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THÈSE

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Présentée par

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Thème

Intégrabilité des systèmes différentiels planaires

Soutenue le : 29-09-2021 **Devant le Jury composé de :**

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Dedications

To my brother ..
you are gone ..
but your soul is not gone ..
I miss you..

To my supervisor Mr. BOUKOUCHA Rachid..
I appreciate you for taking my hand ..
Thank you so much..

To my family..
To the angelic girl B. M ..

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List of contributions

In the setting of this thesis, we have realized the following scientific contributions:

- **Publications:**

  1. S. E. Hamizi and R. Boukoucha; *On a family of planar differential systems*, Nonlinear Studies, vol. **28**, No. 1, (2021), p. 179-188.
  2. S. E. Hamizi and R. Boukoucha, *Stable hyperbolic limit cycles for a class of differential systems*, Russian Mathematics. vol. 65, No. 9 (2021), p. 41-51.
  3. S. E. Hamizi and R. Boukoucha; *A class of planar differential systems with explicit expression for two limit cycles*, Siberian Electronic Mathematical Reports, vol. **17**, (2020), p. 1588-1597.
  4. S. E. Hamizi and R. Boukoucha; *A family of planar differential systems with explicit expression for algebraic and non algebraic limit cycles* , Memoirs on Differential Equations and Mathematical Physics, vol. **83**, (2021), p. 71-81.

- **Communications:**

  1. S. E. Hamizi and R. Boukoucha; *Explicit expression for a hyperbolic limit cycles and invariant algebraic curves for a class of polynomial differential systems (poster)*, Algerian Journal of Engineering, Architecture and Urbanism, (February 05, 2021).
  2. S. E. Hamizi and R. Boukoucha; *A class of planar differential systems with explicit expression for two limit cycles* , African Conference on Dynamical Systems and Ordinary Differential Equations, Bejaia University, Algeria, (March 23, 2021).

General introduction

Since the introduction of differential calculus by Newton and Leibniz, mathematicians had sought to solve differential equations; they were looking for formulas expressing solutions as a function of time. The methods were often clever, but it work only with very specific equations.

With Poincaré, the idea is no longer to solve the differential equation, but he realized that the qualitative properties of the solutions could be investigated without such solutions having to be determined explicitly. He turned to a qualitative approach using geometric and topological techniques. These approach is currently known as qualitative theory of differential equations [33, 84], he observes remarkable situations which can govern global behavior, such as attractive or repulsive fixed points, as well as limit cycles, which are periodic solutions attracting (or repelling) neighboring solutions.

Many systems, especially physical ones, are described by differential equations [52, 88], sometimes their solutions evolve toward limit cycles, the number of which is the issue of the second part of the 16th problem of Hilbert [51, 70, 71], it focuses on polynomial differential equations in the plane; that is a polynomial system of degree n having the form

$$\begin{cases} \dfrac{dx}{dt} & = & P(x,y) \\ \dfrac{dy}{dt} & = & Q(x,y) \end{cases} \tag{1}$$

A limit cycle is a periodic trajectory which is also "isolated", that is the neighboring trajectories are not all periodic. Hilbert's 16th problem in the second part asks : What is the maximum number $H(n)$ of limit cycles that a polynomial system of degree n can have?

The first step in the direction of 16th Hilbert problem was given by H. Dulac [60] in 1923, he published a long article, titled " Sur les cycles limites ", in which he demonstrated a theorem claiming that a polynomial differential equation in the plane has only a finite number of limit cycles. This proof was considered valid for many years. It was not until 1970s that Y. Ilyashenko did prove that it was false [56].

So some years later and independently Y.Ilyashenko and J. Ecalle provided a correct proof. Although the proof given by Dulac was wrong, the idea given by him were very fruitful.

Over years many other works have been done in this direction of 16th Hilbert problem [25, 63]. But even the simplest case, $n = 2$, is still unsolved. N. Bautin [4] (1952) states that $H(2) \geqslant 3$. Later, simultaneously, S. Shi (1979) and L.Chen and M. Wang [81] found an example with $H(2) \geqslant 4$. For the next case, $n = 3$, J. Li and Q. Huan [55, 61, 92] (1987) showed that $H(3) \geqslant 11$. Later (2009), C. Li, C. Liu and J. YANG [58] provided a planar cubic system and demonstrate that it has at least 13 limit cycles.

According with Smale, except for the Riemann hypothesis, the second part of the 16th Hilbert problem seems to be the most elusive of the Hilbert's problems. He said that, first we must consider a special class of simpler polynomial differential equations, and he propose to study the 16th Hilbert problem restricted to the Liénard system [30, 80]

$$\begin{cases} \dfrac{dx}{dt} &= y - f(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{2}$$

where f is a real polynomial of degree n satisfying $f(0) = 0$. The number $H(n)$ for the system (2) remains an open problem.

The existence of limit cycles becomes one of the more difficult objects to study in the qualitative theory of differential equations in the plane. There is a huge literature dedicated to this topic.

If $H(x, y)$ is a real polynomial irreducible in the ring $\mathbb{R}[x, y]$ of all real polynomials in the variables x and y, the zero set $\{H(x, y) = 0\}$ is an algebraic curve. An algebraic limit cycle is a limit cycle contained in an algebraic curve of the plane, otherwise such a limit cycle is called non-algebraic. The degree of an algebraic limit cycle is the degree of the irreducible polynomial which defines the algebraic curve containing the limit cycle. It is well known that the orbits of a polynomial differential system (1) are contained in analytic curves, which usually are not algebraic curves.

In general it is a difficult problem to distinguish if a limit cycle is algebraic or not. The proof that the famous limit cycle exhibited in the Van der Pol equation in 1926 was not algebraic arrived in 1995 [83]. The differential equation of Van der Pol can be written as a polynomial differential system of degree 3 (related to Liénard system)

$$\begin{cases} x' &= y - x^3 + x \\ y' &= -x \end{cases} \tag{3}$$

but we do not know explicitly its limit cycle. More precisely, we do not know the explicit expression of the analytic curve which contains the non-algebraic limit cycle of Van der Pol equation.

The first algebraic limit cycle found in the quadratic polynomial differential systems is due to Qin and to Liu, they proved in 1957 and 1958 that such systems can have algebraic limit cycles of degree 2, and that if a quadratic polynomial differential system has an algebraic limit cycle then it is the unique limit cycle of the system. Later on, it was proved by Evdokimenco [35, 36, 37] that quadratic polynomial differential systems cannot have algebraic limit cycles of degree 3 .

The first family of algebraic limit cycles of degree 4 in the quadratic polynomial differential systems was found in 1966 by Yablonskii. In 1973 Filiptsov [38] found a second family of algebraic limit cycles of degree 4, and other results was appeared later.

New families of algebraic limit cycles of degrees 5 and 6 also for quadratic polynomial differential systems were found by using birational transformation of the plane [28] for some families of algebraic limit cycles of degree 4. Until now we know that the quadratic polynomial differential systems have algebraic limit cycles of degree 6, but it is unknown if these systems can have algebraic limit cycles of degree higher than 6.

Recently, since 2006 up to now, many articles have been showing explicit non-algebraic limit cycles in polynomial differential systems [2, 6, 43, 45, 49], i.e. in those articles the authors provided the explicit expression of the analytic curve containing the limit cycle.

The first paper providing an explicit non-algebraic limit cycle for polynomial differential systems of degree less than 5 was given by Benterki and Llibre [10] in 2012 for a polynomial differential system of degree 3. Later on many other papers have been published providing explicit non-algebraic limit cycles for polynomial differential systems of degree larger than or equal to 3.

In this thesis we deal with some classes of polynomial differential systems of the form (1) by using qualitative techniques, we provide explicit expressions of their limit cycles and first integrals. The work is structured as following:

- Chapter 01 : Concerned for preliminaries and some basic notions in qualitative theory of differential equations.

- Chapter 02 : Devoted to studying two families of planar differentials system having one limit cycle, given explicitly with an expression of a first integral. The results developed on this chapter are already published in

- S. E. Hamizi, R. Boukoucha, *On a family of planar differential systems*, Nonlinear Studies, vol. **28**, No. 1, (2021), p. 179-188.
- S. E. Hamizi and R. Boukoucha, *Stable hyperbolic limit cycles for a class of differential systems*, Russian Mathematics. vol. 65, No. 9 (2021), p. 41-51.

• Chapter 03 : Devoted to studying the coexistence of algebraic and non-algebraic limit cycles for two classes of planar differentials systems, given explicitly with an expression of a first integral. The results developed on this chapter are already published in

- S. E. Hamizi, R. Boukoucha, *A class of planar differential systems with explicit expression for two limit cycles*, Siberian Electronic Mathematical Reports, vol. **17**, (2020), p. 1588-1597.
- S. E. Hamizi and R. Boukoucha; *A family of planar differential systems with explicit expression for algebraic and non algebraic limit cycles* , Memoirs on Differential Equations and Mathematical Physics, vol. **83**, (2021), p. 71-81.

• General conclusion.

Chapter 1

Preliminaries

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $\dfrac{dx}{dt} = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe.

An equation relating an unknown function and one or more of its derivatives is called a differential equation. The order of a differential equation is the largest derivative present in the differential equation.

A solution to a differential equation on an interval $\alpha < t < \beta$ is any function $x(t)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$.

Initial Condition(s) are a condition(s) on the solution that will allow us to determine which solution that we are after, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. Initial conditions are of the form,

$$x(t_0) = x_0 \text{ and / or } x^{(k)}(t_0) = x_k$$

An initial value problem is a differential equation along with an appropriate number of initial conditions.

The most general first order differential equation can be written as,

$$\frac{dx}{dt} = f(t, x)$$

The first special case of first order differential equations is the linear first order differential equation.

1.1 First order linear differential equations

Definition A linear first order differential equations is any differential equation of the form

$$x' + p(t)x = g(t), \tag{1.1}$$

where both p and g are continuous functions.

The solution to a linear first order differential equation (1.1) is

$$x(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + c\right),$$

where c is some real constant and $\mu(t) = exp\left(\int p(t)dt\right)$, called the integrating factor.

1.2 Bernoulli differential equation

Definition A Bernoulli equations is a differential equations of the form,

$$x' + p(t)x = q(t)x^n, \tag{1.2}$$

where p and q are continuous functions.
In order to solve it when n is other than 0 and 1, we divide the differential equation (1.2) by x^n to get,

$$x^{-n}x' + p(t)x^{1-n} = q(t). \tag{1.3}$$

Then we put $v = x^{1-n}$ and taking the derivative gives us,

$$v' = (1-n)x^{-n}x'.$$

Now, plugging this substitution into the differential equation (1.3) gives,

$$\frac{1}{1-n}v' + p(t)v = q(t).$$

This is a linear differential equation that we can solve for v and once we have this we can also get the solution to the original differential equation by plugging v back into our substitution and solving for x.

Example. We want to solve the following initial value problem.

$$\begin{cases} x' + & \frac{4}{t}x & = t^3x^2, \\ x(2) & = & -1, \\ t & > & 0. \end{cases}$$

Dividing everything by x^2, gives

$$x^{-2}x' + \frac{4}{t}x^{-1} = t^3.$$

The substitution and derivative that we will need here is,

$$v = x^{-1} \text{ , so } v' = -x^{-2}x'.$$

With this substitution the differential equation becomes,

$$v' - \frac{4}{t}v = -t^3.$$

This is a linear differential equation that we know how to solve. Using the integrating factor

$$\mu(t) = exp\left(\int -\frac{4}{t}dt\right) = exp\left(-4ln|t|\right) = t^{-4},$$

the solution for v is,

$$\begin{aligned} v(t) =& \frac{1}{\mu(t)}\left(\int -\mu(t)t^3 dt + c\right), \\ =& -t^4 \int t^{-1} dt + ct^4, \\ =& -t^4 ln|t| + ct^4, \\ =& t^4\left(c - ln(t)\right). \end{aligned}$$

So, $x^{-1} = t^4\left(c - ln(t)\right)$. Using the initial condition to determine the value of c

$$(-1)^{-1} = 2^4\left(c - ln(2)\right).$$

Solving for c, we get $c = ln(2) - \dfrac{1}{16}$
So, the solution is

$$x(t) = \frac{1}{t^4\left(ln(2) - \frac{1}{16} - ln(t)\right)};\ t > 0.$$

1.3 Riccati differential equation

Definition The Riccati equation is one of the most interesting nonlinear differential equations of first order. It is written in the form:

$$x' = a(t)x + b(t)x^2 + c(t), \tag{1.4}$$

where $a(t)$, $b(t)$ and $c(t)$ are continuous functions of t.
The differential equation (1.4) is called the general Riccati equation. In general the Riccati equation is not solvable by elementary means. However it can be solved with help of the following theorem:

Theorem 1.1. *If a particular solution x_1 of a Riccati equation is known, the general solution of the equation is given by*

$$x = x_1 + u,$$

Indeed, substituting the solution $x = x_1 + u$ into Riccati equation (1.4), we have

$$x_1' + u' = a(t)x_1 + b(t)x_1^2 + c(t) + a(t)u + b(t)u^2 + 2b(t)x_1 u,$$

we obtain the differential equation for the function $u(t)$

$$u' = b(t)u^2 + (2b(t)x_1 + a(t))\,u,$$

which is a Bernoulli equation that can be converted into a linear differential equation that allows integration.

1.4 Planar differential systems

Definition A planar system of differential equations is a collection of two interrelated differential equations of the form

$$\begin{cases} x' &= f(t,x,y), \\ y' &= g(t,x,y). \end{cases} \tag{1.5}$$

Here the functions f and g are real-valued functions of variables x, y and t.
The system (1.5) is called **autonomous** if none of f and g depends on t.
The system (1.5) is called **polynomial** if it is of the form

$$\begin{cases} x' &= P(x,y), \\ y' &= Q(x,y), \end{cases} \tag{1.6}$$

where P and Q are polynomials with real coefficients. We denote by $n = max\{degP, degQ\}$ the degree of the polynomial system, and we always assume that the polynomials P and Q are relatively prime.
We write equivalently the system (1.6) by using the abbreviated notation

$$X' = F(X), \tag{1.7}$$

where $X = (x,y)$ and $F(X) = F(x,y) = (P(x,y), Q(x,y))$.

1.4.1 Vector field

We regard the right-hand side of equation (1.6) as defining a vector field on $\mathbb{R}^2$. That is, we think of $F(x,y)$ as representing a vector whose x- and y-components are $P(x,y)$ and $Q(x,y)$, respectively. We visualize this vector as being based at the point (x,y).

Definition A vector field on two dimensional space is a function F that assigns to each point (x,y) a vector given by $F(x,y) = (P(x,y), Q(x,y))$. We denote it by a differential operator

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Example. The vector field associated to the system

$$\begin{cases} x' &= y, \\ y' &= -x, \end{cases} \tag{1.8}$$

is displayed in Figure 1.1

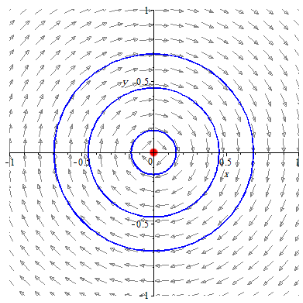


Figure 1.1: The vector field and several solutions for system (1.8).

1.4.2 Solutions of a planar differential system

A solution of system (1.7) is a function $X : J \longrightarrow \mathbb{R}^2$ defined on some interval $J \subset \mathbb{R}$ such that, for all $t \in J$,

$$X'(t) = F(X(t)),$$

that is

$$\begin{cases} x'(t) &= P(x(t), y(t)), \\ y'(t) &= Q(x(t), y(t)). \end{cases}$$

Geometrically, $X(t)$ is a curve in $\mathbb{R}^2$ whose tangent vector $X'(t)$ exists for all $t \in J$ and equals $F(X(t))$. We think of this vector as being based at $X(t)$, so that the map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defines a vector field on $\mathbb{R}^2$.

We denote a solution on the initial value $X_0 = X(t_0)$ by $\varphi(t, X_0)$ or $\varphi_t(X_0)$. This function $\varphi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called the flow associated to the system (1.7). So $\varphi_t(X_0)$ defines a solution curve, trajectory, or orbit through X_0.

