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Approximation of Performance Measures in an M/G/1 Queue with Breakdowns

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Abstract: We present a new perturbation bound in an $M/G/1$ queueing system with breakdowns and repairs. Our analysis is based on bounding the distance of stationary distributions in a suitable functional space. This leads to understand how the breakdowns of the server will affect the system's level of performance. In order to give an idea about the application of our approach in practice, we give a numerical example which would show the difference between explicit analytical estimates of errors and real simulated errors. It also allows for evaluating the potential of our approach. Eventually, we will point out directions of further research.

Keywords: Breakdowns, $M/G/1$ queue, perturbation, strong stability, simulation, stability inequalities.

1. Introduction

Let P denote the transition kernel of a Markov chain X having unique stationary distribution π. Think, for example, of P as the kernel of the imbedded jump chain of the $M/G/1$ queue. What would be the effect on the stationary performance of the queue if we introduced the breakdowns of the server in this queue? Let $\tilde{P}$ denote the transition kernel of the Markov chain $\tilde{X}$ modeling the alternative system, in our example the $M/G/1$ queue with breakdowns, and assume that $\tilde{X}$ has unique stationary distribution $\tilde{\pi}$. The question about the effect of introducing the breakdowns of server on the stationary behavior is expressed by $\pi - \tilde{\pi}$, the difference between the stationary distributions. Obviously, a bound on the effect of the perturbation is of great interest. The study of this type of bounds on perturbations has a long history. See, for example, [14] for an early reference. More specifically, let $\|.\|_\upsilon$ denote the weighted supremum norm, also called υ-norm, where υ is some vector with elements $\upsilon(.) \geq 1$, then the above problem can be phrased as follows : Can $\|\pi - \tilde{\pi}\|_\upsilon$ be approximated or bounded in terms of $\|P - \tilde{P}\|_\upsilon$? This is known as "perturbation analysis" of Markov chains in the literature. However, convergence w.r.t. to the υ-norm allows for only bounding the effect of introducing the breakdowns of server for bounded performance measures only.

In this paper we will establish an upper bound on $\pi - \tilde{\pi}$ in the υ-norm (to be defined presently). This norm is based on a weight function υ and for our analysis we assume that υ is of the form $\upsilon(k) = \beta^k$ for $k \in \mathbb{Z}_+$ and some $\beta > 1$. The two main steps of our analysis are that we first establish a bound of the υ-norm of $\pi - \tilde{\pi}$ in terms of the υ-norm of $P - \tilde{P}$ (the actual perturbation) and in terms of others constants. Secondly, we determine the parameter β such that the υ-norm of our bound is minimized.

We set out to explain the applicability of the strong stability method [9] with the

$M/G/1$ queue with breakdowns, for sufficiently small perturbation of breakdown's rate. For the considered systems π and $\tilde{\pi}$ are known and everything can be computed. This allows for evaluating the potential of this approach. The stationary distribution is generally obtained by using a numerical method to invert the Laplace-Stieltjes transform of the distribution [1]. In fact, no a precision bounds on the quality of the approximation is available. In this paper we follow a different train of thought, and will present an directly computable bound on the effect on the stationary behavior for switching from P to $\tilde{P}$ and thereby establish means to predict $\tilde{\pi}$ by π.

The paper is organized as follows. Section 2 presents the basic results of the strong stability method and the main tools used for our analysis. Section 3 is devoted to establishing the bound on the perturbation. Numerical example is presented in Section 4. A detailed literature review is provided in Section 5. Eventually, we will point out directions of further research.

