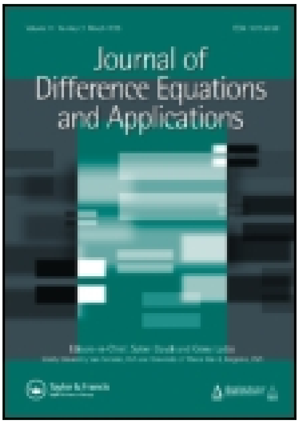

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Homoclinic solutions in a quadratic differential system under discretization

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The present paper is a case study on the question if homoclinic solutions of a differential equation persist under Euler discretization. The motivation for the particular example investigated comes from iterations of the complex polynomial $z^2 + c$.

Keywords: homoclinic orbits; Euler discretization; discrete system; non-hyperbolic stationary point

AMS Classification: 34C25; 34C37; 34A34; 39A05

1. Introduction

Any quadratic complex polynomial can be written in the form $z^2 + c$, where z and c are complex. The study of the iterative behaviour of quadratic complex polynomials has been initiated by Julia and Fatou in the first two decades of the last century. They were interested in the iteration of the family $z^2 + c$ and the sets whose points have a bounded orbit, that is, the Julia sets. In the late 1970s, their work was continued by Mandelbrot and other authors. Julia sets of quadratic complex polynomials serve as examples for fractals and demonstrate that simple iteration systems can produce complicated dynamics. This has given rise to the fractals [2]. For some values of c, many properties concerning the fixed points, periodic points, cycles and more generally invariant sets have been established. The one-parameter quadratic complex iteration $z_{n+1} = z_n^2 + c$ with $n \in \mathbb{N}$ and $c \in \mathbb{C}$ can be rewritten as

$$\begin{cases} X_{n+1} = X_n^2 - Y_n^2 + a', \\ Y_{n+1} = 2X_nY_n + b'. \end{cases} \tag{1}$$

Here $c = a' + ib'$ and $z_n = X_n + iY_n$.

Consider the differential system

$$\begin{cases} \dot{X} = X^2 - Y^2 - X + a', \\ \dot{Y} = \varepsilon\big(2XY - Y + b'\big), \end{cases} \tag{2}$$

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where $\varepsilon > 0$ is a parameter. System (1) is an Euler discretization of (2) when $\varepsilon = 1$. Putting $x = X - 1/2$ and $y = Y$, system (2) simplifies to

$$\begin{cases} \dot{x} = x^2 - y^2 + a, \\ \dot{y} = \varepsilon\big(2xy + b\big), \end{cases} \tag{3}$$

where $a = a' - 1/4$ and $b = b'$. If $b = 0$, the x-axis is invariant and system (3) is symmetric to both axes. Thus, for $b = 0$, the study of system (3) reduces to the half-plane $y \geq 0$. For $a \neq 0$ and $b = 0$, system (3) admits two equilibria. Both are centres for $a > 0$ and $b = 0$ and, up to the one on the equilibrium-free axis, all trajectories are periodic. For $a = b = 0$, the origin is a unique equilibrium point and both eigenvalues of the linearization at the origin are zero. Apart from those on the x-axis, we will prove that all trajectories are homoclinic to the origin.

The aim of this paper is to investigate whether homoclinic solutions of system (3) remain homoclinic under Euler discretization with small stepsize, when the stationary point is non-hyperbolic. Several authors were interested in similar questions such as discretization of invariant manifolds, phase portraits near a stationary point, heteroclinic orbits, saddle-node homoclinic orbits or periodic orbits. In the case of a hyperbolic fixed point of an autonomous equation, Refs [1,5,8] give an affirmative answer to the above question. They give an error estimate of order $O(h^d)$, for the difference between the homoclinic solution of the differential equation and that of the associated discrete equation, where h is the stepsize of the method of discretization and d is its order. They give also the length $l(h)$ of the parameter interval over which the homoclinic orbit persists. The work [8] provides an alternative to the interpolation approach by Fiedler and Scheurle [5]. Still in the hyperbolic case, many results concerning the approximation of the solutions of differential equations by numerical methods are established in Refs [3–6] and the references therein. Besides, the problems of numerical computation of homoclinic and heteroclinic orbits and that of approximation of phase portraits are studied in Refs [5,8] and the references therein. Structural stability results of flows under numerical methods are established in Ref. [7].

In what follows we consider only case $a = b = 0$ of system (3) we rewrite as

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = 2\varepsilon xy. \end{cases} \tag{4}$$

In Section 2, we point out that, apart from those on the x-axis, all trajectories of system (4) are homoclinic to the origin. In Section 3, we prove that, for $r > 0$ small enough and $m > 0$ arbitrary, solutions passing through the set $S_{m,r} = \{x, y \in \mathbb{R}^2 | x^2 + y^2 \leq r^2, y \geq m|x|\}$ remain homoclinic in the discrete system obtained by Euler discretization.

2. Homoclinic orbits in system (4)

In this section, we study the existence of homoclinic solutions of system (4). We have the following proposition.