Remark A point $X_* = (x_*, y_*)$ for which $F(X_*) = 0$ i.e $P(x_*, y_*) = 0$ and $Q(x_*, y_*) = 0$, is called an equilibrium point for the system (1.6). An equilibrium point corresponds to a constant solution $X(t) \equiv X_*$.

1.4.3 Periodic solutions of a planar differential system

A periodic solution for the system (1.7) is a non-equilibrium point X such that:

- $X'(t) = F(X(t))$, for all t ;
- There exists a time $T > 0$ for which $X(t+T) = X(t)$, for all t.

The least such $T > 0$ is called the period of the solution.

Example. The curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\ cos(t) \\ a\ sin(t) \end{pmatrix},$$

for any $a \in \mathbb{R}$ is a solution of the system (1.8). These curves define circles of radius $|a|$ in the plane that are traversed in the clockwise direction as t increases. Hence the solution is periodic of period 2π.
When $a = 0$, the solutions are the constant functions $x(t) \equiv 0 \equiv y(t)$.

1.5 Phase plane and phase portraits

The solutions of $X' = F(X)$ are a functions of time, it can be visualized as trajectories moving on the (x, y) plane, in this context called the phase plane. For several important equations, it is impossible to find an analytical solution and it's useful to develop methods for deducing the behavior of equations without solving them. The motion in the phase plane is determined by a vector field that comes from the planar system $X' = F(X)$. Here X represents a point in the phase plane, and X' is the velocity vector at that point. By flowing along the vector field, a phase point traces out a solution $X(t)$, corresponding to a trajectory winding through the phase plane. Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition. The overall picture of trajectories in phase space is called phase portraits.

Example. The figure 1.2 displays the phase portraits of the system

$$\begin{cases} x'(t) &= x^2 - 1, \\ y'(t) &= -xy + \frac{1}{2}\left(x^2 - 1\right). \end{cases} \tag{1.9}$$

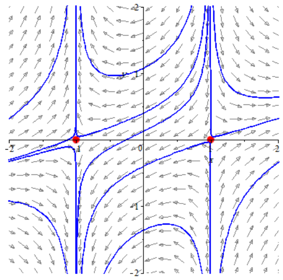


Figure 1.2: Phase portraits of system (1.9).

1.6 Linear differential systems

Definition A two-dimensional linear differential system has the form

$$\left\{\begin{array}{lcl} x' & = & ax+by, \\ y' & = & cx+dy. \end{array}\right. \tag{1.10}$$

where $a, b, c,$ and d are real parameters. Equivalently, in vector notation

$$X' = AX,$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

1.6.1 Equilibrium points

Note that the origin is always an equilibrium point for a linear system (1.10). To find other equilibria, we must solve the linear system of algebraic equations

$$\left\{\begin{array}{lcl} ax+by & = & 0, \\ cx+dy & = & 0. \end{array}\right.$$

This system has a nonzero solution if and only if $detA = 0$. Thus we have:

Proposition 1.1. *The planar linear system $X' = AX$ has:*

1. *A unique equilibrium point* $(0,0)$ *if* $detA \neq 0$.

2. *A straight line of equilibrium points if* $detA = 0$ *(and* A *is not the* 0 *matrix).*

1.6.2 Eigenvalues and eigenvectors

Definition Consider the linear differential system $X' = AX$. A nonzero vector V_0 is called an eigenvector of A if $AV_0 = \lambda V_0$ for some $\lambda \in \mathbb{R}$. The constant λ is called an eigenvalue of A.

There is an important relationship between eigenvalues, eigenvectors, and solutions of linear systems

Theorem 1.2. *Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function*

$$X(t) = V_0 e^{\lambda t},$$

is a solution of the system $X' = AX$.

The collection of all such solutions is called the general solution of $X' = AX$.

Theorem 1.3. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by*

$$X(t) = \alpha V_1 e^{\lambda_1 t} + \beta V_2 e^{\lambda_2 t}.$$

1.6.3 Lyapunov stability

An equilibrium point $X_* \in \mathbb{R}^2$ of the system (1.6) is stable provided that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|X_0 - X_*| < \delta \textit{ implies that } |X(t) - X_*| < \varepsilon, \textit{ for all } t > 0.$$

Otherwise the equilibrium is said to be unstable.
The equilibrium $X_* \in \mathbb{R}^2$ is asymptotically stable if it is stable and there exists $\delta > 0$ such that

$$|X - X_*| < \delta \textit{ implies that } \lim_{t \longrightarrow \infty} X(t) = X_*.$$

1.7 Classification of equilibrium points

Consider $X' = AX$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the eigenvalues of A are given by the characteristic equation

$$det(A - \lambda I) = det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0,$$

where I is the identity matrix.
Expanding the determinant yields

$$\lambda^2 - T\lambda + \Delta = 0,$$

where

$$T = trace(A) = a + d,$$

$$\Delta = det(A) = ad - bc.$$

Then

$$\lambda_1 = \frac{T + \sqrt{T^2 - 4\Delta}}{2}, \lambda_2 = \frac{T - \sqrt{T^2 - 4\Delta}}{2}.$$

Hence, the eigenvalues depend only on the trace and determinant of the matrix A.

1.7.1 Real distinct eigenvalues

Suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$. Assuming that$\lambda_i \neq 0$, there are three cases to consider:

Case 01: $\lambda_1 < 0 < \lambda_2$. In this case the equilibrium point is a saddle. It is always unstable.

Example Take the system

$$X' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} X. \tag{1.11}$$

The matrix has $T = -1$ and $\Delta = -6$, so the characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Hence

$$\lambda_1 = -3, \lambda_2 = 2.$$

The corresponding eigenvectors for $\lambda_1 = -3$ and $\lambda_2 = 2$, are $V_1 = (1, -4)$ and $V_2 = (1, 1)$, respectively.
The general solution is

$$X(t) = \alpha V_1 e^{-3t} + \beta V_2 e^{2t}.$$

Case 02: $\lambda_1 < \lambda_2 < 0$. In this case the equilibrium point is a sink. It is asymptotically stable.

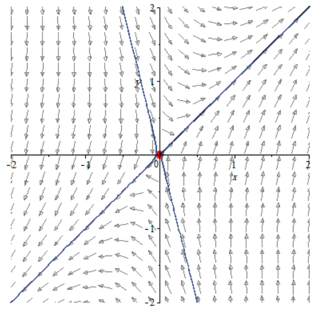


Figure 1.3: Saddle phase portrait of system (1.11).

Example Take the system

$$X' = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} X. \tag{1.12}$$

The matrix has the eigenvalues

$$\lambda_1 = -3, \lambda_2 = -2.$$

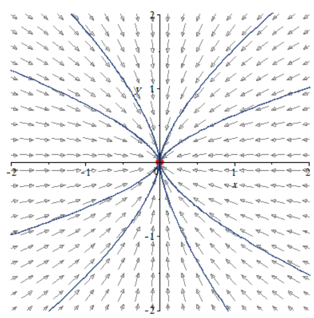


Figure 1.4: Sink phase portrait of system (1.12) .

Case 03: $0 < \lambda_2 < \lambda_1$. In this case the equilibrium point is a source. It is unstable.

Example Take the system

$$X' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} X. \tag{1.13}$$

The matrix has the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$.

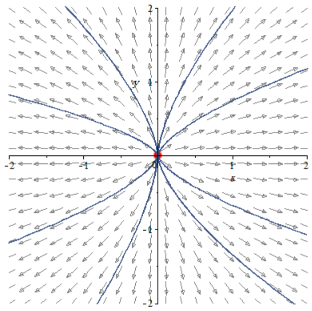


Figure 1.5: Source phase portrait of system (1.13) .

1.7.2 Repeated eigenvalues

Case 01: If $\lambda \neq 0$ and λ has two independent eigenvectors , then the equilibrium point is a star node. It is asymptotically stable if $\lambda < 0$, unstable if $\lambda > 0$.

Example Take the system

$$X' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} X. \tag{1.14}$$

The eigenvalue $\lambda = -2$ is repeated and has two independent eigenvectors $V_1 = (0,1)$ and $V_2 = (1,0)$.

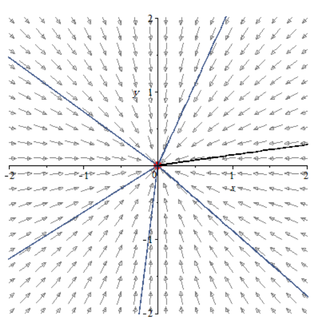


Figure 1.6: Star node phase portrait of system (1.14) .

Case 02: If $\lambda \neq 0$ and λ has one eigenvector , then the equilibrium point is a degenerate node. It is asymptotically stable if $\lambda < 0$, unstable if $\lambda > 0$.

Example Take the system

$$X' = \begin{pmatrix} -2 & 3 \\ 0 & -2 \end{pmatrix} X. \tag{1.15}$$

The eigenvalue $\lambda = -2$ is repeated and has one eigenvectors $V_1 = (1, 0)$.

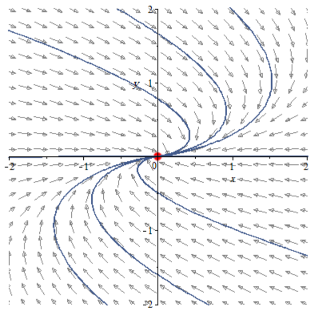


Figure 1.7: Degenerate node phase portrait of system (1.15) .

1.7.3 Complex eigenvalues

Case 01 $\lambda_{1,2} = \pm i\beta$. In this case the equilibrium point is a center.

Example Take the system

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X. \tag{1.16}$$

The characteristic polynomial is $\lambda^2 + 1 = 0$, so the eigenvalues are now the imaginary numbers $\pm i$. The general solution is

$$X(t) = c_1 \begin{pmatrix} cost \\ -sint \end{pmatrix} + c_2 \begin{pmatrix} sint \\ cost \end{pmatrix}.$$

Case 02 $\lambda_{1,2} = \alpha \pm i\beta$. In this case the equilibrium point is a spiral sink (asymptotically stable) if $\alpha < 0$ and spiral source (unstable) (if $\alpha > 0$).

Example Ttake the system

$$X' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} X. \tag{1.17}$$

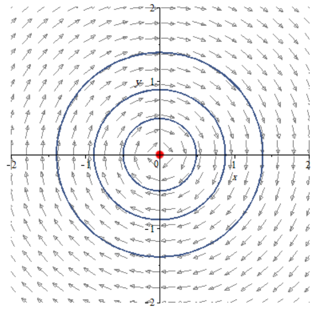


Figure 1.8: Center phase portrait of system (1.16) .

The eigenvalues are $\lambda_{1,2} = 1 \pm i3$. The general solution is

$$X(t) = c_1 e^t \begin{pmatrix} cos3t \\ -sin3t \end{pmatrix} + c_2 e^t \begin{pmatrix} sin3t \\ cos3t \end{pmatrix}.$$

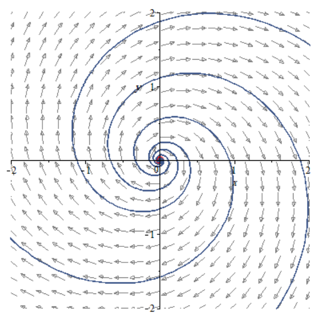


Figure 1.9: Spiral source phase portrait of system (1.17).

1.7.4 Equilibrium points and linearization

The hope is that we can approximate the phase portrait of polynomial differential system near an equilibrium point by that of a corresponding linear system.

Consider the autonomous system (1.10)

$$\begin{cases} x' &= P(x,y), \\ y' &= Q(x,y), \end{cases}$$

suppose that (x_*, y_*) is a fixed point and let

$$u = x - x_*, \; v = y - y_*.$$

The System

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{pmatrix}_{(x_*, y_*)} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.18}$$

is called the linearized system and the matrix

$$A = \begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{pmatrix}_{(x_*, y_*)},$$

is called the Jacobian matrix at the equilibrium point (x_*, y_*).

The Hartman-Grobman theorem

Theorem 1.4. *With the condition that every eigenvalue of the Jacobian matrix A has nonzero real part, there is a homeomorphism H from a neighbourhood of $(0,0)$ to a neighbourhood of (x_*, y_*), which maps the flow of the linearized system to the flow of the original system.*

1.8 Limit cycles

Definition A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.

- If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting.

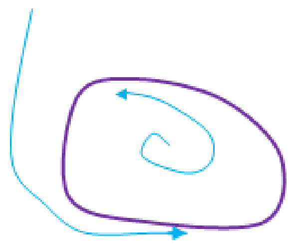


Figure 1.10: Stable limit cycle.

- If all neighboring trajectories spiral away from the limit cycle, we say the limit cycle is unstable.

Figure 1.11: Unstable limit cycle.

- In exceptional cases, we say the limit cycle is half-stable.

Figure 1.12: Half - stable limit cycles.

Example : Van Der Pol oscillator The Van der Pol system is given by

$$\begin{cases} x' &= y - x^3 + x, \\ y' &= -x. \end{cases} \tag{1.19}$$

The Jacobian matrix associated to the system (1.19) is

$$A = \begin{pmatrix} -3x_*^2 + 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Jacobian matrix evaluated at the equilibrium point $(0,0)$ is

$$A_{(0,0)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2},$$

as the real part is $\frac{1}{2} > 0$ the origin is spiral source.

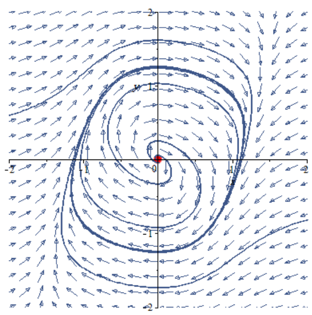


Figure 1.13: The phase portrait of the van der Pol system.

1.9 Existence and non-existence of periodic solutions

1.9.1 The Poincaré map

Suppose that there is a curve or straight line segment, say, Σ, that is crossed transversely (no trajectories are tangential to Σ). Then Σ is called a Poincaré section. Consider a point r_0 lying on Σ. As shown in Figure (1.14), follow the flow of the trajectory until it next meets Σ at a point r_1. This point is known as the first return of the discrete Poincaré map $\mathbf{P} : \Sigma \longrightarrow \Sigma$, defined by

$$r_{n+1} = \mathbf{P}(r_n).$$

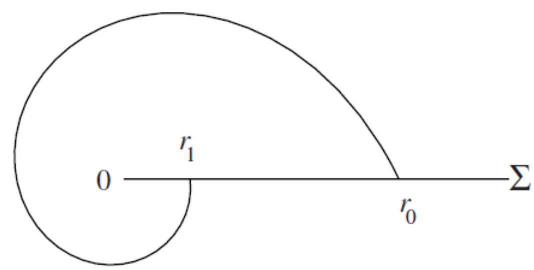


Figure 1.14: A first return on a Poincaré section, Σ.

Definition A point r_* that satisfies the equation $\mathbf{P}(r_*) = r_*$ is called a fixed point of period one.

Theorem 1.5. *Define the characteristic multiplier M to be*

$$M = \left.\frac{d\boldsymbol{P}}{dr}\right|_{r=r_*}.$$

where r_* is a fixed point of the Poincaré map $\boldsymbol{P}$ corresponding to a limit cycle, say, (Γ)*. Then if*

1. $|M| < 1, (\Gamma)$ *is a hyperbolic stable limit cycle,*
2. $|M| > 1, (\Gamma)$ *is a hyperbolic unstable limit cycle,*
3. $|M| = 1$ *and* $\dfrac{d^2\boldsymbol{P}}{dr^2} \neq 0$, *then the limit cycle is half-stable.*

Example Consider the following system

$$\left\{\begin{array}{rcl} x' &=& -y + x\left(1 - \sqrt{x^2+y^2}\right), \\ y' &=& x + y\left(1 - \sqrt{x^2+y^2}\right), \end{array}\right. \tag{1.20}$$

and consider the line segment

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x < \infty, y = 0\right\}.$$

System (1.20) becomes in polar coordinates

$$\left\{\begin{array}{rcl} r' &=& r(1-r), \\ \theta' &=& 1. \end{array}\right. \tag{1.21}$$

The origin is an unstable focus, and there is a limit cycle, say (Γ), of radius 1 centered at the origin. A phase portrait showing two trajectories is given in Figure (1.15) .