2. Strong Stability Method

In this section we introduce necessary notations. For the basic theorems of the strong stability method are given in [9]. The main tool for our analysis is the weighted supremum norm, also called υ-norm, denoted by $\|.\|_\upsilon$, where υ is some vector with elements $\upsilon(k) \geq 1$ for all $k \in \mathbb{Z}_+$, and for any vector f with infinite dimension

$$\|f\|_\upsilon = \sup_{k\geq 0} \frac{|f(k)|}{\upsilon(k)}. \tag{1}$$

Let μ be a probability measure on $\mathbb{Z}_+$, then the υ-norm of μ is defined as

$$\|\mu\|_\upsilon = \sum_{j\geq 0} \upsilon(j) \, |\, \mu_j \,|. \tag{2}$$

The υ-norm is extended to stochastic kernels on $\mathbb{Z}_+$ in the following way: let P the matrix with infinite dimension then

$$\|P\|_\upsilon = \sup_{k\geq 0} \frac{1}{\upsilon(k)} \sum_{j\geq 0} \upsilon(j) \, |\, P_{kj} \,|. \tag{3}$$

Note that υ-norm convergence to 0 implies elementwise convergence to 0.

We associate to each transition kernel P the linear mappings:

$$(\mu P)_k = \sum_{i\geq 0} \mu_i P_{ik}. \tag{4}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki}. \tag{5}$$

The strong stability method [2, 9] considers the problem of the perturbation of general state space Markov chains using operators' theory and with respect to a general class of norms. The basic idea behind the concept of stability is that, for a strongly stable Markov chain, a small perturbation in the transition kernel can lead to only a small deviation of the stationary distribution.

Definition 1: A Markov chain X with transition kernel P and stationary distribution π is said to be strongly stable with respect to the norm $\|.\|_\upsilon$ if $\|P\|_\upsilon < \infty$ and every stochastic kernel Q in some neighborhood $\{Q : \|Q - P\|_\upsilon \leq \varepsilon\}$ admits a unique stationary distribution

ν and

$$\|\nu-\pi\|_{\upsilon}\to 0 \text{ as } \|Q-P\|_{\upsilon}\to 0. \quad (6)$$

In fact, as shown in [2], X is strongly stable if and only if, there exists a positive constant $c=c(P)$ such that

$$\|\nu-\pi\|_{\upsilon}\le c\|Q-P\|_{\upsilon}. \quad (7)$$

In the sequel we use the following results.

Theorem 2： [2] The Markov chain X with the transition kernel P and stationary distribution π is strongly stable with respect to the norm $\|.\|_{\upsilon}$ if and only if there exists a probability measure $\alpha=(\alpha_j)$ and a vector $h=(h_i)$ on $\mathbb{Z}_+$ such that $\pi h>0$, $\alpha \boldsymbol{1}=1$, αh is a positive scalar, and

(a) The matrix $T=P-h\alpha$ is nonnegative, where $h\alpha=(a_{ij})_{ij}$ such that $a_{ij}=h_i\alpha_j$ for $i,j\in\mathbb{Z}_+$.

(b) There exists $\rho<1$ such that $T\upsilon(k)\le\rho\upsilon(k)$ for $k\in\mathbb{Z}_+$.

(c) $\|P\|_{\upsilon}<\infty$.

Here $\boldsymbol{1}$ is the vector having all the components equal to 1.

Theorem 3： [9] Let X be a strongly υ-stable Markov chain that satisfies the conditions of Theorem 2. If ν is the probability invariant measure of a stochastic kernel Q, then for $\|\Delta\|_{\upsilon}<(1-\rho)/c$, we have the estimate

$$\|\nu-\pi\|_{\upsilon}\le\|\Delta\|_{\upsilon}\,c\|\pi\|_{\upsilon}(1-\rho-c\|\Delta\|_{\upsilon})^{-1}, \quad (8)$$

where $\Delta=Q-P$, $c=1+\|\boldsymbol{1}\|_{\upsilon}\|\pi\|_{\upsilon}$ and $\|\pi\|_{\upsilon}\le(\alpha\upsilon)(1-\rho)^{-1}(\pi h)$.

