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APPLICATION OF THE ØPERATOR METHODS TO OBTAIN INEQUALITIES OF STABILITY IN THE G/M/∞ SYSTEM.

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Résumé: L'objet de cette communication est d'obtenir l'estimation de proximité de la distribution stationnaire du processus nombre de demandes dans un système de files d'attente G/M/m, considéré au moment de l'arrivée de la demande suivante, par rapport à la même distribution, pour la chaine de Markov incluse dans un système G/M/∞ (le système perturbé a les mêmes propriétés que le système limite).

1. Introduction: In this paper, we determine the conditions and estimations of the strong stability of chains describing the queue size (demands in service or waiting) in a G/M/∞ system. We then find exact estimations to approximate characteristics of a G/M/m system by identical characteristics of the system limit. In fact, the stationary distribution of the queue size of the G/M/m system is a sufficiently complex functional of the system's parameters even though it can be exactly computed for the G/M/∞ system.

The results of the paper are proved, by using the operator approach of the stability theory, whose concepts are introduced in [1] and the qualitative analysis for the G/M/∞ system is realised in [2]. To apply this approach, we introduce a special class of norm operators (§.2), then we prove the strong stability in the sense cited above (§.3). Finally, we expose the principal results of this work.

Notice that, in [4] (respectively [7]), the perturbation concerns the flux of demands' arrival (respectively -service intensity). Here, we perturbate the system's structure (allowing study of a finite number of servers).

We should notice that, in pratice, for the analysis of these described systems, we never know exactly the systems' parameters. (One estimates only the degree of proximity relatively to those given). That is why the obtention of this kind of inequalities will allow us to numerically estimate the incertainty shown during this analysis [5].

2. Preliminaries and Notations:

Consider a $G/M/m$ queueing system (FIFO, ∞) with H distribution of the period between the time of the arrival of the demands, and with exponential service time distribution with mean γ^{-1}.

The transition chains are defined with the arrival of new demands and let X_n denote the number of demands (in service or waiting) being in the system at the time of the arrival of the n-th demand. The sequence X_n constitutes a Markov chain with transition kernel $P_n = \left\| P_{ij}(m) \right\|_{i,j=o}^{\infty}$

At the same time, we consider a $G/M/\infty$ queueing system with infinite number of servers, having the same distribution as the $G/M/m$ system considered previously. To estimate the difference between the stationary distributions of the chains X_n in the $G/M/m$ and $G/M/\infty$ systems, we introduce in the space $\mathcal{M} = (\mu_n, n \geq o)$ of finite measures on $\mathbb{N}$, a family of norms,

$$\|\mu\|_V = \sum_{n \geq o} V^n |\mu_n| \qquad \text{for any} \quad V > 1,$$

where $\mu = (\mu_n, n \geq o)$ and $|\mu|$ is the variation of the measure μ.

This norm induces a corresponding norm in the space of transition kernels on $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$,

$$\|Q\|_V = \sup_{i \geq o} V^{-i} \sum_{j \geq o} V^j |Q_{ij}|$$

All the notions and notations not defined here, can be found in [2]. In particular, the definitions of uniform ergodicity and strong stability in the sense this research goes, are given in the first paragraph, whereas the expressions of transition kernels P_m and $P_\infty = \| P_{ij}(\infty) \|_{i,j=0}^{\infty}$ are given in paragraph two, or for instance in [3], [5]. Recall that we associate with each transition kernel P_{ij} the linear mapping,

$$P_{ij} : \mathcal{M} \longrightarrow \mathcal{M}$$

, acting on $\mu \in \mathcal{M}$ as follow:

$$(\mu P_s)_k = \sum_{j \geq 0} \mu_j . P_{jk}(s)$$

Secondly, we denote by $\mu(Z)$ and $\nu(Z)$ respectively the generating functions of the measures μ and ν. Furthermore, every integral with unspecified domain of integration is taken all over $\mathbb{R}^+$.

3. Ergodicity and Stability:

Theorem 1: Let P_∞ -be the transition kernel of a Markov chain X in a $G/M/\infty$ system and suppose that the condition

$$(J) \qquad \int dH(t)/t < \infty \quad \text{holds.}$$

Then, the chain X is uniformly ergodic and strongly stable with respect to $\| \cdot \|_v$.