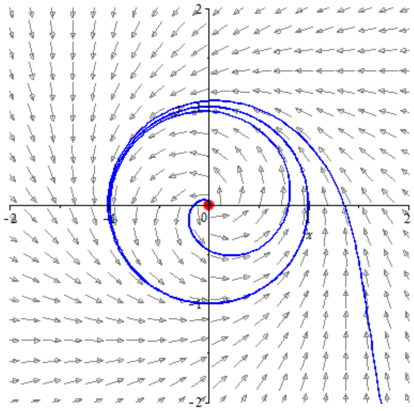


Figure 1.15: Phase portrait of system (1.21).

System (1.21) can be solved since both differential equations are separable. The solutions are given by

$$\begin{cases} r(t) &= \dfrac{1}{1+Ce^{-t}}, \\ \theta(t) &= t+\theta_0, \end{cases}$$

where C and θ_0 are constants. Trajectories flow around the origin with a period of 2π, suppose that a trajectory starts at some $r_0 = r(0)$ on Σ and $\theta(0) = 0$. Then

$$r(t) = \frac{1}{1+Ce^{-\theta(t)}}.$$

The flow is counterclockwise, and the required successive returns occur when $\theta = 2\pi$, 4π... A map defining these points is given by

$$r_n = \frac{1}{1+Ce^{-2n\pi}}, \tag{1.22}$$

where C is a constant. Therefore

$$r_{n+1} = \frac{1}{1+Ce^{-2(n+1)\pi}}. \tag{1.23}$$

Substituting $C = \dfrac{1-r_n}{r_n e^{-2n\pi}}$ from equation (1.22) into (1.23) gives the Poincaré map

$$r_{n+1} = \mathbf{P}(r_n) = \frac{r_n}{r_n + (1-r_n)e^{-2\pi}}.$$

The Poincaré map has two fixed points, one at zero (a trivial fixed point) and the other at $r_* = 1$, corresponding to the critical point at the origin and the limit cycle (Γ), respectively. Now

$$\frac{d\mathbf{P}}{dr} = \frac{e^{-2\pi}}{(r+(1-r)e^{-2\pi})^2},$$

using elementary calculus, and

$$\left.\frac{d\mathbf{P}}{dr}\right|_{r_*=1} = e^{-2\pi} \approx 0.00187 < 1,$$

and so (Γ) is a hyperbolic stable limit cycle.

1.10 Integrability of polynomial differential systems

1.10.1 Invariant curves

An algebraic curve defined by $U(x,y) = 0$ is an invariant curve for the system (1.6) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P\frac{\partial U}{\partial x} + Q\frac{\partial U}{\partial y} = KU.$$

We note that, since the polynomial system has degree n, any cofactor has degree at most $n-1$.

1.10.2 First integrals

The polynomial system (1.6) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant analytic function $H : \Omega \longrightarrow \mathbb{R}$, called a first integral, such that

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} \equiv 0.$$

1.10.3 Algebraic limit cycle

A limit cycle of system (1.6) is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of the system, else it is called non-algebraic.

1.10.4 Darboux integrability

A Darboux function is a function of the form

$$f_1^{\lambda_1}.f_2^{\lambda_2}...f_n^{\lambda_n}.exp\left(\frac{g}{h}\right),$$

where f_i, g and h are real polynomials , the λ_i's are real numbers and $exp\left(\frac{g}{h}\right)$ called the exponential factor. System (1.6) is called Darboux integrable if it has a first integral which is a Darboux function.

1.10.5 Liouville integrability

Liouvillian functions are functions that are built up from elementary functions (using exponentiation, integration, and algebraic functions). If a planar polynomial system has a first integral expressed in term of liouvillian functions , then we say that the system has a liouvillian first integral.

The following theorem says that the method of Darboux finds all liouvillian first integrals.

Theorem 1.6. *(see in [33] p 308) If a polynomial system has a Liouvillian first integral, then the system has a Darbouxian first integral.*

Chapter 2

First integral and limit cycle for some families of polynomial differential systems

This chapter consists of two main parts. The first part deals with the family of the polynomial differential systems of the form

$$\begin{cases} x' = \delta x\left(x^2+y^2\right)^n + (\lambda y - \beta x)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p, \\ y' = \delta y\left(x^2+y^2\right)^n - (\beta y + \lambda x)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p, \end{cases}$$

where a, b, c, d, β, λ, δ are real constants and n, m, $p \in \mathbb{N}$.

In the second part we concentrate our study to a multi-parameter planar polynomial differential systems of the form

$$\begin{cases} x' = x + (\alpha y - \beta x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} \\ y' = y - (\beta y + \alpha x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} \end{cases}$$

where n, λ_i are positive integer and α, β, a_i, b_i, $i = 1,..n$ are real constants.

For each of the two families above, primarily we prove the integrablity, explicit formulas of invariant curve and first integral are introduced. Moreover, we determine sufficient conditions to possess an explicit algebraic or non-algebraic limit cycles. Finally; our study is accompanied with a concrete examples exhibiting the applicability of our results.

2.1 On a family of planar differential systems

In this section, we consider the family of the polynomial differential systems of the form

$$\begin{cases} x' = \delta x\left(x^2+y^2\right)^n + (\lambda y - \beta x)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p, \\ y' = \delta y\left(x^2+y^2\right)^n - (\beta y + \lambda x)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p, \end{cases} \tag{2.1}$$

where a, b, c, d, β, λ, and δ are real constants and n, m, and $p \in \mathbb{N}$.

2.1.1 Main result

Our main result is contained in the following theorem.

Theorem 2.1. *Consider a multi-parameter polynomial differential systems (2.1), then the following statements hold.*

(h_1) *The origin $O(0,0)$ is the unique critical point at finite distance.*

(h_2) *If $m \geq 1$ and $p \geq 1$, then the curve*

$$U(x,y) = -\lambda\left(x^2+y^2\right)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p = 0,$$

is an invariant algebraic of systems (2.1).

(h_3) *If* $\lambda\left(a+\frac{1}{2}b\sin 2w\right)^m\left(c+\frac{1}{2}d\sin 2w\right)^p \neq 0$ *for all* $w \in \mathbb{R}$, *then systems* (2.1) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{m-n+p}\exp\left(\frac{\beta(n-m-p)}{\lambda}\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}} F(w)\,dw,$$

where $F(w) = \dfrac{(2n-2m-2p)\,\delta\exp\left(\frac{(2n-2m-2p)\beta}{\lambda}w\right)}{\lambda\left(a+\frac{1}{2}b\sin 2w\right)^m\left(c+\frac{1}{2}d\sin 2w\right)^p}.$

(h_4) *If* $\lambda\left(a+\frac{1}{2}b\sin 2w\right)^m\left(c+\frac{1}{2}d\sin 2w\right)^p = 0$ *for all* $w \in \mathbb{R}$, *then systems* (2.1) *has the first integral* $H = \dfrac{y}{x}$. *Moreover, the systems* (2.1) *has no limit cycle.*

(h_5) *If* $\lambda > 0, \beta > 0, \delta > 0$, $m > n$, $2a > |b|$, *and* $2c > |d|$, *then systems* (2.1) *has an explicit limit cycle, given in polar coordinates* (r,θ) *by*

$$r(\theta, r_*) = \exp\left(\frac{\beta}{\lambda}\theta\right)\left(r_*^{2m-2n+2p} + \int_0^\theta F(w)\,dw\right)^{\dfrac{1}{2m-2n+2p}},$$

where

$$r_* = \left(\frac{-\int_0^{2\pi} F(w)\,dw}{1-\exp\left(\frac{(4n-4m-4p)\beta\pi}{\lambda}\right)} \right)^{\frac{1}{2m-2n+2p}}$$

Proof of **Theorem 2.1.**

Proof of statement (h_1)

In analysis, $A(x_*, y_*) \in \mathbb{R}^2$ is a critical point of systems (2.1), if,

$$\begin{cases} \delta x_* \left(x_*^2 + y_*^2\right)^n + (\lambda y_* - \beta x_*)\left(a x_*^2 + b x_* y_* + a y_*^2\right)^m \left(c x_*^2 + d x_* y_* + c y_*^2\right)^p = 0, \\ \delta y_* \left(y_*^2 + y_*^2\right)^n - (\beta y_* + \lambda x_*)\left(a x_*^2 + b x_* y_* + a y_*^2\right)^m \left(c x_*^2 + d x_* y_* + c y_*^2\right)^p = 0, \end{cases}$$

that is to say $\lambda\left(x_*^2 + y_*^2\right)\left(a x_*^2 + b x_* y_* + a y_*^2\right)^m \left(c x_*^2 + d x_* y_* + c y_*^2\right)^p = 0$, according to $\lambda \neq 0$, $b^2 - 4a < 0$, and $d^2 - 4c < 0$, hence $x_* = 0, y_* = 0$ is the unique singularity of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of statement (h_1).

Proof of statement (h_2)

We prove that

$$U(x,y) = -\lambda\left(x^2+y^2\right)\left(ax^2+bxy+ay^2\right)^m \left(cx^2+dxy+cy^2\right)^p = 0,$$

is an invariant algebraic curve of the differential systems (2.1). We note that

$$S_1 \equiv S_1(x,y) = ax^2+bxy+ay^2 \text{ and } S_2 \equiv S_2(x,y) = cx^2+dxy+cy^2.$$

Indeed, we have

$$\begin{aligned}
&\frac{\partial U(x,y)}{\partial x} P(x,y) + \frac{\partial U(x,y)}{\partial y} Q(x,y) \\
&= \frac{\partial U(x,y)}{\partial x}\left(\delta x\left(x^2+y^2\right)^n + (\lambda y - \beta x) S_1^m S_2^p\right) + \frac{\partial U(x,y)}{\partial y}\left(\delta y\left(x^2+y^2\right)^n - (\beta y + \lambda x) S_1^m S_2^p\right) \\
&= \left(x\frac{\partial U(x,y)}{\partial x} + y\frac{\partial U(x,y)}{\partial y}\right)\delta\left(x^2+y^2\right)^n + (\lambda y - \beta x) S_1^m S_2^p \frac{\partial U(x,y)}{\partial x} - (\beta y + \lambda x) S_1^m S_2^p \frac{\partial U(x,y)}{\partial y}.
\end{aligned}$$

In what follows, we simplify each member of the last equation above

$$\begin{aligned}
&\left(x\frac{\partial U(x,y)}{\partial x} + y\frac{\partial U(x,y)}{\partial y}\right)\delta\left(x^2+y^2\right)^n \\
&= -\lambda(\lambda y - \beta x) S_1^m S_2^p \left(2x S_1^m S_2^p + m\left(x^2+y^2\right)(2ax+by) S_1^{m-1} S_2^p + p\left(x^2+y^2\right)(2cx+dy) S_1^m S_2^{p-1}\right) \\
&+\lambda y\left(2y S_1^m S_2^p + m\left(x^2+y^2\right)(bx+2ay) S_1^{m-1} S_2^p + p\left(x^2+y^2\right)(dx+2cy) S_1^m S_2^{p-1}\right)\delta\left(x^2+y^2\right)^n \\
&= -2\lambda\left(S_1^m S_2^p + m\left(ax^2+bxy+ay^2\right) S_1^{m-1} S_2^p + p\left(cx^2+dxy+cy^2\right) S_1^m S_2^{p-1}\right)\delta\left(x^2+y^2\right)^{n+1} \\
&= -2\lambda\left(x^2+y^2\right) S_1^m S_2^p (m+p+1)\delta\left(x^2+y^2\right)^n \\
&= 2\delta(m+p+1)\left(x^2+y^2\right)^n U(x,y),
\end{aligned}$$

also, we have

$$
\begin{aligned}
&(\lambda y-\beta x)S_1^m S_2^p\frac{\partial U(x,y)}{\partial x}-(\beta y+\lambda x)S_1^m S_2^p\frac{\partial U(x,y)}{\partial y}=\\
&-\lambda(\lambda y-\beta x)S_1^m S_2^p\left(2xS_1^mS_2^p+m\left(x^2+y^2\right)(2ax+by)S_1^{m-1}S_2^p+p\left(x^2+y^2\right)(2cx+dy)S_1^mS_2^{p-1}\right)+\\
&\lambda(\beta y+\lambda x)S_1^mS_2^p\left(2yS_1^mS_2^p+m\left(x^2+y^2\right)(bx+2ay)S_1^{m-1}S_2^p+p\left(x^2+y^2\right)(dx+2cy)S_1^mS_2^{p-1}\right)\\
&=-\lambda\left(x^2+y^2\right)S_1^mS_2^p\left(-2(m+p+1)\beta S_1^mS_2^p+mb\lambda\left(y^2-x^2\right)S_1^{m-1}S_2^p+d\lambda p\left(y^2-x^2\right)S_1^mS_2^{p-1}\right)\\
&=\left(-2(m+p+1)\beta S_1^mS_2^p+\lambda\left(y^2-x^2\right)\left(mbS_1^{m-1}S_2^p+dpS_1^mS_2^{p-1}\right)\right)U(x,y).
\end{aligned}
$$

Now, by substitution we have

$$
\begin{aligned}
&\frac{\partial U(x,y)}{\partial x}P(x,y)+\frac{\partial U(x,y)}{\partial y}Q(x,y)\\
&=\left(2\delta(m+p+1)\left(x^2+y^2\right)^n-2(m+p+1)\beta S_1^mS_2^p+\lambda\left(y^2-x^2\right)\left(mbS_1^{m-1}S_2^p+dpS_1^mS_2^{p-1}\right)\right)U(x,y)\\
&=U(x,y)K(x,y),
\end{aligned}
$$

where

$$
\begin{aligned}
K(x,y) \;=\;& 2\delta(m+p+1)\left(x^2+y^2\right)^n-2(m+p+1)\beta S_1^mS_2^p+\\
&\lambda\left(y^2-x^2\right)\left(mbS_1^{m-1}S_2^p+dpS_1^mS_2^{p-1}\right).
\end{aligned}
$$

Therefore,

$$
U(x,y)=-\lambda\left(x^2+y^2\right)\left(ax^2+bxy+ay^2\right)^m\left(cx^2+dxy+cy^2\right)^p=0,
$$

is an invariant algebraic curve of the polynomial differential systems (2.1), with the cofactor $K(x,y)$. Hence, statement (h_2) is proved.

Proof of statements (h_3), (h_4) and (h_5)

In order to prove our results (h_3), (h_4) and (h_5) we write the polynomial differential systems (2.1) in polar coordinates (r,θ), defined by $x=r\cos\theta$ and $y=r\sin\theta$, then the systems become

$$
\begin{cases}
r'=\delta r^{2n+1}-\beta\left(a+\frac{1}{2}b\sin 2\theta\right)^m\left(c+\frac{1}{2}d\sin 2\theta\right)^p r^{2m+2p+1},\\[2ex]
\theta'=-\lambda\left(a+\frac{1}{2}b\sin 2\theta\right)^m\left(c+\frac{1}{2}d\sin 2\theta\right)^p r^{2m+2p},
\end{cases}
\tag{2.2}
$$

where $\theta'=\dfrac{d\theta}{dt}$, $r'=\dfrac{dr}{dt}$.