3. Analysis of the Model

3.1. Model Description

Consider an $M/G/1$ $(FIFO,\infty)$ queue with server breakdowns. Denote this queue by $\tilde{\Sigma}$. Customers arrive according to a Poisson process with rate λ and demand independent and identically distributed service times with common distribution function B with mean $1/\mu$. We assume that when a server fails, the time required to repair it has exponential distribution with rate $r>0$.

In this model, we consider the breakdowns with losses. If an arriving customer finds the server idle-up (*i.e.* it is ready to serve), it immediately occupies the server and leaves it after completion service if no breakdown had occurred during this period. Therefore, with probability $\overline{q}=1-q$, the server breaks down while serving a customer, in which case the customer is discarded, while with probability q, the service is successfully completed. We assume that when the server fails, it is under repair and it cannot be occupied. As soon as the repair of the failed server is completed, the server enters an operating state and continues to serve the other customers. In words, the breakdown occurs at the beginning of the service. Otherwise, there will be arrivals before the server is down.

Let $\tilde{X}=\{\tilde{X}_n, n\in\mathbb{Z}_+\}$ be the Markov chain describing the state of $\tilde{\Sigma}$: the number of customers in the queue at the n th time point which is either "end" of a service or "end" of a repair. The transition kernel $\tilde{P}=(\tilde{P}_{ij})_{i,j\ge 0}$ of $\tilde{\Sigma}$ is then given by

$$\tilde{P}_{ij} = \begin{cases} q\int_0^\infty e^{-\lambda x}\dfrac{(\lambda x)^j}{j!}dB(x) + \dfrac{r\overline{q}}{\lambda + r}\left(\dfrac{\lambda}{\lambda + r}\right)^j, & \text{if } i = 0; \\ q\int_0^\infty e^{-\lambda x}\dfrac{(\lambda x)^{j-i+1}}{(j-i+1)!}dB(x) + \dfrac{r\overline{q}}{\lambda + r}\left(\dfrac{\lambda}{\lambda + r}\right)^{j-i+1}, & \text{if } 1 \le i \le j+1; \\ 0, & \text{otherwise.} \end{cases}$$

Consider also an $M/G/1$ $(FIFO, \infty)$ queue without breakdowns. Denote this queue by Σ. It behaves exactly as a classical $M/G/1$ system: Arrivals occur as a Poisson process of rate λ, and it has the same general service time distribution function B. Let P be the transition kernel for the corresponding Markov chain $X = \{X_n, n \in \mathbb{Z}_+\}$, in Σ. We have:

$$P_{ij} = \begin{cases} \int_0^\infty e^{-\lambda x}\dfrac{(\lambda x)^j}{j!}dB(x), & \text{if } i = 0; \\ \int_0^\infty e^{-\lambda x}\dfrac{(\lambda x)^{j-i+1}}{(j-i+1)!}dB(x), & \text{if } 1 \le i \le j+1; \\ 0, & \text{otherwise.} \end{cases}$$

3.2. υ-Strong Stability Conditions

In the following lemma we will identify the range for β that leads to determine the strong υ-stable conditions of the Markov chain X. Indeed, the main work in strong stability method is finding β such that $\|T\|_\upsilon < 1$, where T is a stochastic kernel. For that, we choose the function $\upsilon(k) = \beta^k$, $\beta > 1$, $h_i = \boldsymbol{I}_{i=0}$ and $\alpha_j = P_{0j}$ (see Theorem 2).

Lemma 4: Suppose that the geometric ergodicity and the Cramér conditions in the system Σ hold:

(a) $\lambda \boldsymbol{E}(\xi) < 1$, where ξ is the service time,

(b) $\exists a > 0, \boldsymbol{E}(e^{a\xi}) = \int_0^\infty e^{au}dB(u) < \infty$ and

$$\beta_0 = \sup\{\beta : B^*(\lambda(1-\beta)) < \beta\}, \tag{9}$$

where $B^*(\lambda(1-\beta)) = \int_0^\infty \exp[(\lambda(\beta-1))x]dB(x)$. Then, for all β such that $1 < \beta < \beta_0$, the Markov chain X is strongly stable for the function $\upsilon(k) = \beta^k$.

