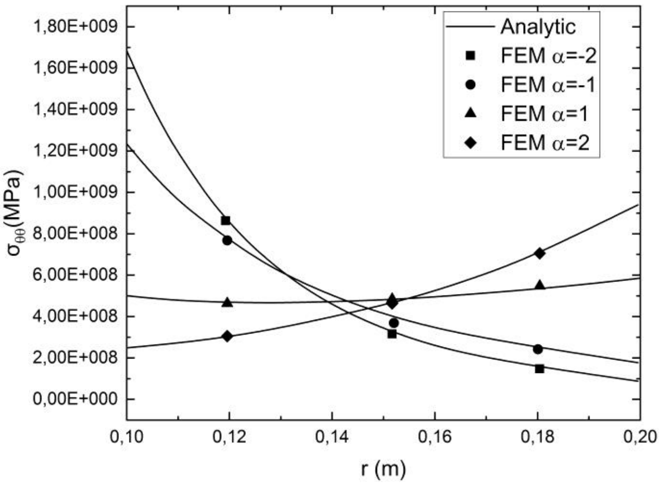

Figure 6. Radial distribution of hoop stress for different values of inhomogeneity parameter α.

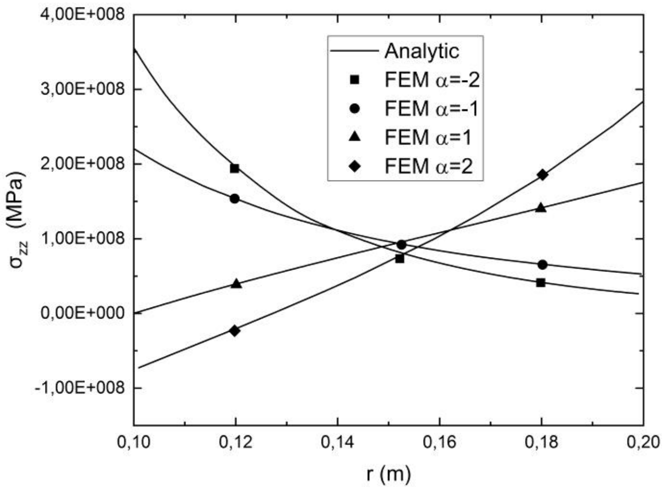

Figure 7. Radial distribution of axial stress for different values of inhomogeneity parameter α.

Fig. 7 shows the axial stress distribution plotted along the radial direction. The axial stress along the radius decreases monotonically respect to r for all different negative values of α considered, while the hoop stress increases respect to r for positive values of α.
It was seen that, for each fixed value of r in r<~ 0.15 m, the axial stress magnitude decrease as α increases. However, for each fixed value of r in r >~ 0.15 m, the hoop stress magnitude increase as α increases. Indeed, higher values of α mean increasing stiffness.

V. Conclusions

In this work, an analytical formulation was presented to find displacements and stress components in thick-walled cylinder subjected to internal pressure. The FGM properties were supposed to be exponentially-varying across the thickness of the cylinder.
For validation, a finite element (FE) model of the pressurized cylinder was constructed, where the radial continuous varying of elastic modulus along the thickness was implemented in a FE code. Then we compared the analytical and numerical solutions. The analytical solution was validating since the comparisons of the numerical results with solutions obtained gives a good agreement.
The results are presented as evolution of the displacement and stress components through the radial direction are plotted for different material inhomogeneity parameter. From above results, it can be concluded that the spatial variation of Young's modulus has a great effect on stresses and radial displacement distribution in FGM cylinder. Thus, the parameter of inhomogeneity is a useful parameter from a design point of view and can be tailored to specific applications to control the stress distributions.

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