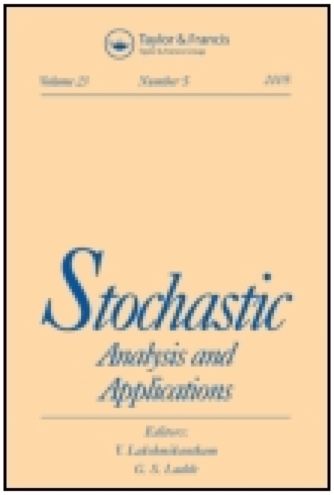

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Strong Stability of the Batch Arrival Queueing Systems

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In this work, we use the strong stability method to study the batch arrival queue after a perturbation of the batch size distribution. We show that, under some hypotheses, the characteristics of the batch arrival queueing system $M^X/M/1$ may be approximated by the correspondent characteristics of the system $M^{Geo}/M/1$.

After clarifying the conditions of approximation, we obtain stability inequalities with an exact computation of constants.

Keywords Approximation; Batch arrival queues; Batch size; Geometric distribution; Perturbation; Strong stability.

Mathematics Subject Classification 60K25; 68M20.

During the investigation of classical problems of the queueing theory, it was assumed that customers arrived one at a time. But, in many situations encountered in practice, customers arrive in batch of random size. These situations can be represented by *queueing models with batch arrivals*. These models have been studied by several researchers such as Medhi [19], Neuts [14], and Gross and Harris [12]. A bibliography on the subject can be found in Chaudhry and Templeton [11]. The batch arrival queueing systems have mostly been studied for modeling the performance of specific systems such as computer systems, communication networks, production systems, transportation systems, manufacturing systems (in the electrical and electronics industry) [2, 12].

To obtain information about the behavior of modeled phenomena, we are interested in the performance measures (response time, utilization rate, etc.). But the study of performance of batch arrival queues is limited in scope because of the complexity of the known results. Therefore, we generally resort to approximation

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methods. In the case of the study of the waiting time distribution [9, 13] gave approximations (for this distribution) in the queue $M^X/G/1$. The approximation of the waiting time distribution of the queue $GI^X/G/1$ was given by Chaudhry and Gupta [10]. Among the approximation approaches, there are also stability methods to approximate the characteristics of the complex system (real) by those of the much simpler system (ideal).

In the stability theory, we establish the domain within which a model may be used as a good approximation or idealization to the real system under consideration (see [4, 17, 25, 26]). In other words, we clarify here the conditions for which the proximity in one way or another of the parameters of the system involves the proximity of the studied characteristics. Such results give the possibility of approximating some systems (very complicated) by other systems more exploitable or much simpler.

There are numerous results on perturbation bounds of Markov chains. General results are summarized by Heidergott and Hordijk [15]. One group of results concerns the sensitivity of the stationary distribution of a finite, homogeneous Markov chain (see [16]), and the bounds are derived using methods of matrix analysis; see the review Cho and Mayer [8] and recent papers of Kirkland [20] and Neumann and Xu [22]. Another group includes perturbation bounds for finite-time and invariant distributions of Markov chains with general state space (see [1, 3, 18, 21, 25]). In these works, the bounds for general Markov chains are expressed in terms of ergodicity coefficients of the iterated transition kernel, which are difficult to compute for infinite state spaces. These results were obtained using operator-theoretic and probabilistic methods.

The strong stability method (also called operator method) has been developed in the early 1980s by Aïssani and Kartashov [1]. It allows both to make qualitative and quantitative analysis of some complex systems. This approach assumes that the perturbation of the transition kernel is small with respect to a certain norm. Such a strict condition allows us to obtain better estimations on the stationary characteristics of the perturbed chain. In addition, using this method, it is possible to obtain inequalities of stability with an exact computation of the constants.

This approach, based on the perturbation of operators, is applicable to all stochastic models which may be governed by a Markov chain. In particular, it has been applied to several queueing models (see [5–7, 24]) and inventory models (see [23]).

In this work, we apply this method to batch arrival queueing systems. We are exactly interested to study the strong stability of the stationary distribution of the imbedded Markov in the batch arrival queue $M^{Geo}/M/1$ after a small perturbation of the batch size distribution. We show that under some hypotheses, the characteristics of the batch arrival queueing system $M^X/M/1$ may be approximated by the correspondent characteristics of the system $M^{Geo}/M/1$. In the next, we obtain the error of approximation on the stationary distribution of the considered Markov chain.

Specify here that the perturbed parameter is the distribution of the batch size. This parameter plays an important role in the batch arrival queueing systems. As the strong stability method supposes that the perturbation is small, then the distribution of the batch size of the real system must be sufficiently close to the geometric law.

This article is organized as follows: First, the preliminaries and notations are given in Section 1. In Section 2, we described the considered queueing systems ($M^X/M/1$ and $M^{Geo}/M/1$). In Section 3, we clarified the domain within the imbedded Markov chain of the system $M^{Geo}/M/1$ is strongly stable after a small perturbation of its size distribution. The deviation of the transition kernel is determined in Section 4. Finally, the inequalities of stability are obtained in the last section and we concluded with a short conclusion. An Appendix is included with the article.

1. Notations and Preliminaries

In this section, we introduce necessary notations adapted to our work. For more details see Kartashov [18] and comments in the Appendix.

Consider the measurable space $(\mathbb{N}, \mathscr{B}(\mathbb{N}))$, where $\mathscr{B}(\mathbb{N})$ is the σ-algebra engendered by all singletons $\{j\}$, $j \in \mathbb{N}$.