Proof: We have $\pi h = \pi_0 > 0$, $\alpha \boldsymbol{1} = 1$ and $\alpha h = \alpha_0 = P_{00} > 0$.

$$T_{ij} = P_{ij} - h_i\alpha_j = \begin{cases} 0, & \text{if } i = 0, \\ P_{ij}, & \text{if } i \ge 1. \end{cases} \tag{10}$$

Hence, the kernel T is nonnegative.

According to Equation (5), we have:

$$T\upsilon(i) = \sum_{j\ge 0}\beta^j T_{ij}. \tag{11}$$

(a) If $i = 0$, then

$$T\upsilon(0) = \sum_{j\geq 0} \beta^j T_{0j} = 0. \tag{12}$$

(b) If $i \geq 1$, then

$$T\upsilon(i) = \sum_{j\geq 0} \beta^j P_{ij} = \sum_{j\geq 0} \beta^{j+i-1} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) \tag{13}$$

$$\leq \beta^{i-1} \int_0^\infty e^{-\lambda x} \sum_{j\geq 0} \frac{(\lambda \beta x)^j}{j!} dB(x) \tag{14}$$

$$\leq \beta^i \frac{B^*(\lambda(1-\beta))}{\beta}. \tag{15}$$

We pose

$$\rho(\beta) = \frac{B^*(\lambda(1-\beta))}{\beta}. \tag{16}$$

From the convexity and the monotony of $\rho(\beta)$, we have $\rho(\beta) < 1$, and obtain

$$T\upsilon(i) \leq \rho\upsilon(i), \forall i \in Z^+. \tag{17}$$

We verify that $\|P\|_\upsilon < \infty$. We have:

$$T = P - h\alpha \Rightarrow P = T + h\alpha \Rightarrow \|P\|_\upsilon \leq \|T\|_\upsilon + \|h\|_\upsilon \|\alpha\|_\upsilon, \tag{18}$$

or, according to equation (3),

$$\|T\|_\upsilon = \sup_{i\geq 0} \frac{1}{\upsilon(i)} \sum_{j\geq 0} \upsilon(j) | T_{ij} | \leq \sup_{i\geq 0} \frac{1}{\upsilon(i)} \rho\upsilon(i) \leq \rho < 1. \tag{19}$$

According to Equations (1) and (2), we have:

$$\|h\|_\upsilon = \sup_{i\geq 0} \frac{1}{\upsilon(i)} | h_i | = 1, \tag{20}$$

and

$$\|\alpha\|_\upsilon = \sum_{j\geq 0} \upsilon(j) | \alpha_j | = \sum_{j\geq 0} \beta^j P_{0j} \tag{21}$$

$$= \sum_{j\geq 0} \beta^j \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x)$$

$$= B^*(\lambda(1-\beta)) < \beta < \beta_0 < \infty. \tag{22}$$

Then, $\|P\|_\upsilon < \infty$.

The Markov chain X being strongly stable then, the $\|\pi - \tilde{\pi}\|_\upsilon$ can be bounded in terms of $\|P - \tilde{P}\|_\upsilon$.

3.3. Bound on Perturbation

To be able to estimate numerically the margin between the stationary distributions of the Markov chains $\tilde{X}$ and X, we estimate the norm of the deviation of the transition kernel P.

Suppose that the mean value of the distribution of service times verifies the following inequality

$$\int_0^\infty x^2 \,|\, B - E \,|\, (dx) < \infty \text{ and } \int_0^\infty x \,|\, B - E \,|\, (dx) < W / \lambda, \tag{23}$$

where E is the repair time distribution and

$$W = W(B, E) = \int_0^\infty |\, B - E \,|\, (dx). \tag{24}$$

Then, there exists $\beta > 1$ such that (see [4])

$$\int_0^\infty e^{\lambda(\beta-1)x} \,|\, B - E \,|\, (dx) < \beta W. \tag{25}$$

Lemma 5：Let $\tilde{P}$ (respectively P) be the transition kernel of the Markov chain $\tilde{X}$ (respectively of the Markov chain X). Then, for all β such that $1 < \beta < \beta_0$, we have:

$$\left\| P - \tilde{P} \right\|_\upsilon \le \overline{q} \beta_0 W, \tag{26}$$

where β_0 and W were already defined in (9) and (24) respectively.