Taking as new independent variable the coordinate θ, this differential systems write

$$
\frac{dr}{d\theta}=\frac{\beta}{\lambda}r+\frac{-\delta}{\lambda\left(a+\frac{1}{2}b\sin 2\theta\right)^m\left(c+\frac{1}{2}d\sin 2\theta\right)^p}r^{2n-2m-2p+1},
\tag{2.3}
$$

which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^{2m-2n+2p}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \frac{(2m-2n+2p)\,\beta}{\lambda}\rho + \frac{-(2m-2n+2p)\,\delta}{\lambda\left(a+\frac{1}{2}b\sin 2\theta\right)^m\left(c+\frac{1}{2}d\sin 2\theta\right)^p}. \tag{2.4}$$

The general solution of the linear equation (2.4) is

$$r(\theta) = \exp\left(\frac{\beta}{\lambda}\theta\right)\left(c + \int_0^\theta F(w)\,dw\right)^{\frac{1}{2m-2n+2p}},$$

where $c \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{m-n+p}\exp\left(\frac{\beta\,(n-m-p)}{\lambda}\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}} F(w)\,dw.$$

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently systems (2.1) are Darboux integrable.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential systems (2.1), in cartesian coordinates are written as

$$\left(x^2+y^2\right)^{m-n+p} = \exp\left(\frac{\beta\,(m-n+p)}{\lambda}\arctan\frac{y}{x}\right)\left(h + \int_0^{\arctan\frac{y}{x}} F(w)\,dw\right),$$

where $h \in \mathbb{R}$. Hence, statement (h_3) is proved.

Proof of statement (h_4)

Assume now that $\lambda\left(a+\frac{1}{2}b\sin 2w\right)^m\left(c+\frac{1}{2}d\sin 2w\right)^p = 0$ for all $w \in \mathbb{R}$. Then, from differential systems (2.2) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential systems (2.1) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the systems. Then since all straight lines through the origin are formed by trajectories, which can be written in Cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$. Consequently, there is no limit cycle.

This completes the proof of statement (h_4) .

Proof of statement (h_5)

According to $\lambda > 0$, $\delta > 0$, $m > n$, $2a > |b|$, and $2c > |d|$, hence

$$-\lambda\left(a+\frac{1}{2}b\sin 2\theta\right)^m\left(c+\frac{1}{2}d\sin 2\theta\right)^p < 0,$$

for all $\theta \in \mathbb{R}$, then θ' is negative for all t, the orbits $r(\theta)$ of the differential equation (2.3) has reversed their orientation with respect to the orbits $(x(t), y(t))$ of the differential systems (2.1).

Notice that systems (2.1) has a periodic orbit if and only if equation (2.3) has a strictly positive 2π periodic solution. This, moreover, is equivalent to the existence of a solution of (2.3) that fulfills $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$.

The solution $r(\theta, r_0)$ of the differential equation (2.3) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \exp\left(\frac{\beta}{\lambda}\theta\right)\left(r_0^{2m-2n+2p} + \int_0^\theta F(w)\,dw\right)^{\frac{1}{2m-2n+2p}},$$

where $r_0 = r(0)$.

A periodic solution of systems (2.1) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to a unique value $r_0 = r_*$, given by

$$r_* = \sqrt[2m-2n+2p]{\frac{-\int_0^{2\pi} F(w)\,dw}{1 - \exp\left(\frac{(4n-4m-4p)\beta\pi}{\lambda}\right)}}.$$

According to $\lambda > 0, \beta > 0, \delta > 0$, $m > n$, $2a > |b|$, and $2c > |d|$, hence $\lambda\beta(4n - 4m - 4p) < 0$ and $F(w) < 0$ for all $w \in \mathbb{R}$, then $r_* > 0$.

After the substitution of these value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \exp\left(\frac{\beta}{\lambda}\theta\right)\left(\frac{-\int_0^{2\pi} F(w)\,dw}{1 - \exp\left(\dfrac{(4n-4m-4p)\beta\pi}{\lambda}\right)} + \int_0^\theta F(w)\,dw\right)^{\frac{1}{2m-2n+2p}}.$$

In what follows it is proved that $r(\theta, r_*) > 0$. Indeed

$$\begin{aligned} r(\theta, r_*) &= \exp\left(\frac{\beta}{\lambda}\theta\right)\left(\frac{-\exp\left(\frac{(4n-4m-4p)\beta\pi}{\lambda}\right)}{1 - \exp\left(\frac{(4n-4m-4p)\beta\pi}{\lambda}\right)} \int_0^{2\pi} F(w)\,dw - \int_\theta^{2\pi} F(w)\,dw\right)^{\frac{1}{2m-2n+2p}} \\ &\geq \exp\left(\frac{\beta}{\lambda}\theta\right)\left(-\int_\theta^{2\pi} F(w)\,dw\right)^{\frac{1}{2m-2n+2p}} > 0, \end{aligned}$$

because $F(w) = \dfrac{\delta(2n-2m-2p)\exp\left(\frac{(2n-2m-2p)\beta}{\lambda} w\right)}{\lambda\left(a + \frac{1}{2}b\sin 2w\right)^m \left(c + \frac{1}{2}d\sin 2w\right)^p} < 0$ for all $w \in \mathbb{R}$.

Moreover, we compute

$$\frac{dr(2\pi, r_0)}{dr_0}\Big|_{r_0 = r_*} = \exp\left(\frac{(4m-4n+4p)\beta\pi}{\lambda}\right) > 1.$$

This is a stable and hyperbolic limit cycle for the differential systems (2.1).

This completes the proof of statement (h_5).

2.1.2 Examples

The following examples are given to illustrate our result.

Example 1 If we take $n = p = 0$ and $a = -b = m = \delta = \lambda = \beta = 1$, then systems (2.1) reads

$$\begin{cases} x' = x + (y - x)\left(x^2 - xy + y^2\right), \\ y' = y - (y + x)\left(x^2 - xy + y^2\right). \end{cases} \tag{2.5}$$

The curve $U(x,y) = -x^4 + x^3y - 2x^2y^2 + xy^3 - y^4 = 0$ is an invariant algebraic curve of the polynomial differential system (2.5), with the cofactor

$$K(x,y) = -3x^2 + 4xy - 5y^2 + 4.$$

The system (2.5) is a cubic system which has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta, r_*) = e^{\theta}\sqrt{r_*^2 - 4\int_0^{\theta}\left(\frac{e^{-2\omega}}{2 - \sin 2\omega}\right)d\omega},$$

where $w \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \sqrt{\frac{2e^{4\pi}}{e^{4\pi} - 1}\int_0^{2\pi}\left(\frac{2}{2 - \sin 2\omega}e^{-2\omega}\right)d\omega} \simeq 1.1912.$$

Moreover,

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} = \exp(4\pi) > 1.$$

This limit cycle is a hyperbolic limit cycle. It is the results presented by J. Llibre and R. Benterki in [10].

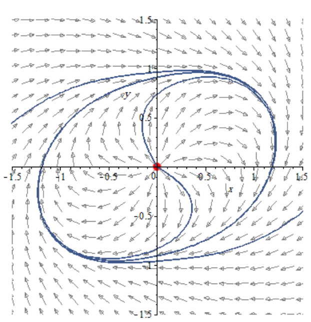


Figure 2.1: Limit cycle of system (2.5).

Example 2 If we take $a = \lambda = \beta = \delta = m = 1$ and $n = p = b = 0$, then systems (2.1) reads

$$\begin{cases} x' = x - x^3 + x^2y - xy^2 + y^3, \\ y' = y - x^3 - x^2y - xy^2 - y^3. \end{cases} \tag{2.6}$$

The curve $U(x,y) = -x^4 - 2x^2y^2 - y^4 = 0$ is an invariant algebraic curve of the polynomial differential system (2.6), with the cofactor

$$K(x,y) = -4x^2 - 4y^2 + 4.$$

System (2.6) is a cubic system which has an algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta, r_*) = 1,$$

where $\theta \in \mathbb{R}$, in cartesian coordinates are written as

$$x^2 + y^2 = 1.$$

Moreover,
$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0=r_*} = \exp(4\pi) > 1.$$

This limit cycle is a hyperbolic limit cycle.

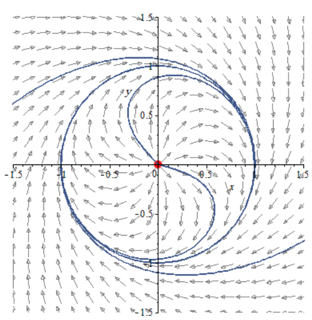


Figure 2.2: Limit cycle of system (2.6).

Example 3 If we take $a = -b = c = -d = m = p = \delta = 1, \lambda = \beta = 4$ and $n = 0$, then systems (2.1) reads

$$\begin{cases} x' = x - 4x^5 + 12x^4y - 20x^3y^2 + 20x^2y^3 - 12xy^4 + 4y^5, \\ y' = y - 4x^5 + 4x^4y - 4x^3y^2 - 4x^2y^3 + 4xy^4 - 4y^5. \end{cases} \tag{2.7}$$

The curve $U(x,y) = -4\left(x^2+y^2\right)\left(x^2-xy+y^2\right)^2 = 0$ is an invariant algebraic curve of the polynomial differential system (2.7), with the cofactor

$$K(x,y) = -16x^4 + 40x^3y - 72x^2y^2 + 56xy^3 - 32y^4 + 6.$$

The system (2.7) is a quintic system has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*) = e^{\theta}\sqrt[4]{\left(r_*^4 - 4\int_0^{\theta}\left(\frac{\exp(-4w)}{(2-\sin 2w)^2}\right)dw\right)},$$

where $w \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \exp(2\pi)\sqrt[4]{\frac{4\int_0^{2\pi}\left(\frac{\exp(-4w)}{(2-\sin 2w)^2}\right)dw}{\exp(8\pi)-1}} \simeq 0.81628.$$

Moreover,

$$\left.\frac{dr(2\pi,r_0)}{dr_0}\right|_{r_0=r_*} = \exp(8\pi) > 1.$$

This limit cycle is a hyperbolic limit cycle.

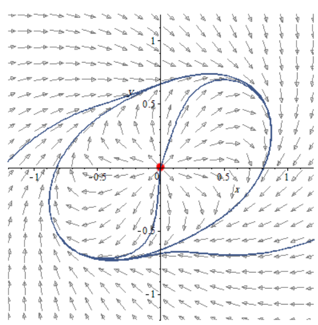


Figure 2.3: Limit cycle of system (2.7).

Example 4 If we take $a=-b=c=-d=1, p=\beta=\lambda=\delta=1, n=0$ and $m=2$, then systems (2.1) reads

$$\begin{cases} x' = x - x^7 + 4x^6y - 9x^5y^2 + 13x^4y^3 - 13x^3y^4 + 9x^2y^5 - 4xy^6 + y^7, \\ y' = y - x^7 + 2x^6y - 3x^5y^2 + x^4y^3 + x^3y^4 - 3x^2y^5 + 2xy^6 - y^7. \end{cases} \tag{2.8}$$

The curve $U(x,y) = \left(x^2+y^2\right)\left(xy-x^2-y^2\right)^3 = 0$ is an invariant algebraic curve of the polynomial differential system (2.8), with the cofactor

$$K(x,y) = -5x^6 + 18x^5y - 42x^4y^2 + 56x^3y^3 - 54x^2y^4 + 30xy^5 - 11y^6 + 8.$$

The system (2.8) has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = e^{\theta} \sqrt[6]{\left(r_*^6 - 6 \int_0^{\theta} \left(\frac{8 \exp(-6w)}{\left(1 - \frac{1}{2}\sin 2w\right)^3} \right) dw \right)},$$

where $w \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \sqrt[6]{\frac{\exp(12\pi)}{-1 + \exp(12\pi)} \int_0^{2\pi} \left(\frac{48 \exp(-6w)}{(2 - \sin 2w)^3} \right) dw} \simeq 1.1189.$$

Moreover,

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r_0 = r_*} = \exp(12\pi) > 1.$$

This limit cycle is a hyperbolic limit cycle.

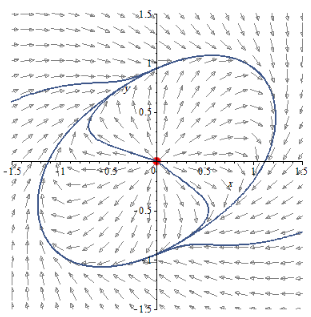


Figure 2.4: Limit cycle of system (2.8).

Example 5 If we take $a = p = \lambda = \delta = n = 1, b = d = -1, c = 4, \beta = 3$ and $\lambda = m = 2$, then systems (2.1) reads

$$\begin{cases} x' = x^3 + xy^2 + (2y - 3x)\left(x^2 - xy + y^2\right)^2 \left(4x^2 - xy + 4y^2\right), \\ y' = x^2 y + y^3 - (3y + 2x)\left(x^2 - xy + y^2\right)^2 \left(4x^2 - xy + 4y^2\right), \end{cases} \tag{2.9}$$

The curve $U(x, y) = \left(-8x^2 + 2xy - 8y^2\right)\left(x^2 + y^2\right)\left(x^2 - xy + y^2\right)^2 = 0$ is an invariant algebraic curve of the polynomial differential system (2.9), with the cofactor

$$K(x, y) = 8x^2 + 8y^2 - 78x^6 + 192x^5 y - 408x^4 y^2 + 456x^3 y^3 - 456x^2 y^4 + 240xy^5 - 114y^6.$$

The system (2.9) has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \exp(3\theta)\left(r_*^4 + \int_0^{\theta} \frac{-4\exp(-12w)}{\left(1-\frac{1}{2}\sin 2w\right)^2\left(4-\frac{1}{2}\sin 2w\right)}dw\right)^{\frac{1}{4}},$$

where $w \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \sqrt[4]{\frac{\int_0^{2\pi} \frac{4\exp(-12w)}{\left(1-\frac{1}{2}\sin 2w\right)^2\left(4-\frac{1}{2}\sin 2w\right)}dw}{1-\exp(-24\pi)}} \simeq 0.567\,83.$$

Moreover,

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0=r_*} = \exp(24\pi) > 1.$$

This limit cycle is a hyperbolic limit cycle.

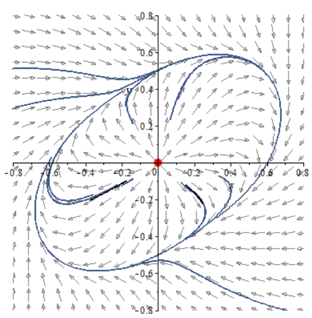


Figure 2.5: Limit cycle of system (2.9).

2.2 Stable hyperbolic limit cycles for a class of differential systems

Now, consider a multi-parameter planar polynomial differential systems of the form

$$\begin{cases} x' = x + (\alpha y - \beta x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} \\ \\ y' = y - (\beta y + \alpha x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} \end{cases} \tag{2.10}$$

where n, λ_i are positive integers and α, β, a_i, b_i, $i = 1,..,n$ are real constants.

2.2.1 Main result

Our first result on the critical point and the expression of invariant algebraic curves of the system (2.10) is the following.

Theorem 2.2. *Consider a multi-parameter planar polynomial differential systems* (2.10), *then the following statements hold.*

(1) *If $n \in \mathbb{N} - \{0\}$, $\lambda_i \in \mathbb{N} - \{0\}$, $b_i^2 - 4a_i^2 < 0$ for $i = 1, ..., n$ then the origin of coordinates $O(0,0)$ is the unique critical point at finite distance. Moreover, $O(0,0)$ is a star node.*

(2) *The curve $U(x,y) = \alpha\left(x^2 + y^2\right)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} = 0$, is an invariant algebraic curve of systems* (2.10) *with cofactor*

$$\begin{aligned} K(x,y) =& 2 + 2\sum_{i=1}^{i=n}\lambda_i - 2\beta\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} + \\ & (\alpha y - \beta x)\frac{\partial}{\partial x}\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} - \\ & (\beta y + \alpha x)\frac{\partial}{\partial y}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i}. \end{aligned}$$

Proof of **Theorem 2.2.**

Proof of statement (1)

We say that $A(x_0, y_0) \in \mathbb{R}^2$ is a critical point of systems (2.10) if

$$\begin{cases} x_0 + (\alpha y_0 - \beta x_0) \prod_{i=1}^{n} \left(a_i x_0^2 + b_i x_0 y_0 + a_i y_0^2 \right)^{\lambda_i} = 0, \\ y_0 - (\beta y_0 + \alpha x_0) \prod_{i=1}^{n} \left(a_i x_0^2 + b_i x_0 y_0 + a_i y_0^2 \right)^{\lambda_i} = 0, \end{cases}$$

then, $\alpha \left(x_0^2 + y_0^2 \right) \prod_{i=1}^{n} \left(a_i x_0^2 + b_i x_0 y_0 + a_i y_0^2 \right)^{\lambda_i} = 0$. According to the conditions $b_i^2 - 4a_i^2 < 0$, for $i = 1, ...n$, we have $x_0 = 0, y_0 = 0$ is the unique solution of this equation. Thus the origin is the unique critical point at finite distance.