Let $\mathscr{M} = \{\mu_j\}$ be the space of finite measures on $\mathscr{B}(\mathbb{N})$ and $\mathscr{N} = \{f(j)\}$ the space of bounded measurable functions on $\mathbb{N}$. We associate to each transition operator P the linear mappings

$$(\mu P)_k = \sum_{j\geq 0} \mu_j P_{jk} \tag{1}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki} \tag{2}$$

Introduce on $\mathscr{M}$ the class of norms of the form

$$\|\mu\|_v = \sum_{j\geq 0} v(j)|\mu_j| \tag{3}$$

where v is an arbitrary measurable function (not necessary finite) bounded from below by a positive constant. This norm induces in the space $\mathscr{N}$ the norm

$$\|f\|_v = \sup_{k\geq 0} \frac{|f(k)|}{v(k)} \tag{4}$$

Let B, the space of bounded linear operators on the space $\{\mu \in \mathscr{M} : \|\mu\|_v < \infty\}$, with norm

$$\|P\|_v = \sup_{k\geq 0} \frac{1}{v(k)} \sum_{j\geq 0} v(j)|P_{kj}| \tag{5}$$

2. Description of the Queueing Systems

In this section, we describe the queueing systems $M^X/M/1$ and $M^{Geo}/M/1$ and clarify their imbedded Markov chains and the correspondent kernels.

2.1. *Description of the Queue $M^X/M/1$*

The queue $M^X/M/1$ may be described in the following manner (see [9, 13]).

The customers arrive in batch of random size C, following a Poisson process with parameter λ, where $P_r\{C = n\} = c_n$, and they are individually served. The service times are independent and distributed following an exponential law with mean $\frac{1}{\mu}$.

Let $C(z)$, a generating function of the batch size defined as follows:

$$C(z) = E(z^c) = \sum_{n=1}^{\infty} c_n z^n \quad (|z| \leq 1)$$

and let A_i, the number of customers arriving at the time instant t_i; then $P_r(A_i = k) = c_k$.

We have $P_r(A_1 + A_2 + \cdots + A_k = n) = \underbrace{c_n \otimes c_n \otimes \cdots \otimes c_n}_{convolution\ k\text{-}product} = C_n^{(k)}$ where $\{C_n^{(k)}\}$ is the convolution k-product of c_n, such that: $C_n^{(1)} = P_r(A_i = n) = c_n$ and $C_n^{(0)} = \begin{cases} 1 \text{ if } & n=0 \\ 0 \text{ if } & n>0 \end{cases}$

Then $P_r\{n$ customers arrive during $(0, t)\} = p_n(t)$, where

$$p_n(t) = \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} C_n^{(k)} \quad (n \geq 0).$$

Consider the following regeneration points:

- the time instant of a customer departure;
- the end of the server idle period.

The random variable X_n, representing the number of customers in the system $M^X/M/1$ immediately after the nth regeneration point, forms a discrete-time Markov chain. Consider the process B_n "the number of customers arriving during the time of the nth service."

The random variables B_n are independent, their common distribution is $k_n = P_r\{n$ arrivals during the service period$\}$.

$$k_n = \sum_{k=0}^{n} C_n^{(k)} \frac{\mu \lambda^k}{(\lambda + \mu)^{k+1}} \tag{6}$$

Then

$$X_{n+1} = \begin{cases} X_n - 1 + B_{n+1} & \text{if } X_n \geq 1 \\ C & \text{if } X_n = 0 \end{cases}$$

This shows that X_{n+1} depends only on X_n and on B_{n+1} and not on the values taken by $X_{n-1}, X_{n-2}, \ldots$ This means that the sequence $\{X_n, n = 1, 2, \ldots\}$ forms an imbedded Markov chain of the process $\{X(t), t \geq 0\}$.

Transitional Regime. The transition probabilities of the imbedded Markov chain $\{X_n, n = 1, 2, \ldots\}$ allow us to describe the general expression of the transition kernel $P = (P_{ij})$, where

$$P_{ij} = \begin{cases} c_j & \text{if } j \geq 1, \ i = 0 \\ k_{j+1-i} & \text{if } 1 \leq i \leq j+1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where k_j is defined in (6).

Remark 2.1. According to the transition matrix, the Markov chain X_n is irreducible and aperiodic, and we can show that it converges to a limiting distribution if $\rho' < 1$, where

$$\rho' = E(B_n) = \sum_{n=1}^{\infty} nk_n = \frac{\lambda}{\mu} E(C).$$

Stationary Regime. Suppose that $\rho' < 1$ and let $\pi = (\pi_0, \pi_1, \dots)$ the stationary distribution of the Markov chain $\{X_n; n = 1, 2, \dots\}$ where

$$\pi_n = \lim_{k\to\infty} P_r\{X_k = n\}.$$

We obtain

$$\prod(z) = \frac{\pi_0[K(z) - zC(z)]}{K(z) - z}, \quad \text{where}$$

$$\pi_0 = \frac{1-\rho'}{1-\rho' + E(C)}, \quad \text{and}$$

$$K(z) = \frac{\mu}{\lambda + \mu - \lambda C(z)},$$

the generating function of the number of customers joining in a service period.