Remark 1: This theorem differs from theorem(3) [2] in the sense that the conditions imposed are not the same as well as the originality of the proof.

To prove this theorem, lemma(2) [2] shows that it is sufficient to obtain the inequality

$$\| \mu \|_v \leq K_{ov} . \| \nu \|_v$$

where μ is the solution of the system:

$$\mu_j - \sum_{i \geq 0} \mu_i . P_{ij}(\infty) = \nu_j \qquad \text{and} \qquad \sum_{i \geq 0} \mu_i = 0$$

this solution is given by the formula (7) of theorem(2) [2] :

$$\mu(Z) = \sum_{j \geq 0} \prod_{i=0}^{j-1} \Psi_i(Z) \int dH(t) \sum_{n \geq 0} \nu_n \left\{ \left[(Z-1) . e^{-\gamma t j} + 1 \right]^{n+1} - 1 \right\}$$

where $$\Psi_i(Z) = 1 - h(i\gamma) + Z.h(i\gamma)$$

and $$h(i\gamma) = \int \exp-(i\gamma t)\, dH(t)$$

4. Estimate of Stability:

Theorem 2: Let P_∞ -be the transition kernel of a chain X in a $G/M/\infty$ system and suppose that the condition (J) $\int \frac{dH(t)}{t} < \infty$ holds.

Also, let P_m -be the transition kernel of a chain Y in a $G/M/m$ system (with analogous characteristics as the $G/M/\infty$ system).

Then,
$$\| P_m - P_\infty \|_V \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty .$$

To prove this theorem, we need the following results:

Lemma 1: Under the assumptions of theorem(2),
$$A_1(m) = \sup_{i \geq m-1} \left\{ V^{-i} \sum_{j=m}^{i+1} V^j \left| P_{ij}(m) - P_{ij}(\infty) \right| \right\} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty .$$

Proof: From the expressions of P_m and P_∞ see formulae (1) and (2) [2], we obtain
$$A_1(m) = \sup_{i \geq m-1} \left\{ V^{-i} \sum_{j=m}^{i+1} V^j \left| \int \binom{i+1}{j} (1 - e^{-\gamma t})^{i+1-j} . e^{-\gamma t j} . dH(t) - \right.\right.$$
$$\left.\left. - \int \frac{1}{(i+1-j)!} . e^{-\gamma m t} . (\gamma m t)^{i+1-j} . dH(t) \right| \right\}$$

$$\leq \sup_{i \geq m-1} \left\{ \int \left[1 + (V-1) . e^{-\gamma t} \right] . V^{-i} . \left[1 + (V-1) . e^{-\gamma t} \right]^i dH(t) + \right.$$
$$\left. + V \int e^{-\gamma m t} . \sum_{k=0}^{i+1} \frac{1}{V^k} . \frac{(\gamma m t)^k}{k!} dH(t) \right\} \qquad (1)$$

It is easy to show that
$$\sum_{k=0}^{i+1} \left(\frac{\gamma m t}{V} \right)^k / k! \leq \exp(\gamma m t / V)$$

Now, let us show that both terms in (1) go to zero as m goes to infinity. For this, it is sufficient to remark that the secund term in (1) is not greater than

$$\int V.\exp\left(-m\left[\gamma t - \frac{\gamma t}{V}\right]\right).dH(t)$$

Since $\gamma t - \dfrac{\gamma t}{V} > 0$, then $\exp\left[-m(\gamma t - \dfrac{\gamma t}{V})\right] < 1$

There fore,

$$\sum_{m \geq 0} \exp\left[-m(\gamma t - \frac{\gamma t}{V})\right] = 1/\left[1 - \exp(\gamma t - \frac{\gamma t}{V})\right] \qquad (2)$$

Finally, using the formule (4) and (5) in the proof of lemma(1) [2], we can easily show that the first term in (1) tends to zero. From here, and in consideration of (2) and (1), the lemma is proved. ■