Proof：From Equation (3), we have:

$$\left\| P - \tilde{P} \right\|_\upsilon = \sup_{k \ge 0} \frac{1}{\beta^k} \sum_{j \ge 0} \beta^j \,|\, P_{kj} - \tilde{P}_{kj} \,|, \tag{27}$$

(a) For $k = 0$:

$$\left\| P - \tilde{P} \right\|_\upsilon = \sum_{j \ge 0} \beta^j \,|\, P_{0j} - \tilde{P}_{0j} \,| \tag{28}$$

$$\le \overline{q} \int e^{-\lambda x} \sum_{j \ge 0} \frac{(\lambda x \beta)^j}{j!} \,|\, B - E \,|\, (dx) \tag{29}$$

$$\le \overline{q} \int e^{\lambda(\beta-1)x} \,|\, B - E \,|\, (dx). \tag{30}$$

(b) For $k \ge 1$:

$$\left\| P - \tilde{P} \right\|_\upsilon = \overline{q} \sup_{k \ge 1} \frac{1}{\beta^k} \sum_{j \ge 0} \beta^j \,|\, P_{kj} - \tilde{P}_{kj} \,| \tag{31}$$

$$\le \overline{q} \frac{1}{\beta} \int e^{-\lambda x} \sum_{j \ge 0} \frac{(\lambda \beta x)^j}{j!} \,|\, B - E \,|\, (dx) \tag{32}$$

$$= \frac{\overline{q}}{\beta} \int e^{\lambda(\beta-1)x} \,|\, B - E \,|\, (dx). \tag{33}$$

Then, we have:

$$\left\|P-\tilde{P}\right\|_{\upsilon} \leq \frac{\overline{q}}{\beta} \int e^{\lambda(\beta-1)x} \mid B-E \mid (dx). \tag{34}$$

From Equation (25), we have :

$$\left\|P-\tilde{P}\right\|_{\upsilon} < \overline{q}\beta_0 W. \tag{35}$$

We summarize our analysis in the following result.

Theorem 6 : Let $\tilde{\pi}$ (respectively π) be the stationary distribution of the Markov chain in $\tilde{\Sigma}$ (respectively of the Markov chain in Σ), then for all $1<\beta<\beta_0$, we have :

$$\left\|\pi-\tilde{\pi}\right\|_{\upsilon} \leq \overline{q}\beta_0 W c_0 c(1-\rho-c\overline{q}\beta_0 W)^{-1}, \tag{36}$$

where c_0 and c are given respectively by

$$c_0 = \frac{(1-\lambda m)(\beta-1)}{1-\rho}\rho \tag{37}$$

and $m=\boldsymbol{E}(\xi)=1/\mu$ (ξ is the service time),

$$c = 1+\left\|\pi\right\|_{\upsilon}. \tag{38}$$

Proof : By definition,

$$\left\|\pi\right\|_{\upsilon} = \sum_{j\geq 0}\upsilon(j)\pi_j = \sum_{j\geq 0}\beta^j\pi_j = \Pi(\beta). \tag{39}$$

This is the Pollaczek-Khinchin generating function for B. Hence,

$$\begin{aligned}\Pi(\beta) &= \frac{(\beta-1)(1-\lambda m)B^*(\lambda(1-\beta))}{\beta-B^*(\lambda(1-\beta))}\\ &\leq \frac{(\beta-1)(1-\lambda m)\rho\beta}{\beta-\rho\beta}\\ &= c_0,\end{aligned}$$

where $c_0 = [(1-\lambda m)(\beta-1)/(1-\rho)]\rho$.