2.2. *Description of the Queue $M^{Geo}/M/1$*

The system $M^{Geo}/M/1$ may be described in the following manner. The customers arrive in batch of random size $\widetilde{C}$, following a Poisson process with parameter λ, and they are individually served. The service times are independent and distributed following an exponential law with mean $\frac{1}{\mu}$. The batch size $\widetilde{C}$ follows a geometric distribution with parameter q where,

$$P_r\{\widetilde{C} = k\} = \tilde{c}_k = (1-q)q^{k-1}, \quad 0 < q < 1 \quad (k \geq 1), \tag{8}$$

and let $\widetilde{C}(z)$ the generating function:

$$\widetilde{C}(z) = E(z^{\tilde{c}}) = \sum_{n=1}^{\infty} \tilde{c}_n z^n \quad (|z| \leq 1). \tag{9}$$

Let $\widetilde{A}_i$, the number of customers arriving at the time instant t_i. $P_r(\widetilde{A}_i = k) = \tilde{c}_k$.
We have

$$P_r(\widetilde{A}_1 + \widetilde{A}_2 + \cdots + \widetilde{A}_k = n) = \underbrace{\tilde{c}_n \otimes \tilde{c}_n \otimes \cdots \otimes \tilde{c}_n}_{\text{convolution } k\text{-product}} = \widetilde{C}_n^{(k)},$$

where $\{\widetilde{C}_n^{(k)}\}$ is the convolution k-product of $\tilde{c}_n$.

Such that:

$$\widetilde{C}_n^{(1)} = P_r(\widetilde{A}_i = n) = \tilde{c}_n \quad \text{and} \quad \widetilde{C}_n^{(0)} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

$P_r\{n$ customers arriving during $(0, t)\} = \tilde{p}_n(t)$

$$\tilde{p}_n(t) = \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \widetilde{C}_n^{(k)} \quad (n \geq 0).$$

Consider the following regeneration points:

- the time instant of a customer departure;
- the end of the server idle period.

The random variable $\widetilde{X}_n$ representing the number of customers in the system just after the nth regeneration point forms a discrete-time Markov chain.

Consider the process $\widetilde{B}_n$ "the number of customers arriving during the time of the (n)th service." The random variables $\widetilde{B}_n$ are independent, and their common distribution is:

$$\begin{aligned} \tilde{k}_n &= P_r\{n \text{ arrivals during the period of service}\} \\ &= \sum_{k=0}^{n} \widetilde{C}_n^{(k)} \frac{\mu \lambda^k}{(\lambda + \mu)^{k+1}} \end{aligned} \tag{10}$$

Then

$$\widetilde{X}_{n+1} = \begin{cases} \widetilde{X}_n - 1 + \widetilde{B}_{n+1} & \text{if } \widetilde{X}_n \geq 1 \\ \widetilde{C} & \text{if } \widetilde{X}_n = 0 \end{cases}$$

This shows that $\widetilde{X}_{n+1}$ depends only on $\widetilde{X}_n$ and on $\widetilde{B}_{n+1}$ and not on values taken by $\widetilde{X}_{n-1}, \widetilde{X}_{n-2}, \ldots$ This means that the sequence $\{\widetilde{X}_n, n = 1, 2, \ldots\}$ forms an imbedded Markov chain of the process $\{\widetilde{X}(t), t \geq 0\}$.

Transitional Regime. The transition probabilities of the imbedded Markov chain $\{\widetilde{X}_n, n = 1, 2, \ldots\}$ allow us to describe the general expression of the transition kernel $\widetilde{P} = (\widetilde{P}_{ij})$, where

$$\widetilde{P}_{ij} = \begin{cases} \tilde{c}_j & \text{if } j \geq 1, \; i = 0 \\ \tilde{k}_{j+1-i} & \text{if } 1 \leq i \leq j+1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $\widetilde{k}_i$ is defined in (10).

Remark 2.2. According to the transition matrix, the Markov chain $\widetilde{X}_n$ is irreducible and aperiodic, and we can show that it converges to a limiting distribution if $\tilde{\rho}' < 1$, where

$$\tilde{\rho}' = E(\widetilde{B}_n) = \sum_{n=1}^{\infty} n\tilde{k}_n = \frac{\lambda}{\mu} E(\widetilde{C}). \tag{12}$$

Stationary Regime. Suppose that $\tilde{\rho}' < 1$ and let $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1, \dots)$ the stationary distribution of the Markov chain $\{\widetilde{X}_n; n = 1, 2, \dots\}$ where

$$\tilde{\pi}_n = \lim_{k \to \infty} P\{\widetilde{X}_k = n\}.$$

We have

$$\widetilde{\prod}(z) = \frac{\tilde{\pi}_0[\widetilde{K}(z) - z\widetilde{C}(z)]}{\widetilde{K}(z) - z}, \quad \text{where} \tag{13}$$

$$\tilde{\pi}_0 = \frac{\mu(1-q) - \lambda}{\mu(1-q) - \lambda + \mu}, \quad \text{and} \tag{14}$$

$$\widetilde{K}(z) = \frac{\mu}{\lambda + \mu - \lambda \widetilde{C}(z)}. \tag{15}$$

3. Strong Stability in the System $M^{Geo}/M/1$

In this section, we determine the conditions under which, it is possible to approximate the characteristics of the system $M^X/M/1$ by those of the system $M^{Geo}/M/1$.

Definition 3.1 (cf. [1, 18]). We say that the Markov chain $\widetilde{X}_n$ with transition kernel $\widetilde{P}$ verifying $\|\widetilde{P}\|_v < \infty$ and invariant measure $\tilde{\pi}$ is strongly v-stable, if every stochastic transition kernel P in the neighborhood $\{P : \|P - \widetilde{P}\|_v < \varepsilon\}$ admits a unique stationary vector π such that $\|\tilde{\pi} - \pi\|_v \to 0$ when $\|\widetilde{P} - P\|_v \to 0$.