Lemma 2: Under the assumption of theorem(2),

$$A_2(m) = \sup_{i \geq m-1} \left\{ V^{-i} \sum_{j=0}^{m-1} V^j \left| P_{ij}(m) - P_{ij}(\infty) \right| \right\} \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Proof of theorem 2: From the definition of $\|\cdot\|_V$,

$$\| P_m - P_\infty \|_V = \sup_{i \geq 0} V^{-i} \sum_{j \geq 0} V^j \left| P_{ij}(m) - P_{ij}(\infty) \right|$$

Notice that $P_{ij}(m) = P_{ij}(\infty)$ for $j > i+1$ and $j \leq i+1 \leq m$

therefore,

$$\| P_m - P_\infty \|_V = \sup_{i \geq m-1} V^{-i} \left\{ \sum_{j=0}^{m-1} V^j \left| P_{ij}(m) - P_{ij}(\infty) \right| + \right.$$

$$\left. + \sum_{j=m}^{i+1} V^j \left| P_{ij}(m) - P_{ij}(\infty) \right| \right\}$$

Using lemma (1) and (2), the theorem is proved. ■

Let π^m denote the stationary distribution of number of demands in a G/M/m system, considered at the time of the arrival of the next demand.

π^∞ denote the analogous stationary distribution in a G/M/∞ sys.

The following claim is a consequence of the strong stability of the chain X in a G/M/∞ system with respect to $\|\cdot\|_V$.

Theorem 3: Suppose the condition (J) $\int dH(t)/t < \infty$ holds, then for any $V > 1$,

$$\|\pi^m - \pi^\infty\|_v = \sum_{i \geq 0} v^i |\pi_i^m - \pi_i^\infty| \longrightarrow 0 \text{ as } m \longrightarrow \infty .$$

The proof of this theorem can be done, for instance, using theorem(3) [6]. When $\|P_m - P_\infty\|_v \longrightarrow 0$ as $m \longrightarrow \infty$, then it is easy to see that $\|\pi^m - \pi^\infty\|_v \longrightarrow 0$ as $m \longrightarrow \infty$.

Remark 2: In essence, in the proof of the results of this paragraph, the quantitative estimates of the stability of systems are given as in [4].

REFERENCES

[1]. Aïssani D. and Kartashov N. V., Ergodicity and Stability of Markov chains with respect to operator topology in the space of transition kernels. Compte Rendus Academie Sciences U.S.S.R, Ser A, Math-Phys-Tech, N°11, pp. 3-5, 1983.

[2]. Aïssani D., Stability forte de la chaine de Markov incluse dans un systeme G/M/00. Review of Advanced Technics N°1, pp.1-13, 1988.

[3]. Karlin S., A first course in stochastic processes, Acad. Press, 1968 .

[4]. Aïssani D., Application de la methode Øperationnelle pour l'obtention des inégalités de stabilité dans un système M/G/1. Proceeding du 1er Colloque Maghrebin sur les M.N.I, Alger, Volume 1, pp.6-12, 1987 .

[5]. Aïssani D., Modélisation des systèmes Informatiques, Cours de Post-Graduation, U.S.T.H.B, Bab Ezzouar, 1988.

[6]. Kartashov N.V., Strong stability of Markov chains. In Problems of stability of stachastics models", Seminar VNISSI, pp.54-59, Sov. Math, 1981 .

[7]. Aïssani D., Stabilité forte de la chaine de Markov incluse dans un système M(2)/G(2)/1 avec priorité relative. Actes du Colloque National "Micro-ordinateur et système", H.C.R, Arzew, pp.37-43, 1988.

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RESUME DU VOLUME 1

Ce premier volume parmi les deux constituants des actes du 2 ème Colloque Maghrébin sur les Modèles Numériques de l'Ingénieur, tenu à Rabat du 22 au 24 Novembre 1989, présente les principales recherches, efféctuées essentiellement par des chercheurs Maghrébins au Maghreb et à l'etranger, sur les techniques mathématiques, numériques et informatiques, utilisées par l'ingénieur pour les tâches d'analyse et de conception d'une part, et sur l'application de ces techniques aux domaines du traitement du signal, d'automatique, d'éléctronique, de télécommunications, de propagation d'ondes et d'acoustique d'autre part. Une partie du volume est consacrée à la présentation des logiciels soumis au colloque sur les différents aspects de l'utilisation des ordinateurs pour résoudre les problèmes posés à l'ingénieur.