Let us

$$c = 1+\left\|\mathbf{1}\right\|_{\upsilon}\left\|\pi\right\|_{\upsilon}, \tag{40}$$

where

$$\left\|\mathbf{1}\right\|_{\upsilon} = \sup_{k\geq 0}\frac{1}{\beta^k} = 1. \tag{41}$$

Then

$$c = 1+\left\|\pi\right\|_{\upsilon}. \tag{42}$$

4. Numerical Example

In this section we will apply our bound put forward in Theorem 6.

4.1. Approximation Algorithm of the $M/G/1$ System with Breakdowns

In this subsection we elaborate an algorithm **STRO-STAB-BREAK** which allows us to get the domain of the approximation of $\tilde{\Sigma}$ by Σ and to determine the error on the stationary distribution due to the approximation.

Algorithm STRO-STAB-BREAK

STEP 1. Definition of the inputs :

- The service density function $b(x)$;
- The arrival mean rate λ;
- The repair mean rate r;
- The probability for that the customer does not leave the system q;
- The precision error ε;

STEP 2. Determination of the service mean rate :

$$\mu \leftarrow \frac{1}{\int_0^\infty u b(u) du}; \tag{43}$$

STEP 3. Verification of the stability :

if $\lambda/\mu \geq 1$ **then** $\prec *$ the system is unstable $* \succ$ **go to *STEP 7***;

else put :

$$B^* \leftarrow B^*(\lambda(1-\beta)) \leftarrow \int_0^\infty e^{(\lambda(\beta-1))u} dB(u), \tag{44}$$

go to *STEP 4*;

STEP 4. $\beta_0 \leftarrow \max(\beta : 1 < \beta \text{ and } B^*/\beta < 1)$;

STEP 5. $\beta_{\min} \leftarrow \min(\beta : 1 < \beta \text{ and } (1 - \rho - c\bar{q}\beta_0 W)^{-1}(\bar{q}\beta_0 W c_0 c) < \infty)$;

STEP 6. $\beta_{\max} \leftarrow \max(\beta : 1 < \beta \text{ and } (1 - \rho - c\bar{q}\beta_0 W)^{-1}(\bar{q}\beta_0 W c_0 c) < \infty)$;

STEP 7. end.

4.2. Numerical Validation

The primary objective of this subsection is to compare our expected approximation error against results obtained from simulations. For this, we implement the algorithm and simulator on a concrete case. Indeed, we apply the **STRO-STAB-BREAK** algorithm to determine the made error (on stationary distribution) due to the approximation (when the approximation is possible) as well as the norm from which the error is obtained. This norm will be introduced into the simulator to simulate an error (on stationary distribution) with respect to the same norm.

For the simulation of the error, we used the discrete events approach and elaborated the program in the Matlab environment according to the following steps :

(1) Simulate the stationary distribution $\tilde{\pi} = (\tilde{\pi}_i, i \geq 0)$ of $\tilde{\Sigma}$;

(2) Simulate the stationary distribution $\pi = (\pi_i, i \geq 0)$ of Σ;

(3) Calculate $\sum_{i \geq 0} \beta^i \, | \pi_i - \tilde{\pi}_i |$.

In order to appreciate the performance of this approach, we supposed that the service times of the considered models, $\tilde{\Sigma}$ and Σ, are distributed according to the exponential law with parameter $\mu = 0.6$. The arrival mean rate $\lambda = 0.2$ and the repair mean rate $r = 0.4$. The density function of the service time is given by: $b(x) = \mu e^{-\mu x}$, $x > 0$.

We interest to know if the $M/M/1$ model with breakdowns $(\tilde{\Sigma})$ can be approximated by the $M/M/1$ model (Σ) and to determine the made error, in the case when the approximation is validated.