We compute the Jacobian matrix of systems (2.10) evaluated at $O(0,0)$, we have

$$J = \left. \begin{pmatrix} \frac{\partial P}{\partial x}(x,y) & \frac{\partial P}{\partial y}(x,y) \\ \frac{\partial Q}{\partial x}(x,y) & \frac{\partial Q}{\partial y}(x,y) \end{pmatrix} \right|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$P(x,y) = x + (\alpha y - \beta x) \prod_{i=1}^{n} \left(a_i x^2 + b_i xy + a_i y^2 \right)^{\lambda_i}$$

and

$$Q(x,y) = y - (\beta y + \alpha x) \prod_{i=1}^{n} \left(a_i x^2 + b_i xy + a_i y^2 \right)^{\lambda_i}.$$

This matrix has repeated positive real eigenvalues $\lambda = 1 > 0$, then $O(0,0)$ is the unstable node of systems (2.10) .

This completes the proof of statement (1).

Proof of statement (2)

We prove that $U(x,y) = \alpha \left(x^2 + y^2 \right) \prod_{i=1}^{i=n} \left(a_i x^2 + b_i xy + a_i y^2 \right)^{\lambda_i} = 0$ is an invariant algebraic curve of the differential systems (2.10).

A computation we have

$$\frac{\partial U(x,y)}{\partial x} = 2\alpha x \prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} + \alpha\left(x^2+y^2\right)\frac{\partial}{\partial x}\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i},$$

$$\frac{\partial U(x,y)}{\partial y} = 2\alpha y \prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} + \alpha\left(x^2+y^2\right)\frac{\partial}{\partial y}\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i},$$

$$P(x,y) = x + (\alpha y - \beta x)\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} \text{ and}$$

$$Q(x,y) = y - (\beta y + \alpha x)\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i}.$$

After the substitution of $P(x,y)$, $Q(x,y)$, $\dfrac{\partial U(x,y)}{\partial x}$ and $\dfrac{\partial U(x,y)}{\partial y}$ in the linear partial differential equation , we obtain

$$\frac{\partial U(x,y)}{\partial x}P(x,y) + \frac{\partial U(x,y)}{\partial y}Q(x,y) = \frac{\partial U(x,y)}{\partial x}\left(x + (\alpha y - \beta x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i}\right)$$
$$+\frac{\partial U(x,y)}{\partial y}\left(y - (\beta y + \alpha x)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i}\right).$$

Then, taking into account that

$$x\frac{\partial}{\partial x}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} + y\frac{\partial}{\partial y}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} = 2\left(\sum_{i=1}^{i=n}\lambda_i\right)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i},$$

we have

$$\frac{\partial U(x,y)}{\partial x}P(x,y) + \frac{\partial U(x,y)}{\partial y}Q(x,y) = (2 + 2\sum_{i=1}^{i=n}\lambda_i - 2\beta\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} +$$
$$(\alpha y - \beta x)\frac{\partial}{\partial x}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} - (\beta y + \alpha x)\frac{\partial}{\partial y}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i})U(x,y)$$

Therefore, $U(x,y) = \alpha\left(x^2+y^2\right)\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} = 0$ is an invariant algebraic curve of the polynomial differential systems (2.10) with the cofactor

$$\begin{aligned} K(x,y) =& 2 + 2\sum_{i=1}^{i=n}\lambda_i - 2\beta\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} + \\ & (\alpha y - \beta x)\frac{\partial}{\partial x}\prod_{i=1}^{i=n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i} - \\ & (\beta y + \alpha x)\frac{\partial}{\partial y}\prod_{i=1}^{n}\left(a_i x^2 + b_i xy + a_i y^2\right)^{\lambda_i}. \end{aligned}$$

Our second result on the existence of a first integral and explicit expression of a limit cycles of systems (2.10) is the following.

Theorem 2.3. *Consider a multi-parameter planar polynomial differential systems* (2.10), *then the following statements hold.*

(1) *The systems* (2.10) *has the first integral*

$$H(x,y) = \frac{\left(x^2+y^2\right)^{\lambda_1+\ldots+\lambda_n}}{\exp\left(\frac{2\beta}{\alpha}(\lambda_1+\ldots+\lambda_n)\arctan\frac{y}{x}\right)} + F\left(\arctan\frac{y}{x}\right),$$

where $F(\theta) = \int_0^\theta \left(\frac{2(\lambda_1+\ldots+\lambda_n)\exp(-2(\lambda_1+\ldots+\lambda_n)w)}{\alpha\prod_{i=1}^n\left(a_i+\frac{1}{2}b_i\sin 2w\right)^{\lambda_i}}\right)dw.$

(2) *If* $n \in \mathbb{N}-\{0\}, \alpha > 0,\ \beta > 0,\ \lambda_i \in \mathbb{N}-\{0\},\ 2a_i > |b_i|$, *for* $i = 1,\ldots,n$ *then systems* (2.10) *has non-algebraic limit cycle* (Γ), *explicitly given in polar coordinates* (r,θ) *by the equation*

$$r(\theta,r_*) = \left(\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1+\ldots+\lambda_n)\right)}{-1+\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1+\ldots+\lambda_n)\right)}F(2\pi) - F(\theta)\right)^{\frac{1}{2(\lambda_1+\ldots+\lambda_n)}} \exp\left(\frac{\beta}{\alpha}\theta\right),$$

where

$$r_* = \left(\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1+\ldots+\lambda_n)\right)}{-1+\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1+\ldots+\lambda_n)\right)}F(2\pi)\right)^{\frac{1}{2(\lambda_1+\ldots+\lambda_n)}}.$$

Moreover, this limit cycle is a stable hyperbolic limit cycle.

Proof of **Theorem 2.3.**

In order to prove our results (1) and (2), we write the polynomial differential systems (2.10) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the systems become

$$\begin{cases} r' = r - \left(\beta\prod_{i=1}^n\left(a_i+\frac{1}{2}b_i\sin 2\theta\right)^{\lambda_i}\right)r^{2(\lambda_1+\ldots+\lambda_n)+1}, \\ \theta' = -\left(\alpha\prod_{i=1}^n\left(a_i+\frac{1}{2}b_i\sin 2\theta\right)^{\lambda_i}\right)r^{2(\lambda_1+\ldots+\lambda_n)}, \end{cases} \tag{2.11}$$

We have $n \in \mathbb{N}-\{0\}$, $\lambda_i \in \mathbb{N}-\{0\}$, $\alpha > 0$, $2a_i > |b_i|$, for $i = 1,...,n$ then θ' is negative for all $t \in \mathbb{R}$, the orbits $(r(t),\theta(t))$ of systems (2.11) have the opposite orientation with respect to those $(x(t),y(t))$ of systems (2.10).

Taking θ as an independent variable, we obtain the Bernoulli equation

$$\frac{dr(\theta)}{d\theta} = \frac{\beta}{\alpha}r - \frac{r^{1-2(\lambda_1+...+\lambda_n)}}{\alpha\prod_{i=1}^{n}\left(a_i+\frac{1}{2}b_i\sin 2\theta\right)^{\lambda_i}}. \tag{2.12}$$

Via the change of variables $\rho = r^{2(\lambda_1+...+\lambda_n)}$, this Bernoulli equation (3.4) is transformed into the linear equation

$$\frac{d\rho(\theta)}{d\theta} = \frac{2\beta}{\alpha}(\lambda_1+...+\lambda_n)\rho - \frac{2(\lambda_1+...+\lambda_n)}{\alpha\prod_{i=1}^{n}\left(a_i+\frac{1}{2}b_i\sin 2\theta\right)^{\lambda_i}}. \tag{2.13}$$

The general solution of linear equation (2.13) is

$$\rho(\theta) = (h-F(\theta))\exp\left(\frac{2\beta}{\alpha}(\lambda_1+...+\lambda_n)\theta\right),$$

where $h \in \mathbb{R}$ and $F(\theta) = \displaystyle\int_0^\theta \left(\frac{2(\lambda_1+...+\lambda_n)\exp(-2(\lambda_1+...+\lambda_n)w)}{\alpha\prod_{i=1}^{n}\left(a_i+\frac{1}{2}b_i\sin 2w\right)^{\lambda_i}}\right)dw.$

Consequently, the general solution of (2.12) is

$$r(\theta) = (h-F(\theta))^{\frac{1}{2(\lambda_1+...+\lambda_n)}}\exp\left(\frac{\beta}{\alpha}\theta\right),$$

where $h \in \mathbb{R}$.

From this solution we obtain a first integral in the variables (x,y) of the form

$$H(x,y) = \frac{\left(x^2+y^2\right)^{(\lambda_1+...+\lambda_n)}}{\exp\left(\frac{2\beta}{\alpha}(\lambda_1+...+\lambda_n)\arctan\frac{y}{x}\right)} + F\left(\arctan\frac{y}{x}\right).$$

Hence, statement (1) is proved.

Notice that systems (2.10) has a periodic orbit if and only if equation (2.12) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (2.12) that fulfills $r(0,r_*) = r(2\pi,r_*)$ and $r(\theta,r_*) > 0$ for any θ in $(0,2\pi)$.

The solution $r(\theta,r_0)$ of the differential equation (2.12) such that $r(0,r_0) = r_0$ is

$$r(\theta,r_0) = \left(r_0^{2(\lambda_1+...+\lambda_n)} - F(\theta)\right)^{\frac{1}{2(\lambda_1+...+\lambda_n)}}\exp\left(\frac{\beta}{\alpha}\theta\right),$$

where $r_0 = r(0)$.

A periodic solution of system (2.10) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to a unique value $r_0 = r_*$, given by

$$r_* = \left(\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}{-1 + \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)} F(2\pi) \right)^{\frac{1}{2(\lambda_1 + ... + \lambda_n)}}.$$

The r_* is the intersection of the periodic orbit with the OX_+ axes.

After the substitution of this value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \left(\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}{-1 + \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)} F(2\pi) - F(\theta) \right)^{\frac{1}{2(\lambda_1 + ... + \lambda_n)}} \exp\left(\frac{\beta}{\alpha}\theta\right).$$

In what follows it is proved that $r(\theta, r_*) > 0$. Indeed

$$\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}{-1 + \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)} F(2\pi) - F(\theta) = \frac{-F(2\pi)}{1 - \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}$$
$$+ \int_\theta^{2\pi} \left(\frac{2(\lambda_1 + ... + \lambda_n)\exp(-2(\lambda_1 + ... + \lambda_n)w)}{\alpha \prod_{i=1}^n \left(a_i + \frac{1}{2} b_i \sin 2w\right)^{\lambda_i}} \right) dw > 0$$

According to the conditions $n \in \mathbb{N} - \{0\}$, $\lambda_i \in \mathbb{N} - \{0\}, \alpha > 0$, $\beta > 0$, $2a_i > |b_i|$, for $i = 1, ..., n$, hence $a_i + \dfrac{1}{2} b_i \sin 2w > 0$ for all $\theta \in (0, \pi)$, then we have

$$\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}{-1 + \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)} F(2\pi) - F(\theta),$$

this ensures that r_* and $r(\theta, r_*)$ are well defined for all $\theta \in (0, \pi)$, therefore we have $r_* > 0$ and $r(\theta, r_*) > 0$ for all $\theta \in (0, \pi)$ and the limit cycle do not pass through the equilibrium point $O(0, 0)$ of systems (2.10). This is the limit cycle for the differential systems (2.10), we note it by (Γ).

This limit cycle (Γ) is not algebraic, more precisely, in cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\dfrac{y}{x}\right)$, the curve (Γ) defined by this limit cycle is (Γ): $L(x, y) = 0$ where

$$\begin{aligned} L(x,y) &= \left(x^2 + y^2\right)^{\lambda_1 + ... + \lambda_n} - \exp\left(2\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\arctan\frac{y}{x}\right) \\ &\quad \left(\frac{\exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)}{-1 + \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1 + ... + \lambda_n)\right)} F(2\pi) - F\left(\arctan\frac{y}{x}\right) \right). \end{aligned}$$

According to the conditions, we have $\frac{\beta}{\alpha}(\lambda_1+...+\lambda_n) \neq 0$, then the non-algebraic expression $\exp\left(2\frac{\beta}{\alpha}(\lambda_1+...+\lambda_n)\arctan\frac{y}{x}\right)$ appears in the $L(x,y)$, hence the expression $L(x,y)$ is not algebraic. Consequently, (Γ): $L(x,y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

In order to prove the hyperbolicity of the limit cycle it is sufficient to use the Poincaré return map. A computation shows that

$$\left.\frac{dr(2\pi,r_0)}{dr_0}\right|_{r_0=r_*} = \exp\left(4\pi\frac{\beta}{\alpha}(\lambda_1+...+\lambda_n)\right) > 1,$$

Therefore the limit cycle (Γ) of the differential systems (2.10) is stable hyperbolic limit cycle.

This completes the proof of statement (2) .

2.2.2 Examples

The following examples are given to illustrate our results.

Example 1 If we take $\alpha = \beta = n = a_1 = \lambda_1 = 1$ and $b_1 = -1$, then systems (2.10) reads

$$\begin{cases} x' = x + (y-x)\left(x^2 - xy + y^2\right), \\ y' = y - (y+x)\left(x^2 - xy + y^2\right). \end{cases} \tag{2.14}$$

The curve $U(x,y) = \left(x^2+y^2\right)\left(x^2-xy+y^2\right) = 0$, is an invariant algebraic curve of system (2.14) with cofactor $K(x,y) = -3x^2 + 4xy - 5y^2 + 4$.

System (2.14) is a cubic system that has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*) = e^{\theta}\sqrt{r_*^2 - 4\int_0^{\theta}\left(\frac{e^{-2w}}{2-\sin 2w}\right)dw},$$

where $w \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \sqrt{\frac{2e^{4\pi}}{e^{4\pi}-1}\int_0^{2\pi}\left(\frac{2}{2-\sin 2w}e^{-2w}\right)dw} \simeq 1.1912.$$

Moreover,

$$\left.\frac{dr(2\pi,r_0)}{dr_0}\right|_{r_0=r_*} = \exp(4\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle. It is the results presented by J. Llibre and R. Benterki in [10].

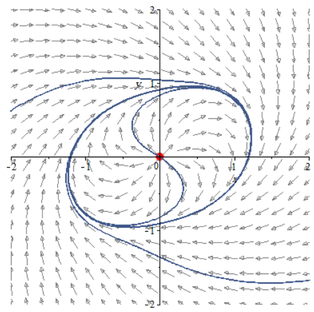


Figure 2.6: Limit cycle of system (2.14).