To prove the fact of strong v-stability of our system (result obtained in the following Theorem 3.1), we apply the Theorem 5.2 given in Appendix. For this, we first choose

$$v(k) = \frac{1}{\beta^k} + \frac{\lambda}{\mu} k, \quad \beta > 1,$$
$$h_i = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases} \tag{16}$$

and

$$\alpha_j = \widetilde{P_{1j}} = \widetilde{k_j} = \sum_{k=0}^{j} \widetilde{C}_j^{(k)} \frac{\mu \lambda^k}{(\lambda + \mu)^{k+1}}. \tag{17}$$

where $\widetilde{P_{1j}}$ is defined in (11), and prove the following lemma.

Lemma 3.1. *Consider the following inequality system*:

$$\begin{cases} \dfrac{\beta-1}{\beta^2} < \dfrac{\lambda}{\mu}(1-\tilde{\rho}') \quad \text{and} \\ \beta > \dfrac{1}{1-\tilde{\rho}'}. \end{cases}$$

This system admits solutions given as follows:

- $\beta \in]\beta_3, +\infty[$ *if* $\frac{\lambda}{\mu}(1-\tilde{\rho}') > \frac{1}{4}$ *and*
- $\beta \in]\beta_1, +\infty[$ *if* $\frac{\lambda}{\mu}(1-\tilde{\rho}') \leq \frac{1}{4}$, *with*

$$\tilde{\rho}' = \frac{\lambda}{\mu}E(\widetilde{C}), \quad \beta_3 = \frac{1}{1-\tilde{\rho}'} \quad \text{and} \quad \beta_1 = \frac{\mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}}{2\lambda(1-\tilde{\rho}')}.$$

Proof 3.1. Consider the function f defined by:

$$f :]1, +\infty[\to \mathbb{R}, \\ \beta \mapsto f(\beta) = \frac{\beta - 1}{\beta^2}.$$

The study of the function f gives us $f(\beta) \leq \frac{1}{4}, \forall \beta > 1$.

To solve the first inequality of the system, we have two possible cases:

- *Case* 1: if $\frac{\lambda}{\mu}(1-\tilde{\rho}') > \frac{1}{4}$, thus $f(\beta) < \frac{\lambda}{\mu}(1-\tilde{\rho}')$.
  Then, the system admits solutions for all $\beta \in]\beta_3, +\infty[$.
- *Case* 2: if $\frac{\lambda}{\mu}(1-\tilde{\rho}') \leq \frac{1}{4}$, thus $\exists \beta_0 \in]1, 2]$ and $\beta_1 \in [2, \infty[$ such that

$$\frac{1}{4} \geq f(\beta_0) = f(\beta_1) = \frac{\lambda}{\mu}(1-\tilde{\rho}') > 0.$$

So, for $\beta \in]1, \beta_0[\cup]\beta_1, \infty[$ we have, $f(\beta) < f(\beta_0) = f(\beta_1) = \frac{\lambda}{\mu}(1-\tilde{\rho}')$.
Now compute the values of β_0 and β_1. For this, we have to solve the following equation:

$$\frac{\lambda}{\mu}(1-\tilde{\rho}') = \frac{\beta-1}{\beta^2}.$$

which admits two roots:

$$\beta_0 = \frac{\mu - \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}}{2\lambda(1-\tilde{\rho}')},$$

and

$$\beta_1 = \frac{\mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}}{2\lambda(1-\tilde{\rho}')}. \tag{18}$$

To determine the domain of β for this second case, it is necessary to situate the value of β_3.

- Compare β_3 and β_1. It is easy to show that $\beta_3 - \beta_1 < 0$; indeed,

$$\beta_3 - \beta_1 = \frac{2\lambda}{2\lambda(1-\tilde{\rho}')} - \frac{\mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}}{2\lambda(1-\tilde{\rho}')}.$$

The sign of $\beta_3 - \beta_1$ is the sign of

$$[2\lambda(1-\tilde{\rho}')][2\lambda - \mu - \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}].$$

We know that $2\lambda(1-\tilde{\rho}') > 0$, then it remains to study the sign of

$$2\lambda - \mu - \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}.$$

We have

$$\begin{aligned} E(\widetilde{C}) &> 1, \\ \Rightarrow \frac{\lambda}{\mu}E(\widetilde{C}) &> \frac{\lambda}{\mu}, \\ \Rightarrow 4\lambda\mu\tilde{\rho}' &> 4\lambda^2, \\ \Rightarrow \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')} &> (2\lambda - \mu). \end{aligned}$$

Finally, $2\lambda - \mu - \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')} < 0$, from where $\beta_3 - \beta_1 < 0$.