PROPOSITION 1. *For any $y_0 \neq 0$, the orbit of system* (4) *through* (x_0, y_0) *is homoclinic.*

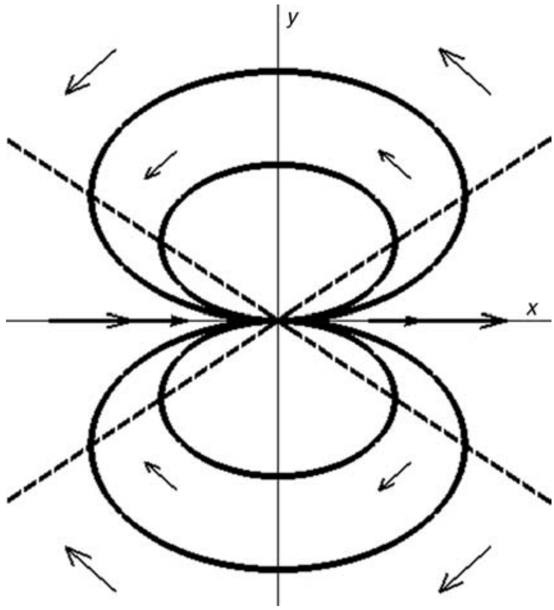

Figure 1. Homoclinic orbits of system (4).

Proof. When $x \neq 0$ and $y \neq 0$, system (4) reduces to the Bernoulli equation

$$2x\frac{dx}{dy} = \frac{1}{\varepsilon}\left(\frac{x^2}{y} - y\right), \tag{5}$$

whose solutions are given by

$$\begin{cases} x^2 - c_1 y^{(1/\varepsilon)} + \frac{y^2}{2\varepsilon - 1} = 0, & \text{if } \varepsilon \neq \frac{1}{2}, \\ x^2 + 2y^2 \log|y| - c_2 y^2 = 0, & \text{if } \varepsilon = \frac{1}{2}, \end{cases} \tag{6}$$

where c_1 and c_2 are constants in $\mathbb{R}$.

The desired result follows from analysing formula (6). Homoclinicity is counter-clockwise in the half-plane $y > 0$. Trajectories tend to the origin from the left as $t \longrightarrow \infty$ and escape the origin from the right as $t \longrightarrow -\infty$ (Figure 1). □

3. Homoclinic orbits in the discrete system associated with (4)

Discretization by Euler method with stepsize $h > 0$ yields to the iteration system

$$\begin{cases} x_{n+1} = x_n + h\left(x_n^2 - y_n^2\right), \\ y_{n+1} = y_n + 2h\varepsilon x_n y_n. \end{cases} \tag{7}$$

In this section, we deal with the behaviour of the solutions of system (7). Together with (4), (7) is also symmetric with respect to the x-axis. Our main result is the following:

THEOREM 1. *Given $m > 0$ arbitrarily, there exists a $h_0 > 0$ and an $r > 0$ such that for any $h \in (0, h_0]$ and $(x_0, y_0) \in S_{m,r}$, the trajectories of system (7) through (x_0, y_0) are homoclinic to the origin.*

Note that the trajectories starting near the x-axis are, in general, not homoclinic to the origin.

The proof of Theorem 1 is based on a sequence of Lemmas. The first Lemma follows directly from the inverse function theorem.

LEMMA 1. *There exists a $h_0 > 0$ and an $R > 0$ such that for any $h \in (0, h_0]$, the map*

$$\varphi_h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x, y) \mapsto \left(x + h\left(x^2 - y^2\right), y + 2hxy\right)$$

is a diffeomorphism on the ball $\mathcal{B}_R = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq R^2\}$ and $\varphi_h(\mathcal{B}_R) \supset \mathcal{B}_{R/2}$.

For simplicity, we assume that $\varepsilon = 1$ and $m = 1$. Thus $S_{m,R} = S_{1,R} = B \cup C$ where

$$B = \left\{(x, y) \in S_{1,R} | y \geq x \geq 0\right\}, \quad C = \left\{(x, y) \in S_{1,R} | -y \leq x \leq 0\right\}.$$

For later use, we set

$$A = \left\{(x, y) \in S_{1,R} | x \geq y \geq 0\right\}, \quad D = \left\{(x, y) \in S_{1,R} | x \leq -y \leq 0\right\}.$$

LEMMA 2. *There exists a $h_0 > 0$ such that for any $h \in (0, h_0]$, D is invariant under system (7) and $(x_0, y_0) \in D$ implies that $(x_n, y_n)_{n \in \mathbb{N}} \longrightarrow (0, 0)$ as $n \longrightarrow \infty$.*

Proof. Pick $(x_0, y_0) \in D$ arbitrarily and observe that, for h small enough,

$$\begin{aligned} &0 \leq y_1 = y_0(1 + 2hx_0) \leq y_0, \quad x_0 \leq x_1 = x_0 + h\left(x_0^2 - y_0^2\right), \\ &y_1 + x_1 = \left(y_0 + x_0\right)\left(1 + h\left(x_0 - y_0\right)\right) + 2hx_0y_0 \leq 0. \end{aligned}$$