Example 2 If we take $n = 2$, $\alpha = \beta = \lambda_2 = 1$, $\lambda_1 = 2$, $a_1 = a_2 = 3, b_1 = -1$ and $b_2 = -2$, then systems (2.10) reads

$$\begin{cases} x' = x + (y - x)\left(3x^2 - xy + 3y^2\right)^2\left(3x^2 - 2xy + 3y^2\right), \\ y' = y - (y + x)\left(3x^2 - xy + 3y^2\right)^2\left(3x^2 - 2xy + 3y^2\right). \end{cases} \tag{2.15}$$

The curve $U(x, y) = \left(x^2 + y^2\right)\left(3x^2 - xy + 3y^2\right)^2\left(3x^2 - 2xy + 3y^2\right) = 0$, is an invariant algebraic curve of system (2.15) with cofactor

$$K(x, y) = 8 - 2\left(6x^2 - 3xy + 7y^2\right)\left(5x^2 - 3xy + 6y^2\right)\left(3x^2 - xy + 3y^2\right).$$

System (2.15) is a quintic system that has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \left(\frac{\exp(12\pi)}{-1 + \exp(12\pi)} F(2\pi) - F(\theta)\right)^{\frac{1}{6}} \exp(\theta).,$$

where $\theta \in \mathbb{R}$, $F(\theta) = \int_0^\theta \left(\frac{6\exp(-6w)}{\left(3 - \frac{1}{2}\sin 2w\right)^2 (3 - \sin 2w)}\right) dw$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \left(\frac{\exp(12\pi)}{-1 + \exp(12\pi)} \int_0^{2\pi} \left(\frac{6\exp(-6w)}{\left(3 - \frac{1}{2}\sin 2w\right)^2 (3 - \sin 2w)}\right) dw\right)^{\frac{1}{6}} \simeq 0.60031.$$

Moreover,

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} = \exp(12\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

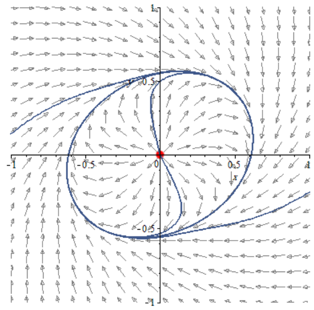


Figure 2.7: Limit cycle of system (2.15).

Example 3 If we take $n = 3$, $\alpha = 3$, $\beta = 2$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$, $a_1 = 10$, $b_1 = 12$, $a_2 = 7$, $b_2 = 5$, $a_3 = 4$ and $b_3 = 1$, then systems (2.10) reads

$$\begin{cases} x' = x + (3y - 2x)\left(10x^2 + 12xy + 10y^2\right)^3 \\ \left(7x^2 + 5xy + 7y^2\right)^4 \left(4x^2 + xy + 4y^2\right)^5, \\ \\ y' = y - (2y + 3x)\left(10x^2 + 12xy + 10y^2\right)^3 \\ \left(7x^2 + 5xy + 7y^2\right)^4 \left(4x^2 + xy + 4y^2\right)^5, \end{cases} \tag{2.16}$$

The curve

$$U(x,y) = \left(x^2 + y^2\right)\left(10x^2 + 12xy + 10y^2\right)^3 \left(7x^2 + 5xy + 7y^2\right)^4 \left(4x^2 + xy + 4y^2\right)^5 = 0,$$

is an invariant algebraic curve of systems (2.16) with cofactor

$$K(x,y) = 26 - 8\left(6xy + 5x^2 + 5y^2\right)^2 \left(5xy + 7x^2 + 7y^2\right)^3 \left(xy + 4x^2 + 4y^2\right)^4 M(x,y),$$

where $M(x,y) = 10\,517x^6 + 19\,959x^5y + 35\,881x^4y^2 + 33\,072x^3y^3 + 27\,247x^2y^4 + 11\,553xy^5 + 4043y^6$.

This system (2.16) has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta, r_*) = \left(\frac{\exp(32\pi)}{-1 + \exp(32\pi)} F(2\pi) - F(\theta)\right)^{\frac{1}{24}} \exp\left(\frac{2}{3}\theta\right),$$

where $\theta \in \mathbb{R}$ and $F(\theta) = \displaystyle\int_0^\theta \left(\frac{24\exp(-24w)}{3(10 + 6\sin 2w)^3 \left(7 + \frac{5}{2}\sin 2w\right)^4 \left(4 + \frac{1}{2}\sin 2w\right)^5}\right) dw$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \left(\frac{\exp(32\pi)}{-1 + \exp(32\pi)} \int_0^{2\pi} \left(\frac{24\exp(-24w)}{3(10 + 6\sin 2w)^3 \left(7 + \frac{5}{2}\sin 2w\right)^4 \left(4 + \frac{1}{2}\sin 2w\right)^5}\right) dw\right)^{\frac{1}{24}} \simeq 0.383\,65.$$

Moreover,
$$\left.\frac{dr(2\pi,r_0)}{dr_0}\right|_{r_0=r_*}=\exp(32\pi)>1.$$
This limit cycle is a stable hyperbolic limit cycle.

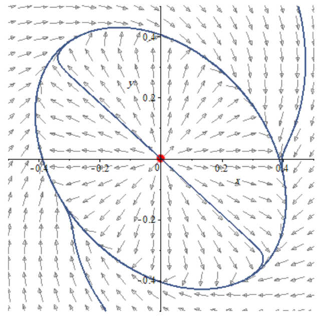


Figure 2.8: Limit cycle of system (2.16).

Example 4 If we take $n=2$, $\alpha=\beta=1$, $\lambda_1=4$, $\lambda_2=3$, $a_1=5$, $b_1=-1$, $a_2=4$ and $b_2=-3$, then system (2.10) reads

$$\begin{cases} x'=x+(y-x)\left(5x^2-xy+5y^2\right)^4\left(4x^2-3xy+4y^2\right)^3, \\ \\ y'=y-(y+x)\left(5x^2-xy+5y^2\right)^4\left(4x^2-3xy+4y^2\right)^3. \end{cases} \tag{2.17}$$

The curve
$$U(x,y)=\left(x^2+y^2\right)\left(5x^2-xy+5y^2\right)^4\left(4x^2-3xy+4y^2\right)^3=0,$$
is an invariant algebraic curve of system (2.17) with cofactor
$$K(x,y)=16+\left(4x^2-3xy+4y^2\right)^2\left(xy-5x^2-5y^2\right)^3 M(x,y),$$
where $M(x,y)=259x^4-283x^3y+688x^2y^2-325xy^3+381y^4$.

This system (2.17) has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is
$$r(\theta,r_*)=\left(\frac{\exp(28\pi)}{-1+\exp(28\pi)}F(2\pi)-F(\theta)\right)^{\dfrac{1}{14}}\exp(\theta),$$
where $\theta\in\mathbb{R}$ and $F(\theta)=\displaystyle\int_0^\theta\left(\frac{14\exp(-14w)}{\left(5-\frac{1}{2}\sin 2w\right)^4\left(4-\frac{3}{2}\sin 2w\right)^3}\right)dw$, and the intersection of the limit cycle with the OX_+ axes is the point having r_*
$$r_*=\left(\frac{\exp(28\pi)}{-1+\exp(28\pi)}\int_0^{2\pi}\left(\frac{14\exp(-14w)}{\left(5-\frac{1}{2}\sin 2w\right)^4\left(4-\frac{3}{2}\sin 2w\right)^3}\right)dw\right)^{\dfrac{1}{14}}\simeq 0.477\,65.$$

Moreover,

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0=r_*} = \exp(28\pi) > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

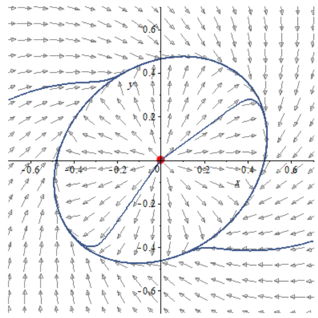


Figure 2.9: Limit cycle of system (2.17).

Chapter 3

On the coexistence of algebraic and non-algebraic limit cycles for some classes of planar differential systems

Limit cycles of planar polynomial differential systems are not, in general, algebraic. It is not easy work to decide if a limit cycle is algebraic or not. In this chapter we deal with the situation of coexistence of algebraic and non-algebraic limit cycles, we introduce tow families: the first one is of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = xS_4(x,y) + P_7(x,y) + xR_8(x,y), \\ \\ y' = \dfrac{dy}{dt} = yS_4(x,y) + Q_7(x,y) + yR_8(x,y), \end{cases}$$

where $S_4(x,y), P_7(x,y), Q_7(x,y)$ and $R_8(x,y)$ are real polynomials.

The second family is of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + \left(x^2+y^2\right)^2 \left(P_3(x,y) - x\left(x^2+y^2\right)^3 R_2(x,y)\right), \\ \\ y' = \dfrac{dy}{dt} = y + \left(x^2+y^2\right)^2 \left(Q_3(x,y) - y\left(x^2+y^2\right)^3 R_2(x,y)\right), \end{cases}$$

where $P_3(x,y), Q_3(x,y)$ and $R_2(x,y)$ are real polynomials.

For each of the families, at first we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable, with an explicit expression of a first integral. Moreover, we provide sufficient conditions for the polynomial differential systems to possess two limit cycles explicitly given: one is algebraic and the other is shown to be non-algebraic. Finally, some concrete examples are introduced to illustrate the applicability of our results.

3.1 A class of planar differential systems with explicit expression for two limit cycles

Consider a multi-parameter planar polynomial differential systems of degree nine of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = xS_4(x,y) + P_7(x,y) + xR_8(x,y), \\ \\ y' = \dfrac{dy}{dt} = yS_4(x,y) + Q_7(x,y) + yR_8(x,y), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} P_7(x,y) &= \frac{1}{3}\left(x^2+y^2\right)^2\left((2a-b)x^3 + (15d-6c)x^2y + (2b-a)xy^2 + (6d-3c)y^3\right), \\ Q_7(x,y) &= -\frac{1}{3}\left(x^2+y^2\right)^2\left((6d-3c)x^3 + (b-2a)x^2y - 3dxy^2 + (a-2b)y^3\right), \\ S_4(x,y) &= \alpha x^4 + \lambda x^3 y + \delta x^2y^2 + \lambda xy^3 + \eta y^4, \\ \text{and } R_8(x,y) &= -\frac{1}{3}\left(x^2+y^2\right)^2((3\alpha+2a-b)x^4 + (3\lambda-3c+9d)x^3y + (3\lambda-3c+9d)xy^3 \\ &\quad + (a+b+3\delta)x^2y^2 + (2b-a+3\eta)y^4), \end{aligned}$$

in which a, b, c, d, α, δ, λ and η are real constants.

We define the trigonometric functions

$$\begin{aligned} F(\theta) &= \frac{1}{8}(3\alpha+\delta+3\eta) + \frac{1}{2}\lambda\sin 2\theta + \frac{1}{2}(\alpha-\eta)\cos 2\theta + \frac{1}{8}(\alpha-\delta+\eta)\cos 4\theta, \\ G(\theta) &= \frac{1}{6}(a+b) + \frac{1}{2}(a-b)\cos 2\theta + \frac{1}{2}(3d-c)\sin 2\theta, \\ K(\theta) &= -\frac{1}{6}a - \frac{1}{6}b - \frac{3}{8}\alpha - \frac{1}{8}\delta - \frac{3}{8}\eta + \frac{1}{2}(c-3d-\lambda)\sin 2\theta + \frac{1}{8}(\delta-\alpha-\eta)\cos 4\theta \\ &\quad + \frac{1}{2}(b-a-\alpha+\eta)\cos 2\theta, \\ M(\theta) &= \int_0^\theta \left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt, \\ \text{and } N(\theta) &= \exp\left(\int_0^\theta \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right). \end{aligned}$$

3.1.1 Main result

Our main result is contained in the following Theorem.

Theorem 3.1. *Consider a multi-parameter planar polynomial differential systems* (3.1), *then the following statements hold.*

(1) If $2d-c \neq 0$, then the origin of coordinates $O(0,0)$ is the unique critical point at finite distance.

(2) The curve $U(x,y) = x^2 + y^2 - 1$, is an invariant algebraic curve of systems (3.1) *with cofactor*

$$\begin{aligned} K(x,y) &= -\frac{2}{3}\left(x^2+y^2\right)((2a-b+3\alpha)x^6 + 3\alpha x^4 + 3\eta y^4 + (9d-3c+3\lambda)xy\left(x^2+y^2\right)^2 + \\ &\quad 3x^2y^2\left((a+\alpha+\delta)x^2 + (b+\delta+\eta)y^2 + \delta\right) + 3\lambda xy\left(x^2+y^2\right) + (2b-a+3\eta)y^6). \end{aligned}$$

(3) The systems (3.1) *has the first integral*

$$H(x,y) = \frac{N\left(\arctan\frac{y}{x}\right) + \left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1}.$$

(4) The systems (3.1) *has an explicit limit cycle, given in cartesian coordinates by*

$$(\Gamma_1): x^2+y^2-1=0.$$

(5) If

$$\begin{gathered} \frac{2}{3}a + \frac{2}{3}b + 3\alpha + \delta + 3\eta > 2|c-3d-2\lambda| + 2\left|b-a-2\alpha+2\eta\right| + \left|\delta-\alpha-\eta\right|, \\ -\frac{1}{3}a - \frac{1}{3}b - \frac{3}{4}\alpha - \frac{1}{4}\delta - \frac{3}{4}\eta > |c-3d-\lambda| + \frac{1}{4}\left|\delta-\alpha-\eta\right| + \left|b-a-\alpha+\eta\right|, \\ \delta \neq \alpha+\eta \text{ and } c < 2d, \end{gathered} \tag{3.2}$$

then the systems (3.1) *has another limit cycle* (Γ_2)*, explicitly given in polar coordinates* (r,θ) *by*

$$r(\theta, r_*) = \sqrt{\frac{(N(2\pi)-1)(N(\theta)+M(\theta)) + M(2\pi)}{(N(2\pi)-1)M(\theta)+M(2\pi)}}.$$

Moreover, the limit cycle (Γ_1) *lies inside the limit cycle* (Γ_2).

Proof.

Proof of statement (1).
By definition, $A(x_*, y_*) \in \mathbb{R}^2$ is a critical point of systems (3.1) if

$$\begin{cases} x_* S_4(x_*, y_*) + P_7(x_*, y_*) + x_* R_8(x_*, y_*) = 0, \\ y_* S_4(x_*, y_*) + Q_7(x_*, y_*) + y_* R_8(x_*, y_*) = 0, \end{cases}$$

we have $y_* P_7(x_*, y_*) - x_* Q_7(x_*, y_*) = (2d - c)\left(x_*^2 + y_*^2\right)^4 = 0$. According to the condition $2d - c \neq 0$, then $x_* = 0, y_* = 0$ is the unique of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of statement (1) of Theorem 1.

Proof of statement (2).

A computation shows that $U(x,y) = x^2+y^2-1$ satisfies the linear partial differential equation (**??**), the associated cofactor being

$$\begin{aligned} K(x,y) \;\; = \;\; & -\frac{2}{3}\left(x^2+y^2\right)((2a-b+3\alpha)x^6 + 3\alpha x^4 + 3\eta y^4 + (9d-3c+3\lambda)xy\left(x^2+y^2\right)^2 + \\ & 3x^2y^2\left((a+\alpha+\delta)x^2 + (b+\delta+\eta)y^2 + \delta\right) + 3\lambda xy\left(x^2+y^2\right) + (2b-a+3\eta)y^6), \end{aligned}$$

then the curve $U(x,y) = 0$ is an invariant algebraic curve of systems (3.1) with cofactor $K(x,y)$.

This completes the proof of statement (2) .

Proof of statements (3), (4) and (5)

In order to prove our results (3), (4) and (5) we write the polynomial differential systems (3.1) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the systems become

$$\begin{cases} r' = \dfrac{dr}{dt} = F(\theta)r^5 + G(\theta)r^7 + K(\theta)r^9, \\ \\ \theta' = \dfrac{d\theta}{dt} = (c-2d)r^6. \end{cases} \tag{3.3}$$

where the trigonometric functions $F(\theta)$, $G(\theta)$ and $K(\theta)$ are given in introduction.

According to $c < 2d$, we get θ' is negative for all $t \in \mathbb{R}$, the orbits $(r(t),\theta(t))$ of systems (3.3) have the opposite orientation with respect to those $(x(t),y(t))$ of systems (3.1).

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = \frac{F(\theta)}{c-2d}\frac{1}{r} + \frac{G(\theta)}{c-2d}r + \frac{K(\theta)}{c-2d}r^3. \tag{3.4}$$

Via the change of variables $\rho = r^2$, this equation (3.4) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = \frac{2F(\theta)}{c-2d} + \frac{2G(\theta)}{c-2d}\rho + \frac{2K(\theta)}{c-2d}\rho^2. \tag{3.5}$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $\rho = z+1$ we obtain the Bernoulli equation

$$\frac{dz}{d\theta} = \left(\frac{2G(\theta)+4K(\theta)}{c-2d}\right)z + \frac{2K(\theta)}{c-2d}z^2. \tag{3.6}$$

We note that $z = 0$ is solutions for (3.6), assume now that $z \neq 0$ by introducing the standard change of variables $y = \dfrac{1}{z}$ we obtain the linear equation

$$\frac{dy}{d\theta} = \left(\frac{2G(\theta) + 4K(\theta)}{2d - c}\right) y + \frac{2K(\theta)}{2d - c}. \tag{3.7}$$

The general solution of linear equation (3.7) is

$$y(\theta) = \frac{\mu + M(\theta)}{N(\theta)},$$

where $\mu \in \mathbb{R}$.