- Compare β_3 and β_0. We have $\beta_3 - \beta_0 > 0$.
  Indeed,

$$\beta_3 - \beta_0 = \frac{2\lambda}{2\lambda(1-\tilde{\rho}')} - \frac{\mu - \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}}{2\lambda(1-\tilde{\rho}')}.$$

The sign of $\beta_3 - \beta_0$ returns to the sign of the expression

$$[2\lambda(1-\tilde{\rho}')][2\lambda - \mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}].$$

It is clear that $2\lambda(1-\tilde{\rho}') > 0$ so, study the sign of

$$2\lambda - \mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')}.$$

From

$$E(\widetilde{C}) > 1,$$

we obtain, after multiplying by λ/μ,

$$\frac{\lambda}{\mu}E(\widetilde{C}) > \frac{\lambda}{\mu}.$$

We also have

$$4\lambda\mu\tilde{\rho}' > 4\lambda^2,$$

hence

$$\sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')} > (\mu - 2\lambda).$$

From where

$$2\lambda - \mu + \sqrt{\mu^2 - 4\lambda\mu(1-\tilde{\rho}')} > 0.$$

Then $\beta_3 - \beta_0 > 0$.
Since $\beta_3 \in]\beta_0, \beta_1[$, then the system of inequalities is verified for all $\beta \in]\beta_1, +\infty[$.

Using the previous results, we clarify in the following theorem the conditions under which the Markov chain $\widetilde{X_n}$ is strongly v-stable.

Theorem 3.1. *Suppose that the ergodicity condition* $\frac{\lambda}{\mu}E(\widetilde{C}) < 1$ *holds. Then, the Markov chain* $\widetilde{X_n}$ *is strongly* v*-stable for the function*

$$v(k) = \frac{1}{\beta^k} + \frac{\lambda}{\mu}k, \quad \beta > \beta_1.$$

Proof 3.2. To prove the strong v-stability of the Markov chain $\widetilde{X_n}$, for the function

$$v(k) = \frac{1}{\beta^k} + \frac{\lambda}{\mu}k, \quad \text{with } \beta > \beta_1,$$

we apply the strong stability criteria (Theorem 5.2 in Appendix). We first check the conditions: $\tilde{\pi}h > 0$, $\alpha 1 = 1$, $\alpha h > 0$.

- $\tilde{\pi}h = \sum_{i\geq 0} \widetilde{\pi}_i h_i = \widetilde{\pi}_1$.

where h is defined in (16) and α in (17). It is known that

$$\widetilde{\pi}_j = \sum_{i\geq 0} \widetilde{P}_{ij}\widetilde{\pi}_i, \quad \text{where } \widetilde{P}_{ij} \text{ is defined in (11).}$$

We obtain $\widetilde{\pi}_j = \widetilde{\pi}_0\widetilde{c}_j + \sum_{i=1}^{j+1} \tilde{k}_{j-i+1}\widetilde{\pi}_i,$

then $\widetilde{\pi}_0 = \widetilde{\pi}_0\widetilde{c}_0 + \widetilde{k}_0\widetilde{\pi}_1,$

and $\widetilde{\pi}_1 = \dfrac{\widetilde{\pi}_0}{\widetilde{k}_0}.$

From (10),

$$\tilde{k}_0 = \frac{\mu}{\lambda + \mu} > 0, \quad (\lambda > 0, \mu > 0),$$

and from (14)

$$\tilde{\pi}_0 = \frac{\mu(1-q) - \lambda}{\mu(1-q) - \lambda + \mu} \geq 0. \tag{19}$$

Thus, this shows that $\tilde{\pi}_0 \neq 0$.

We have these equivalences,

$$\tilde{\pi}_0 \neq 0 \Leftrightarrow [\mu(1-q) - \lambda] \neq 0, \\ \Leftrightarrow \frac{\lambda}{\mu(1-q)} \neq 1, \\ \Leftrightarrow \tilde{\rho}' \neq 1.$$

As $\tilde{\rho}' < 1$ thus $\tilde{\rho}' \neq 1 \Rightarrow \tilde{\pi}_0 \neq 0$.

Then $\tilde{\pi}_0 > 0$. From where

$$\widetilde{\pi_1} = \frac{\widetilde{\pi_0}}{\widetilde{k_0}} > 0.$$

From (17) and (16),

- $\alpha 1 = \sum_{j\geq 0} \alpha_j = \sum_{j\geq 0} \widetilde{P}_{1j} = 1$,
- $\alpha h = \sum_{i\geq 0} \alpha_i h_i = \alpha_1 = \widetilde{P}_{11} = \tilde{k}_1$.

From (10), we have

$$\tilde{k}_1 = \sum_{k=0}^{1} \widetilde{C}_1^{(k)} \frac{\mu\lambda^k}{(\lambda+\mu)^{k+1}}, \\ = (1-q)\frac{\mu\lambda}{(\lambda+\mu)^2} > 0.$$

Now, verify the conditions *a*, *b* and *c* of Theorem 5.2.

1. We first verify the condition *b*.

   - If $i = 1$ then,

   $$T_{1j} = \widetilde{P}_{1j} - \widetilde{P}_{1j} = 0.$$

   - If $i = 0$ then,

   $$T_{0j} = \widetilde{P}_{0j} = \tilde{c}_j = (1-q)q^{j-1} \geq 0, \quad \text{with } 0 < q < 1, \ (j \geq 1).$$

   - If $i \geq 2$ then,

   $$T_{ij} = \widetilde{P}_{ij} = \tilde{k}_{j+1-i} \geq 0.$$

   From where, T_{ij} is non negative.
2. Now, we aim to show that there exists some constant $\rho < 1$ such that $Tv(k) \leq \rho v(k)$, for all $k \in \mathbb{N}$ (condition a of Theorem 5.2).
   According to Equation (2), $Tv(k) = \sum_{j\geq 0} v(j) T_{kj}$. We observe three cases.