In particular, $x_1 \leq 0$ and $y_1^2 + x_1^2 \leq y_0^2 + x_0^2 \leq R^2$. The estimates above show that x_n is increasing and y_n is decreasing. Thus, $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$ for some $(x^*, y^*) \in D$. Letting $n \longrightarrow \infty$ in (7) leads to $x^* = y^* = 0$. □

LEMMA 3. *There exists $h_0 > 0$, $r \in (0, \frac{R}{6}]$ with the property as follows. For any $h \in (0, h_0]$ and $(x_0, y_0) \in B$ with $y_0 \leq r$, there exists a $p \in \mathbb{N}^*$ such that $y_p \leq 5r$ and (x_0, y_0), $(x_1, y_1), \ldots, (x_p, y_p) \in B$, $(x_{p+1}, y_{p+1}) \in C$.*

Proof. We begin by observing that $(x_1, y_1) \notin B$ implies that $(x_1, y_1) \in C$. In fact, $y_1 = y_0(1 + 2hx_0) \geq y_0 > 0$ and, for h and y_0 sufficiently small, $-y_1 \leq x_1 = x_0 + h(x_0^2 - y_0^2) \leq x_0$. Now assume that $(x_n, y_n) \in B$ for all $n \in \mathbb{N}$. We distinguish two cases according as $y_n \leq 2r$ for all $n \in \mathbb{N}$ or not. If $y_n \leq 2r$ for all $n \in \mathbb{N}$, then $x_n \longrightarrow 0$ because x_n is decreasing. Otherwise, x_n is separated from 0 and by $y_{n+1} = y_n(1 + 2hx_n) > y_n$ the sequence y_n is unbounded. (There is nothing to prove if $y_0 = 0$, i.e. $y_0 = x_0 = 0$) But then $0 \leq x_{n+1} = x_n + h(x_n^2 - y_n^2)$ is a contradiction for large n. (This follows via $0 \leq -hy_0^2$ by letting $n \longrightarrow \infty$. (Here again, $y_0 \neq 0$.)) If $y_n > 2r$ for some n, we may assume that

$0 \le x_n \le r$ and $2r \le y_n \le 3r$. Suppose that (x_n, y_n), $(x_{n+1}, y_{n+1}), \ldots, (x_{n+k}, y_{n+k}) \in B$ for some $k \in \mathbb{N}$. Then $y_{n+1} = y_n(1 + 2hx_n) \ge y_n \ge 2r$, $x_{n+1} = x_n + h(x_n^2 - y_n^2) \le r - 3hr^2$ and by induction, $x_{n+k} \le r - 3hkr^2$. Hence $k \le m = [\frac{1}{3hr}] + 1$ the integer part of $\frac{1}{3hr}$ plus 1. It follows that $y_{n+k} \le y_n(1 + 2hr)^m \le 3re^{2/3}(1 + 2hr) \le 5r$. Thus, p can be chosen for the largest integer k satisfying (x_n, y_n), $(x_{n+1}, y_{n+1}), \ldots, (x_{n+k}, y_{n+k}) \in B$. □

Lemma 4. *There exists $h_0 > 0$ such that for any $h \in (0, h_0]$, $(x_0, y_0) \in C$ implies that (x_0, y_0), $(x_1, y_1), \ldots, (x_r, y_r) \in C$, $(x_{r+1}, y_{r+1}) \in D$ for some $r \in \mathbb{N}$.*

Proof. As long as $(x_n, y_n) \in C$, both x_n and y_n are decreasing and bounded. There is nothing to prove if $y_0 = 0$. If $(x_n, y_n) \in C$ for all n, then $x_n \to x^* < 0$ and $y_n \to y^* \ge -x^* > 0$ for some $(x^*, y^*) \in C$. Thus, $y^* = y^*(1 + 2hx^*)$ and $y^* = 0$, a contradiction. Thus (x_0, y_0), $(x_1, y_1), \ldots, (x_r, y_r) \in C$ but $(x_{r+1}, y_{r+1}) \notin C$ for some $r \in \mathbb{N}$. For h small enough, $(x_{r+1}, y_{r+1}) \in D$. □

Now, we are in a position to give the proof of the main result.

Proof of Theorem 1. Let (x_0, y_0) be a point in $B' = \{(x, y) \in S_{1,R/6} | y \ge x \ge 0\}$ and let $(x_n, y_n)_{n\in\mathbb{N}}$ be a solution of system (7) starting from (x_0, y_0). Using Lemmas 3 and 4, the solution $(x_n, y_n)_{n\in\mathbb{N}}$ enters first the set C and then the set D. By Lemma 2, this solution will never leave D and the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ converges to the unique stationary point $(0, 0)$ of system (7). In the same way, it can be proved that set A is invariant under the sequence $(x_{-n}, y_{-n})_{n\in\mathbb{N}}$ which is well defined by Lemma 1. By using the same arguments as above, it appears that this sequence tends to $(0, 0)$ when n tends to $+\infty$.

If (x_0, y_0) is in C, the proof is similar. □

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