Consequently, the general solution of equation (3.6) is

$$z(\theta) = 0, \quad z(\theta) = \frac{N(\theta)}{\mu + M(\theta)},$$

where $\mu \in \mathbb{R}$.

Then the general solution of equation (3.5) is

$$\rho(\theta) = 1, \quad \rho(\theta) = \frac{\mu + N(\theta) + M(\theta)}{\mu + M(\theta)},$$

where $\mu \in \mathbb{R}$.

Consequently, the general solution of (3.4) is

$$r(\theta, \mu) = 1, \quad r(\theta, \mu) = \left(\frac{\mu + N(\theta) + M(\theta)}{\mu + M(\theta)}\right)^{\frac{1}{2}},$$

where $\mu \in \mathbb{R}$.

From this solution we obtain a first integral in the variables (x, y) of the form

$$H(x, y) = \frac{N\left(\arctan \frac{y}{x}\right) + \left(1 - x^2 - y^2\right) M\left(\arctan \frac{y}{x}\right)}{x^2 + y^2 - 1}.$$

Hence, statement (3) is proved.

The curves $H = \mu$ with $\mu \in \mathbb{R}$, which are formed by trajectories of the differential systems (3.1), in cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 &= 1, \\ x^2 + y^2 &= \frac{\mu + N\left(\arctan \frac{y}{x}\right) + M\left(\arctan \frac{y}{x}\right)}{\mu + M\left(\arctan \frac{y}{x}\right)}, \end{aligned}$$

where $\mu \in \mathbb{R}$.

Notice that system (3.1) has a periodic orbit if and only if equation (3.4) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (3.4) that fulfills $r(0,r_*) = r(2\pi,r_*)$ and $r(\theta,r_*) > 0$ for any θ in $[0,2\pi]$.

The solution $r(\theta,r_0)$ of the differential equation (3.4) such that $r(0,r_0) = r_0$ is

$$r(\theta,r_0) = \sqrt{\frac{N(\theta)+M(\theta)+\frac{1}{-1+r_0^2}}{M(\theta)+\frac{1}{-1+r_0^2}}}$$

where $r_0 = r(0)$.

We have the particular solution $\rho(\theta) = 1$ of the differential equation (3.4), from this solution we obtain $r^2(\theta,1) = 1 > 0$, for all $\theta \in [0,\pi]$ is a particular solution of the differential equation (3.4). This is the limit cycle for the differential systems (3.1), corresponding of course to an invariant algebraic curve $U(x,y) = x^2+y^2-1 = 0$.

More precisely, in cartesian coordinates $r^2 = x^2+y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, the curve (Γ_1) defined by this limit cycle is (Γ_1): $x^2+y^2-1 = 0$.

Hence, statement (4) is proved.

A periodic solution of systems (3.1) must satisfy the condition $r(2\pi,r_0) = r(0,r_0)$, which leads to unique value $r_0 = r_*$, given by

$$r_* = \sqrt{\frac{N(2\pi)+M(2\pi)-1}{M(2\pi)}},$$

r_* is the intersection of the periodic orbit with the OX_+ axes.

After the substitution of this value of r_* into $r(\theta,r_0)$ we obtain

$$r(\theta,r_*) = \sqrt{\frac{(N(2\pi)-1)(N(\theta)+M(\theta))+M(2\pi)}{(N(2\pi)-1)M(\theta)+M(2\pi)}}.$$

In what follows it is proved that $r(\theta,r_*) > 0$. Indeed

$$\begin{aligned} M(2\pi)-M(\theta) &= \int_0^{2\pi}\left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t\left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt \\ &\quad + \int_\theta^0\left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t\left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt \\ &= \int_\theta^{2\pi}\left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t\left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt \end{aligned}$$

According to the conditions (3.2), hence $\frac{G(\theta)+2K(\theta)}{2d-c} < 0$ and $\frac{K(\theta)}{2d-c} > 0$ for all $\theta \in (0,\pi)$, then we have $M(2\pi)-M(\theta) > 0$ and $N(2\pi) > 1$, this ensures that r_* and $r(\theta,r_*)$ are well defined for all $\theta \in (0,\pi)$, therefore we have $r_* > 0$ and $r(\theta,r_*) > 0$ for all $\theta \in [0,\pi]$ and the limit cycle do not pass through the equilibrium point $O(0,0)$ of system (3.1). This is the second limit cycle for the differential systems (3.1), we note it by (Γ_2).

According to the conditions (3.2), we get

$$\begin{aligned} M(\theta) &= \int_0^{\theta}\left(\frac{2K(t)}{2d-c}\exp\left(\int_0^{t}\left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt > 0 \\ \text{and } N(\theta) &= \exp\left(\int_0^{\theta}\left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right) > 1, \end{aligned}$$

for all $\theta \in [0,\pi]$, then we have $r_* = \sqrt{1+\frac{N(2\pi)-1}{M(2\pi)}} > 1$. Moreover,

$$r(\theta, r_*) = \sqrt{1+\frac{(N(2\pi)-1)N(\theta)}{(N(2\pi)-1)M(\theta)+M(2\pi)}} > 1,$$

this justified that the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

We conclude that system (3.1) has two limit cycles (Γ_1) and (Γ_2).

This completes the proof of statement (5). □

3.1.2 Examples

The following examples are given to illustrate our results.

Example 1 If we take $a=b=-50, c=-3, d=-1, \alpha=\eta=10, \lambda=1$ and $\delta=28$, then systems (3.1) reads

$$\begin{cases} x' = x\left(10x^4+x^3y+28x^2y^2+xy^3+10y^4\right)+\frac{1}{3}(3y-50x)\left(x^2+y^2\right)^3 \\ -\frac{1}{3}x\left(x^2+y^2\right)^2\left(-20x^4+3x^3y-16x^2y^2+3xy^3-20y^4\right), \\ \\ y' = y\left(10x^4+x^3y+28x^2y^2+xy^3+10y^4\right)-\frac{1}{3}(3x+50y)\left(x^2+y^2\right)^3 \\ -\frac{1}{3}y\left(x^2+y^2\right)^2\left(-20x^4+3x^3y-16x^2y^2+3xy^3-20y^4\right), \end{cases} \tag{3.8}$$

The curve $x^2+y^2-1=0$ is an invariant algebraic curve of system (3.8) with cofactor

$$\begin{aligned} K(x,y) &= 2\left(x^2+y^2\right)(8\left(x^6+y^6\right)-12\left(x^4+y^4\right)+3xy\left(x^4+x^2+y^2+y^4\right) \\ &\quad +2x^2y^2\left(11x^2+3xy+11y^2-13\right)). \end{aligned}$$

The system (3.8) has the first integral

$$H(x,y) = \frac{N\left(\arctan\frac{y}{x}\right)+\left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1},$$

where $N(\theta) = \exp\left(1 + \frac{32}{3}\theta - \cos 2\theta - \sin 4\theta\right)$ and

$$M(\theta) = -\frac{1}{2}\exp\left(\frac{32}{3}\theta - \cos 2\theta - \sin 4\theta\right) + \frac{50e}{3}\int_0^{\theta} \exp\left(\frac{32}{3}t - \cos 2t - \sin 4t\right)dt.$$

The system (3.8) has the limit cycle (Γ_1) whose expression is (Γ_1): $x^2 + y^2 - 1 = 0$.

The system (3.8) has another limit cycle (Γ_2) whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \sqrt{\frac{(N(2\pi) - 1)(N(\theta) + M(\theta)) + M(2\pi)}{(N(2\pi) - 1)M(\theta) + M(2\pi)}},$$

where $\theta \in \mathbb{R}$. The intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_* = \sqrt{\frac{e^{\frac{64}{3}\pi} + 2.4047 \times 10^{29} - 1}{2.4047 \times 10^{29}}} = 1.2376$$

We conclude that system (3.8) has two limit cycles (Γ_1) and (Γ_2). Since $r_* = 1.2376 > 1$, the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

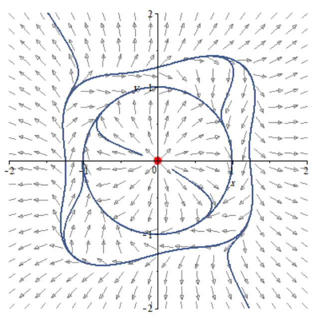


Figure 3.1: Limit cycles of system (3.8).

Example 2 If we take $a = b = -60, c = 3, d = 1, \alpha = \eta = 12, \lambda = -3$ and $\delta = 26$, then systems (3.1) reads

$$\begin{cases} x' = \frac{dx}{dt} = x\left(12x^4 - 3x^3y + 26x^2y^2 - 3xy^3 + 12y^4\right) - \left(x^2 + y^2\right)^3(20x + y) + \\ x\left(x^2 + y^2\right)^2\left(8x^4 + 8y^4 + 3xy^3 + 3x^3y + 14x^2y^2\right), \\ \\ y' = \frac{dy}{dt} = y\left(12x^4 - 3x^3y + 26x^2y^2 - 3xy^3 + 12y^4\right) + \left(x^2 + y^2\right)^3(x - 20y) + \\ y\left(x^2 + y^2\right)^2\left(8x^4 + 8y^4 + 3xy^3 + 3x^3y + 14x^2y^2\right), \end{cases} \tag{3.9}$$

The curve $U(x, y) = x^2 + y^2 - 1 = 0$ is an invariant algebraic curve of system (3.9) with cofactor

$$\begin{aligned} K(x,y) &= -\frac{2}{3}\left(x^2+y^2\right)(20\left(x^6+y^6\right)-30\left(x^4+y^4\right)- \\ &\quad 3xy\left(x^2+y^2+1\right)\left(x^2+y^2\right)+36x^2y^2\left(x^2+y^2-\frac{7}{3}\right)). \end{aligned}$$

The system (3.9) has the first integral

$$H(x,y)=\frac{N\left(\arctan\frac{y}{x}\right)+\left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1},$$

where $N(\theta)=\exp\left(3-9\theta+\dfrac{1}{4}\sin 4\theta-3\cos 2\theta\right)$ and

$$M(\theta)=-\frac{e^2}{2}\exp\left(-9\theta+\frac{1}{4}\sin 4\theta-3\cos 2\theta\right)-20e^3\int_0^\theta \exp\left(-9t+\frac{1}{4}\sin 4t-3\cos 2t\right)dt.$$

The system (3.9) has the limit cycle (Γ_1) whose expression is (Γ_1): $x^2+y^2-1=0$.

The system (3.9) has another limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\sqrt{\frac{(N(2\pi)-1)(N(\theta)+M(\theta))+M(2\pi)}{(N(2\pi)-1)M(\theta)+M(2\pi)}},$$

where $\theta\in\mathbb{R}$. The intersection of the limit cycle with the OX_+ axes is the point having r_*

$$r_*=\sqrt{\frac{2.762\times 10^{-25}-2.6042-1}{-2.6042}}=1.1764$$

We conclude that system (3.9) has two limit cycles (Γ_1) and (Γ_2). Since $r_*=1.1764>1$, the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

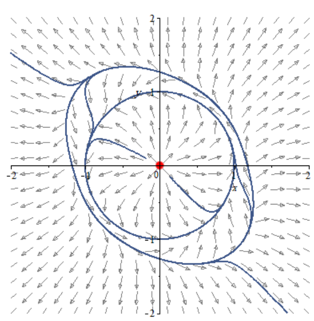


Figure 3.2: Limit cycles of system (3.9).

3.2 A family of planar differential systems with explicit expression for algebraic and non algebraic limit cycles

Consider a multi-parameter planar polynomial differential systems of degree thirteen of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + \left(x^2+y^2\right)^2 \left(P_3(x,y) - x\left(x^2+y^2\right)^3 R_2(x,y)\right), \\ \\ y' = \dfrac{dy}{dt} = y + \left(x^2+y^2\right)^2 \left(Q_3(x,y) - y\left(x^2+y^2\right)^3 R_2(x,y)\right), \end{cases} \tag{3.10}$$

where

$$\begin{aligned} &P_3(x,y) = ax^3 + bx^2y + cxy^2 - dy^3, \\ &Q_3(x,y) = ax^2y + dx^3 + (b+2d)xy^2 + cy^3 \text{ and} \\ &R_2(x,y) = (a+1)x^2 + (b+d)xy + (c+1)y^2, \end{aligned}$$

in which a, b, c and d are a real constants.

We define the trigonometric functions

$$\begin{aligned} &G(\theta) = \frac{a+c}{2} + \frac{a-c}{2}\cos 2\theta + \frac{b+d}{2}\sin 2\theta, \\ &A(\theta) = \int_0^\theta \frac{6+6G(t)}{d} \exp\left(\int_0^t \frac{-12-6G(\omega)}{d}\, d\omega\right) dt, \\ &\text{and } B(\theta) = exp\left(\int_0^\theta \frac{-12-6G(\omega)}{d}\, d\omega\right). \end{aligned}$$

3.2.1 Main result

Our main result is contained in the following theorem.

Theorem 3.2. *For the systems (3.10) the following statements hold.*

1. *If $d \neq 0$, then the origin of coordinates $O(0,0)$ is the unique critical point of system (3.10) at finite distance.*

2. *The curve $U(x,y) = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1$ is an invariant algebraic curve of systems (3.10) with cofactor*

$$K(x,y) = -6\left(x^2+y^2\right)^3 \left(1 + \left(x^2+y^2\right)^2 \left((a+1)x^2 + (b+d)xy + (c+1)y^2\right)\right).$$

3. *The systems (3.10) has the first integral*

$$H(x,y) = \frac{\left(1-\left(x^2+y^2\right)^3\right) A\left(arctan\frac{y}{x}\right) + B\left(arctan\frac{y}{x}\right)}{(x^2+y^2)^3 - 1}.$$

4. *The systems (3.10) has an explicit limit cycle, given in cartesian coordinates by*

$$(\Gamma_1): x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1 = 0$$

.

5. *If $d < 0, -2-(a+c) > |b+d| + |c-a|$ and $4+a+c > |b+d| + |c-a|$, then systems (3.10) has non-algebraic limit cycle (Γ_2), explicitly given in polar coordinates (r,θ) by*

$$r(\theta, r_*) = \left(\frac{(B(\theta)+A(\theta))(B(2\pi)-1)+A(2\pi)}{A(\theta)(B(2\pi)-1)+A(2\pi)}\right)^{\frac{1}{6}}.$$

Moreover, the algebraic limit cycle (Γ_1) lies inside the non-algebraic limit cycle (Γ_2).

Proof.

Proof of statement (1).

By definition $A(x_0, y_0) \in \mathbb{R}^2$ is a critical point of systems (3.10) if

$$\begin{cases} x_0 + \left(x_0^2+y_0^2\right)\left(P_3(x_0,y_0) - x_0\left(x_0^2+y_0^2\right)^3 R_2(x_0,y_0)\right) = & 0, \\ y_0 + \left(x_0^2+y_0^2\right)\left(Q_3(x_0,y_0) - y_0\left(x_0^2+y_0^2\right)^3 R_2(x_0,y_0)\right) = & 0, \end{cases}$$

we have

$$\left(x_0^2+y_0^2\right)^2(y_0P_3(x_0,y_0) - x_0Q_3(x_0,y_0)) = -d\left({x_0}^2+{y_0}^2\right)^4.$$

Since $d \neq 0$ then $(x_0, y_0) = (0,0)$ is the unique solution of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of the statement 1.

Proof of statement (2).