   - *Case* 1: $k = 1$

   $$Tv(1) = \sum_{j\geq 0} v(j) T_{1j} = 0.$$

- *Case* 2: $k = 0$

$$\begin{aligned}
Tv(0) &= \sum_{j\geq 0} v(j)T_{0j}, \\
&= \sum_{j\geq 1}\left(\frac{1}{\beta^j}+\frac{\lambda}{\mu}j\right)\tilde{c}_j, \\
&\leq \sum_{j\geq 1}\left(\frac{1}{\beta}+\frac{\lambda}{\mu}j\right)\tilde{c}_j, \\
&= \frac{1}{\beta}\sum_{j\geq 1}\tilde{c}_j+\frac{\lambda}{\mu}\sum_{j\geq 1} j\tilde{c}_j, = \frac{1}{\beta}+\tilde{\rho}'.
\end{aligned}$$

Let us choose

$$\rho = \frac{1}{\beta}+\tilde{\rho}'.$$

Then $\rho < 1$, under the condition

$$\beta > \frac{1}{1-\tilde{\rho}'}. \tag{20}$$

- *Case* 3: $k \geq 2$

$$\begin{aligned}
Tv(k) &= \sum_{j\geq 0} v(j)T_{kj}, \\
&= \sum_{j+1-k\geq 0}\left(\frac{1}{\beta^j}+\frac{\lambda}{\mu}j\right)\sum_{l=0}^{j+1-k}\widetilde{C}_{j+1-k}^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}}, \\
&\leq \sum_{n\geq 0}\left[\frac{1}{\beta^{k-1}}+\frac{\lambda}{\mu}(n+k-1)\right]\sum_{l=0}^{n}\widetilde{C}_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}}, \\
&\leq \beta^{1-k}+\frac{\lambda}{\mu}(k-1)+\frac{\lambda}{\mu}\tilde{\rho}', \\
&= \left[\beta^{1-k}+\frac{\lambda}{\mu}(k-1+\widetilde{\rho}')\right]\times\frac{\beta^{-k}+\frac{\lambda}{\mu}k}{\beta^{-k}+\frac{\lambda}{\mu}k}.
\end{aligned}$$

Choose

$$\rho = \frac{\beta^{1-k}+\frac{\lambda}{\mu}k-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}\tilde{\rho}'}{\beta^{-k}+\frac{\lambda}{\mu}k}$$

and show that $\rho < 1$. We have the following equivalences

$$\begin{aligned}
\rho < 1 &\Leftrightarrow \beta\beta^{-k}+\frac{\lambda}{\mu}k-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}\tilde{\rho}' < \beta^{-k}+\frac{\lambda}{\mu}k, \\
&\Leftrightarrow \beta^{-k}(\beta-1) < \frac{\lambda}{\mu}(1-\tilde{\rho}').
\end{aligned}$$

Then, determine the domain of β holding:

$$\frac{(\beta-1)}{\beta^k}<\frac{\lambda}{\mu}(1-\tilde{\rho}'),\quad k\geq 2.$$

For $k \geq 2$, the following inequality holds

$$\frac{(\beta-1)}{\beta^k}\leq\frac{(\beta-1)}{\beta^2}.$$

So, it suffices to show that

$$\frac{(\beta-1)}{\beta^2}<\frac{\lambda}{\mu}(1-\tilde{\rho}'). \tag{21}$$

Therefore, β must verify the following system $\begin{cases} \frac{\beta-1}{\beta^2}<\frac{\lambda}{\mu}(1-\tilde{\rho}') \text{ and} \\ \beta>\frac{1}{1-\tilde{\rho}'}. \end{cases}$
However, from the Lemma 3.1, this system admits at least one solution for all $\beta > \beta_1$, where β_1 is defined in (18). Hence, $\exists 0 < \rho < 1$, defined as follows

$$\rho=\min\left\{\widetilde{\rho}'+\frac{1}{\beta},\frac{\beta^{-k}\beta+\frac{\lambda}{\mu}k-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}\tilde{\rho}'}{\beta^{-k}+\frac{\lambda}{\mu}k}\right\}\quad\text{for all } \beta>\beta_1, \tag{22}$$

such that the condition c is verified.

3. It remains to show that $\|\widetilde{P}\|_v < \infty$ (condition a).

$$T=\widetilde{P}-h\circ\alpha\Rightarrow\widetilde{P}=T+h\circ\alpha,$$

and

$$\|\widetilde{P}\|_v=\|T+h\circ\alpha\|_v\leq\|T\|_v+\|h\|_v\times\|\alpha\|_v.$$

From (5)

$$\begin{aligned}\|T\|_v&=\sup_{k\geq 0}\frac{1}{v(k)}\sum_{j\geq 0}v(j)|T_{kj}|,\\&\leq\sup_{k\geq 0}\frac{1}{v(k)}\rho v(k),=\rho<1.\end{aligned}$$

Also, from (3)

$$\begin{aligned}\|\alpha\|_v&=\sum_{j\geq 0}v(j)|\alpha_j|,\\&=\sum_{j\geq 0}\left(\frac{1}{\beta^j}\right)\widetilde{k}_j+\sum_{j\geq 0}\frac{\lambda}{\mu}j\widetilde{k}_j,\\&=\widetilde{K}\left(\frac{1}{\beta}\right)+\frac{\lambda}{\mu}\tilde{\rho}'<\infty.\end{aligned}$$

Finally from (4)

$$\|h\|_v = \sup_{k\geq 0} \frac{1}{v(k)} = 1.$$

From where

$$\|\widetilde{P}\|_v < \infty.$$

4. Deviation of the Transition Kernel

To estimate numerically the difference between the stationary distributions of the Markov chain states $\widetilde{X}_n$ and X_n, let us beforehand estimate the deviation norm of the transition kernels P and $\widetilde{P}$.