- Verification of the stability condition : $\lambda / \mu = 0.2 / 0.6 = 0.3333 < 1$.
- We fix a value from the approximation domain : $\beta = 1.5$ $(\in [\beta_{\min}, \beta_{\max}])$.
- We fix the simulation time $t_{\max} = 1000$ units of time.
- We fix the precision $\varepsilon = 0.001$.

Introduce this value $(\beta = 1.5)$ in the simulator. The obtained results for different values of q are presented in Table 1. From these numerical results, it is easy to see that, the error decreases as the probability q increases $(q \to 1)$. Besides, the values of the both errors (algorithmic and numeric) tend to coincide in the neighborhood of lower bound $(q \to 1)$. This can be explain by the way that it represents the frontier (critical value) of the stability domain. Indeed, it is completely logical that the $M/M/1$ queueing system with breakdowns is close to the classical $M/M/1$ system with the same arrival flux and distribution of service time when the breakdown rate tends to zero (or $q \to 1$). Nevertheless, we can notice the remarkable sensitivity of the strong stability method in the variation of the rate of breakdowns with regard to the simulation. The bound obtained by the method of strong stability is much smaller than that obtained by the simulation when the rate of breakdowns tends to zero. This means that the numerical error is really the point of the error which we can do when switching from $\tilde{\Sigma}$ to Σ.

Table 1. Errors comparative table.

q	β_0	$\beta_{\min}$	$\beta_{\max}$	Algorithmic error	Simulated error
0.9200	8.3301	1.0400	6.8201	4.0791	0.7473
0.9250	8.3301	1.0400	6.8901	3.1763	0.4874
0.9300	8.3301	1.0300	6.9601	2.5351	0.6331
0.9350	8.3301	1.0300	7.0301	2.0562	0.4819
0.9400	8.3301	1.0300	7.1101	1.6848	0.3647
0.9450	8.3301	1.0300	7.1801	1.3885	0.5132
0.9500	8.3301	1.0300	7.2501	1.1465	0.4378
0.9550	8.3301	1.0200	7.3301	0.9451	0.3117
0.9600	8.3301	1.0200	7.4001	0.7750	0.2631
0.9650	8.3301	1.0200	7.4801	0.6294	0.2849
0.9700	8.3301	1.0200	7.5601	0.5033	0.1599
0.9750	8.3301	1.0200	7.6501	0.3930	0.2319
0.9800	8.3301	1.0100	7.7401	0.2958	0.1673
0.9850	8.3301	1.0100	7.8301	0.2095	0.1752
0.9900	8.3301	1.0100	7.9401	0.1323	0.1223

5. Literature Review

There exists numerous results on perturbation bounds of Markov chains. General results are summarized by Heidergott and Hordijk [6]. One group of results concerns the sensitivity of the stationary distribution of a finite, homogeneous Markov chain (see Heidergott *et al.* [8]), and the bounds are derived using methods of matrix analysis; see the review of Cho and Meyer [5] and recent papers of Kirkland [10], and Neumann and Xu [12]. Another group includes perturbation bounds for finite-time and invariant distributions of Markov chains with general state space; see Anisimov [3], Rachev [13], Aïssani and Kartashov [2], Kartashov [9], Mitrophanov [11]. In these works, the bounds for general Markov chains are expressed in terms of ergodicity coefficients of the iterated transition kernel, which are difficult to compute for infinite state spaces. These results were obtained using operator-theoretic and probabilistic methods.

6. Further Research

An alternative method for computing bounds on perturbations of Markov chains is the series expansion approach to Markov chains (SEMC). The general approach of SEMC has been introduced in [6]. SEMC for discrete time finite Markov chains is discussed in [8], and SEMC for continuous time Markov chains is developed in [7]. The key feature of SEMC is that a bound for the precision of the approximation can be given. Unfortunately, SEMC requires (numerical) computation of the deviation matrix, which limits the approach in essence to Markov chains with finite state space. Perturbation analysis via the SEMC approach overcomes this drawback, however, in contrast to SEMC, no measure on the quality of the approximation can be given.

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