Theorem 4.1. *Let $\widetilde{P}$ and P the transition kernels of the imbedded Markov chains of systems $M^{Geo}/M/1$ and $M^X/M/1$. Then, for all β such that $\beta > \beta_1$ where β_1 is defined in* (18), *we have*

$$\|P - \widetilde{P}\|_v \leq \frac{2\mu}{\lambda} + \rho' + \tilde{\rho}'.$$

Proof 4.1. From the expression (5) we have

$$\|P - \widetilde{P}\|_v = \sup_{k\geq 0} \frac{1}{v(k)} \sum_{j\geq 0} v(j)|P_{kj} - \widetilde{P}_{kj}|,$$

$$\|P - \widetilde{P}\|_v = \sup\left\{ \sum_{j\geq 0} v(j)|P_{0j} - \widetilde{P}_{0j}|, \sup_{k\geq 1} \frac{1}{v(k)} \sum_{j\geq 0} v(j)|P_{kj} - \widetilde{P}_{kj}| \right\}.$$

Estimate first, the first expression under the supremum.

$$\begin{aligned}\sum_{j\geq 0} v(j)|P_{0j} - \widetilde{P}_{0j}| &= \sum_{j\geq 1} \left(\frac{1}{\beta^j} + \frac{\lambda}{\mu} j\right)|c_j - \tilde{c}_j|,\\ &\leq C\left(\frac{1}{\beta}\right) + \widetilde{C}\left(\frac{1}{\beta}\right) + \frac{\lambda}{\mu}E(C) + \frac{\lambda}{\mu}E(\widetilde{C}),\\ &\leq C\left(\frac{1}{\beta}\right) + \widetilde{C}\left(\frac{1}{\beta}\right) + \rho' + \tilde{\rho}'.\end{aligned}$$

Estimate now, the second expression under the supremum.

$$\begin{aligned}&\sup_{k\geq 1} \frac{1}{v(k)} \sum_{j\geq 0} v(j)|P_{kj} - \widetilde{P}_{kj}|\\ &\quad = \sup_{k\geq 1} \frac{1}{\beta^{-k} + \frac{\lambda}{\mu}k} \sum_{j+1-k\geq 0} \left(\beta^{-j} + \frac{\lambda}{\mu} j\right)\\ &\qquad \times \left| \sum_{l=0}^{j+1-k} C_{j+1-k}^{(l)} \frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} - \sum_{l=0}^{j+1-k} \widetilde{C}_{j+1-k}^{(l)} \frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} \right|,\end{aligned}$$

$$\leq \sup_{k\geq 1} \frac{1}{\beta^{-k} + \frac{\lambda}{\mu}k}\Bigg\{\beta^{1-k}\Bigg(\sum_{n\geq 0}\sum_{l=0}^{n} C_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} + \sum_{n\geq 0}\sum_{l=0}^{n} \widetilde{C}_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}}\Bigg)$$

$$+ \frac{\lambda}{\mu}(k-1)\Bigg(\sum_{n\geq 0}\sum_{l=0}^{n} C_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} + \sum_{n\geq 0}\sum_{l=0}^{n} \widetilde{C}_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}}\Bigg)$$

$$+ \frac{\lambda}{\mu}\sum_{n\geq 0} n\sum_{l=0}^{n} C_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} + \frac{\lambda}{\mu}\sum_{n\geq 0} n\sum_{l=0}^{n} \widetilde{C}_n^{(l)}\frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}}\Bigg\}.$$

From where, we have:

$$\sup_{k\geq 1}\frac{1}{v(k)}\sum_{j\geq 0} v(j)|P_{kj} - \widetilde{P}_{kj}|$$

$$\leq \sup_{k\geq 1} \frac{1}{\beta^{-k} + \frac{\lambda}{\mu}k}\left\{2\beta^{1-k} + \frac{2\lambda}{\mu}(k-1) + \frac{\lambda}{\mu}(\rho' + \tilde{\rho}')\right\}.$$

$$\sup_{k\geq 1}\frac{1}{v(k)}\sum_{j\geq 0} v(j)|P_{kj} - \widetilde{P}_{kj}| \leq \sup_{k\geq 1}\left\{\frac{2\beta\mu}{\beta^k k\lambda} + \frac{2(k-1)}{k} + \frac{1}{k}(\rho' + \tilde{\rho}')\right\}.$$

The sup is reached by 1, then,

$$\sup_{k\geq 1}\frac{1}{v(k)}\sum_{j\geq 0} v(j)\big|P_{kj} - \widetilde{P}_{kj}\big| \leq \frac{2\mu}{\lambda} + \rho' + \tilde{\rho}'.$$

From where,

$$\|P - \widetilde{P}\|_v \leq \sup\left\{C\left(\frac{1}{\beta}\right) + \widetilde{C}\left(\frac{1}{\beta}\right) + \rho' + \tilde{\rho}',\, \frac{2\mu}{\lambda} + \rho' + \tilde{\rho}'\right\}.$$

Finally

$$\|P - \widetilde{P}\|_v \leq \frac{2\mu}{\lambda} + \rho' + \tilde{\rho}' \quad \text{since } \frac{\mu}{\lambda} > 1.$$

5. Inequalities of Stability

The inequalities of stability provide an estimation of the difference between the stationary distributions of the Markov chains X_n and $\widetilde{X}_n$.

Consider the two imbedded Markov chains X_n and $\widetilde{X}_n$ in the queueing systems $M^X/M/1$ and $M^{Geo}/M/1$ respectively, with transition kernels P and $\widetilde{P}$. Let π and $\tilde{\pi}$ their stationary probabilities. We have shown (in Theorem 3.1) that $\widetilde{X}_n$ is strongly v-stable. Applying the Theorem 5.3 (see the Appendix), we obtain the following result.

Theorem 5.1. *Suppose that the imbedded Markov chain $\widetilde{X}_n$ of the system $M^{Geo}/M/1$ is strongly v-stable. Then under the condition $\|\Delta\|_v < \frac{(1-\rho)}{C}$, and for all $\beta > \beta_1$, the following inequality is verified*

$$\|\pi - \tilde{\pi}\|_v \leq (1-\rho)\|\tilde{\pi}\|_v(1 - \rho - C\|\Delta\|_v)^{-1}.$$

Where ρ *is defined in* (22),

$$C = 1 + \|\mathbb{1}\|_v \|\tilde{\pi}\|_v,$$
$$\Delta = P - \widetilde{P}$$

and $\|\tilde{\pi}\|_v = \widetilde{\Pi}\left(\frac{1}{\beta}\right) + \frac{\lambda}{\mu} E(\widetilde{X}_n)$.

Proof 5.1. From the expression (4), we have

$$\|\mathbb{1}\|_v = \sup_{k\geq 0} \frac{1}{v(k)} = 1.$$

And from (3)

$$\|\tilde{\pi}\|_v = \sum_{j\geq 0} v(j)\tilde{\pi}_j, = \sum_{j\geq 0} \left(\frac{1}{\beta}\right)^j \tilde{\pi}_j + \frac{\lambda}{\mu} \sum_{j\geq 0} j\tilde{\pi}_j.$$

From where

$$\|\tilde{\pi}\|_v = \widetilde{\Pi}\left(\frac{1}{\beta}\right) + \frac{\lambda}{\mu} E(\widetilde{X}_n).$$

with,

$\widetilde{\Pi}\left(\frac{1}{\beta}\right)$, the generating function of $\widetilde{X}_n$ at point $\left(\frac{1}{\beta}\right)$.
$E(\widetilde{X}_n)$, the expectation of $\widetilde{X}_n = \widetilde{\Pi}'(1)$.

$$E(\widetilde{X}_n) = \frac{2E(\widetilde{C}) + \widetilde{C}''(1) + E(\widetilde{C})\widetilde{K}''(1)}{2(1-\rho)},$$
$$\widetilde{C}''(1) = \frac{2q}{(1-q)^2},$$
$$\widetilde{K}''(1) = 2\left(\frac{\lambda}{\mu} V(\widetilde{C}) + \tilde{\rho}^2\right),$$
$$V(\widetilde{C}) = \frac{q}{(1-q)^2}.$$

Impose the condition $\|P - \widetilde{P}\|_v < \frac{1-\rho}{c}$, i.e., $\|\Delta\|_v < \frac{1-\rho}{c}$ and replace the constants by their values, we will then have:

$$\|\pi - \tilde{\pi}\|_v \leq (1-\rho)\|\tilde{\pi}\|_v (1 - \rho - C\|\Delta\|_v)^{-1}.$$

6. Conclusion

We have clarified the conditions under which it will be possible to approximate the characteristics of the queueing system $M^X/M/1$ by those of the system $M^{Geo}/M/1$. We also have obtained the inequalities of stability with an exact computation of constants. These results can be a concrete practical application. To do this, it is necessary to quantify the approximation error by applying the algorithmic approach proposed in [6].

Appendix: Strong Stability Criterion

The following theorem gives sufficient conditions for the strong stability of a Harris Markov chain.

Theorem 5.2 ([1, 18]). *The Harris Markov chain X_n, with transition operator P and invariant measure π is v-strongly stable, if and only if there exists a measure α on $\mathcal{B}(\mathbb{N})$ and a non-negative measurable function h on $\mathbb{N}$ such that*: $\pi h > 0$, $\alpha\mathbb{1} = 1$, $\alpha h > 0$ *and*

a. $\|P\|_v < \infty$
b. *The operator* $T = P - h o \alpha$ *is non-negative*
c. $\exists \rho < 1$ *such that* $Tv(k) \leq \rho v(k)$, $\forall k \in \mathbb{N}$

where o denotes the convolution between a measure and a function and $\mathbb{1}$ *is the identity function.*

When approximating a system by another, it is important to give an idea about the approximation error. Usually, stability methods provide quantitative estimates. One of the characteristic of the strong stability method is the obtaining of the inequalities with an exact computation of constants. The following Theorem 5.3 allows us to obtain the deviation of the stationary distribution of the Markov chain X_n.

Theorem 5.3 ([18]). *Let a Markov chain X_n, with transition operator P and invariant measure π, verifying conditions of Theorem 5.2. Then, for a transition operator Q, with invariant measure μ, in the neighborhood of P and for* $\|\Delta\|_v < \frac{(1-\rho)}{c}$, *we have the estimation*

$$\|\mu - \pi\|_v \leq \|\Delta\|_v c \|\pi\|_v (1 - \rho - c\|\Delta\|_v)^{-1},$$

where

$$\Delta = Q - P,$$
$$c = 1 + \|\mathbb{1}\|_v \|\pi\|_v,$$

and

$$\|\pi\|_v \leq (\alpha v)(1 - \rho)^{-1}(\pi h).$$

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