A computation shows that

$$U(x,y) = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1,$$

satisfies the linear partial differential equation

$$\frac{\partial U(x,y)}{\partial x}P(x,y) + \frac{\partial U(x,y)}{\partial y}Q(x,y) = U(x,y)K(x,y),$$

the associated cofactor being

$$K(x,y) = -6\left(x^2+y^2\right)^3\left(1+\left(x^2+y^2\right)^2\left((a+1)x^2+(b+d)xy+(c+1)y^2\right)\right).$$

This completes the proof of statement 2.

Proof of statement (3).

To proving statement (3), we need to convert the systems (3.10) in polar coordinates (r,θ) given by $x = r cos\theta$ and $y = r sin\theta$, then the systems (3.10) become

$$\begin{cases} r' = \dfrac{dr}{dt} = r + G(\theta)r^7 + (-G(\theta)-1)r^{13}, \\ \\ \theta' = \dfrac{d\theta}{dt} = dr^6. \end{cases} \tag{3.11}$$

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = \frac{1}{d}r^{-5} + \frac{G(\theta)}{d}r + \frac{-G(\theta)-1}{d}r^7. \tag{3.12}$$

By using the change of variable $\rho = r^6$, the equation (3.12) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = \frac{6}{d} + \frac{6G(\theta)}{d}\rho + \frac{-6G(\theta)-6}{d}\rho^2. \tag{3.13}$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.
By introducing the standard change of variables $z = \rho - 1$ we obtain the Bernoulli equation

$$\frac{dz}{d\theta} = \frac{-6-6G(\theta)}{d}z^2 + \frac{-12-6G(\theta)}{d}z. \tag{3.14}$$

We note that $z = 0$ is the solution for (3.14), by introducing the standard change of variables $y = \dfrac{1}{z}$ we obtain the linear equation

$$\frac{dy}{d\theta} = -\frac{6+6G(\theta)}{d} - \frac{12+6G(\theta)}{d}y. \tag{3.15}$$

The general solution of linear equation (3.15) is

$$y(\theta) = \frac{\alpha + A(\theta)}{B(\theta)},$$

where $\alpha \in \mathbb{R}$. Then the general solution of equation (3.14) is

$$z(\theta) = 0, \qquad\qquad z(\theta) = \frac{B(\theta)}{\alpha + A(\theta)}, \text{ where } \alpha \in \mathbb{R}.$$

Then the general solution of equation (3.13) is

$$\rho(\theta) = 1, \qquad\qquad \rho(\theta) = \frac{\alpha + A(\theta) + B(\theta)}{\alpha + A(\theta)}, \text{ where } \alpha \in \mathbb{R}.$$

Consequently, the general solution of (3.12) is

$$r(\theta) = 1, \qquad r(\theta) = \left(\frac{\alpha + A(\theta) + B(\theta)}{\alpha + A(\theta)}\right)^{\frac{1}{6}}, \text{ where } \alpha \in \mathbb{R}.$$

From this solution we obtain a first integral in the variables (x, y) of the form

$$H(x,y) = \frac{\left(1 - \left(x^2+y^2\right)^3\right) A\left(arctan\frac{y}{x}\right) + B\left(arctan\frac{y}{x}\right)}{(x^2+y^2)^3 - 1}.$$

Hence, statement 3 is proved.

Proof of statement (4).

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential systems (3.10), in cartesian coordinates are written as

$$x^2 + y^2 = 1, \qquad \left(x^2+y^2\right)^3 = \frac{\alpha + A(\theta) + B(\theta)}{\alpha + A(\theta)},$$

where $\alpha \in \mathbb{R}$.

Notice that systems (3.10) has a periodic orbit if and only if equation (3.12) has a strictly positive 2π-periodic solution. This is equivalent to the existence of a solution of (3.12) that fulfills $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$.

The solution $r(\theta, r_0)$ of the differential equation (3.12) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \left(\frac{\frac{1}{r_0^6 - 1} + A(\theta) + B(\theta)}{\frac{1}{r_0^6 - 1} + A(\theta)}\right)^{\frac{1}{6}},$$

where $r_0 = r(0)$.

We have the particular solution $\rho(\theta) = 1$ of the differential equation (3.13); from this solution we obtain $r^6(\theta) = 1 > 0$, for all θ in $[0, 2\pi]$ is a particular solution of the differential equation (3.12).

This is an algebraic limit cycle for the differential systemss (3.10), corresponding of course to an invariant algebraic curve $U(x, y) = 0$.

More precisely, in cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = arctan\left(\frac{y}{x}\right)$ the curve (Γ_1) defined by this limit cycle is $(\Gamma_1): x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1 = 0$.

Hence, statement 4 is proved.

Proof of statement (5).

A periodic solution of systems (3.10) must satisfy the condition $r(0,r_*) = r(2\pi,r_*)$, which leads to a unique value $r_0 = r_*$, given by

$$r_* = \left(\frac{A(2\pi)+B(2\pi)-1}{A(2\pi)}\right)^{\frac{1}{6}}.$$

The r_* is the intersection of the periodic orbit with the OX_+ axes. After the substitution of this value r_* into $r(\theta,r_0)$ we obtain

$$r(\theta,r_*) = \left(\frac{(B(\theta)+A(\theta))(B(2\pi)-1)+A(2\pi)}{A(\theta)(B(2\pi)-1)+A(2\pi)}\right)^{\frac{1}{6}}.$$

In what follows it is proved that $r(\theta,r_*) > 0$. Indeed

$$A(2\pi)-A(\theta) = \int_\theta^{2\pi} \frac{6+6G(t)}{d} exp\left(\int_0^t \frac{-12-6G(\omega)}{d}\,\mathrm{d}\omega\right)\mathrm{d}t.$$

According to $d<0, -2-(a+c) > |b+d|+|c-a|$ and $4+a+c > |b+d|+|c-a|$, hence $\dfrac{-2-G(\theta)}{d}$ and $\dfrac{1+G(\theta)}{d} > 0$ for all θ in $[0,2\pi]$, then we have $A(2\pi)-A(\theta) > 0$ and $B(2\pi) > 1$; therefore we have $r_* > 0$ and $r(\theta,r_*) > 0$ for all θ in $[0,2\pi]$. This is the second limit cycle for the differential systems (3.10), we note it by (Γ_2). This limit cycle is not algebraic, due to the expression

$$B(\theta) = exp\left(\int_0^\theta \frac{-12-6G(\omega)}{d}\,\mathrm{d}\omega\right).$$

More precisely, in cartesian coordinates $r^2 = x^2+y^2$ and $\theta = arctan\left(\dfrac{y}{x}\right)$ the curve defined by this limit cycle (Γ_2) is: $F(x,y) = 0$, where

$$F(x,y) = \left(x^2+y^2\right)^3 - \frac{\left(B\left(arctan\frac{y}{x}\right)+A\left(arctan\frac{y}{x}\right)\right)(B(2\pi)-1)+A(2\pi)}{A\left(arctan\frac{y}{x}\right)(B(2\pi)-1)+A(2\pi)}.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x,y)$ in the variables x and y satisfies that there is a positive integer n such that $\dfrac{\partial^n F(x,y)}{\partial x^n} = 0$, and this is not the case, therefore the curve $(\Gamma_2): F(x,y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

According to $d<0, -2-(a+c) > |b+d|+|c-a|$ and $4+a+c > |b+d|+|c-a|$, we get

$$r_* = \left(1+\frac{B(2\pi)-1}{A(2\pi)}\right)^{\frac{1}{6}} > 1,$$

and

$$r(\theta, r_*) = \left(1 + \frac{B(\theta)}{\frac{1}{r_*^6 - 1} + A(\theta)}\right)^{\frac{1}{6}} > 1,$$

We conclude that systems (3.10) has two limit cycles, the algebraic (Γ_1) lies inside the non-algebraic one (Γ_2).

This completes the proof of statement 5 . □

3.2.2 Examples

Example 1 If We take $a = c = -\frac{6}{5}, d = -5$ and $b = \frac{51}{10}$, then systems (3.10) reads

$$\begin{cases} x' = & x + \left(x^2+y^2\right)^2\left(-\frac{6}{5}x^3 + \frac{51}{10}x^2y - \frac{6}{5}xy^2 + 5y^3\right) \\ & -x\left(x^2+y^2\right)^5\left(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\right), \\ y' = & y + \left(x^2+y^2\right)^2\left(-\frac{6}{5}x^2y - 5x^3 - \frac{49}{10}xy^2 - \frac{6}{5}y^3\right) \\ & -y\left(x^2+y^2\right)^5\left(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\right). \end{cases} \tag{3.16}$$

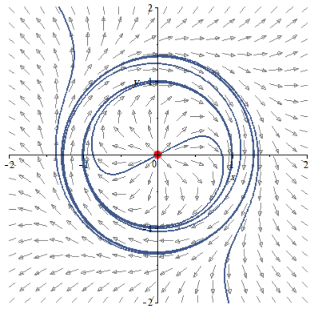
In this case we get

$$A(\theta) = -\frac{3}{50}\int_0^\theta (sin(2t) - 4)\, exp\left(\frac{3}{100} + \frac{24}{25}t - \frac{3}{100}cos(2\theta)\right)dt,$$

$$B(\theta) = exp\left(-\frac{3}{100}cos(2\theta) + \frac{24}{25}\theta + \frac{3}{100}\right).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axes is the point

$$r_* = \left(\frac{116.8 + exp\left(\frac{48\pi}{25}\right) - 1}{116.8}\right)^{\frac{1}{6}} \simeq 1.2876.$$



Limit cycles of system (3.16)

Example 2 If We take $a = \frac{-11}{10}, c = \frac{-115}{100}, d = -7$ and $b = \frac{141}{20}$, then systems (3.10) reads

$$\begin{cases} x' = & x + \left(x^2+y^2\right)^2 \left(\frac{-11}{10}x^3 + \frac{141}{20}x^2y - \frac{23}{20}xy^2 + 7y^3\right) \\ & -x\left(x^2+y^2\right)^5 \left(-\frac{1}{10}x^2 + \frac{1}{20}xy - \frac{3}{20}y^2\right), \\ y' = & y + \left(x^2+y^2\right)^2 \left(-\frac{11}{10}x^2y - 7x^3 - \frac{139}{20}xy^2 - \frac{23}{20}y^3\right) - \\ & y\left(x^2+y^2\right)^5 \left(-\frac{1}{10}x^2 + \frac{1}{20}xy - \frac{3}{20}y^2\right). \end{cases} \tag{3.17}$$

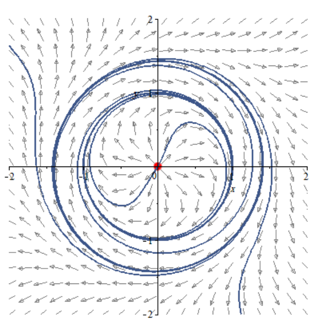
In this case we get

$$A(\theta) = -\frac{3}{140}\int_0^\theta (cos(2t)+sin(2t)-5)\,exp\left(\frac{3}{280}+\frac{3}{280}sin(2t)-\frac{3}{280}cos(2t)+\frac{3}{4}\right)dt,$$

$$B(\theta) = exp\left(-\frac{3}{280}sin(2\theta) - \frac{3}{280}cos(2\theta) + \frac{3}{4}\theta + \frac{3}{280}\right).$$

The intersection of the non-algebraic (Γ_2) limit cycle with the OX_+ axes is the point

$$r_* = \left(\frac{16.509 + exp\left(\frac{2\pi}{3}\right) - 1}{16.509}\right)^{\frac{1}{6}} \simeq 1.4047.$$



Limit cycles of system (3.17)

Example 3 If We take $a = \frac{-101}{100}, c = \frac{-105}{100}, d = -1$ and $b = \frac{151}{150}$, then systems (3.10) reads

$$\begin{cases} x' = & x + \left(x^2+y^2\right)^2\left(-\frac{101}{100}x^3 + \frac{151}{50}x^2y - \frac{21}{20}xy^2 + y^3\right) \\ & -x\left(x^2+y^2\right)^5\left(-\frac{1}{100}x^2 + \frac{1}{50}xy - \frac{1}{20}y^2\right), \\ y' = & y + \left(x^2+y^2\right)^2\left(-\frac{101}{100}x^2y - x^3 + \frac{149}{150}xy^2 - \frac{21}{20}y^3\right) \\ & -y\left(x^2+y^2\right)^5\left(-\frac{1}{100}x^2 + \frac{1}{150}xy - \frac{1}{20}y^2\right). \end{cases} \tag{3.18}$$

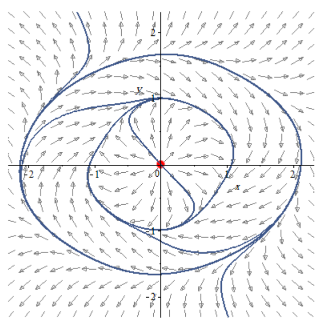
In this case we get

$$A(\theta) = -\frac{1}{50}\int_0^\theta (6cos(2t) + sin(2t) - 9)\,exp\left(\frac{1}{100} + \frac{3}{50}sin(2t) - \frac{1}{100}cos(2t) + \frac{291}{50}t\right)dt,$$

$$B(\theta) = exp\left(\frac{3}{50}sin(2\theta) - \frac{1}{100}cos(2t) + \frac{291}{50}\theta + \frac{1}{100}\right).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axes is the point

$$r_* = \left(\frac{1.019\times 10^{14} + exp\left(\frac{291\pi}{25}\right) - 1}{1.019\times 10^{14}}\right)^{\frac{1}{6}} \simeq 2.0566.$$



Limit cycles of system (3.18)

Example 4 If We take $a = \frac{-107}{100}, c = \frac{-109}{100}, d = -5$ and $b = \frac{507}{100}$, then systems (3.10) reads

$$\begin{cases} x' = & x + \left(x^2+y^2\right)^2\left(-\frac{107}{100}x^3 + \frac{507}{100}x^2y - \frac{109}{100}xy^2 + 5y^3\right) \\ & -x\left(x^2+y^2\right)^5\left(-\frac{7}{100}x^2 + \frac{7}{100}xy - \frac{9}{100}y^2\right), \\ y' = & y + \left(x^2+y^2\right)^2\left(-\frac{107}{100}x^2y - 5x^3 - \frac{493}{100}xy^2 - \frac{109}{100}y^3\right) \\ & -y\left(x^2+y^2\right)^5\left(-\frac{7}{100}x^2 + \frac{7}{100}xy - \frac{9}{100}y^2\right). \end{cases} \tag{3.19}$$

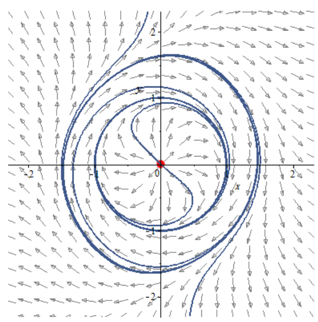
In this case we get

$$A(\theta) = -\frac{3}{500}\int_0^{\theta} (2cos(2t) + 7sin(2t) - 16)\, exp\left(\frac{21}{1000} + \frac{3}{500}sin(2t) - \frac{21}{1000}cos(2t) + \frac{138}{125}t\right)dt,$$

$$B(\theta) = exp\left(\frac{3}{500}sin(2\theta) - \frac{21}{1000}cos(2t) + \frac{138}{25}\theta + \frac{21}{1000}\right).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axes is the point

$$r_* = \left(\frac{104.804 + exp\left(\frac{276\pi}{125}\right) - 1}{104.804}\right)^{\frac{1}{6}} \simeq 1.4870.$$



Limit cycles of system (3.19)

General conclusion and perspectives

In this thesis we are interested in the qualitative study of the planar differential polynomial systems as well as that of the planar differential systems. It is important for a differential system to know if it admits or not a first integral, a periodic solution, moreover if this periodic solution is isolated, one speaks by definition of a limit cycle. On the other hand, the calculation of the first integral of a planar differential system completely determines the phase portrait of the system. For models resulting from practice, it is important to study these questions: first integral, periodic solution, limit cycle, phase portrait. The results obtained in this thesis revolve around these questions.

In the first chapter we presented some basic notions, concerning the qualitative theory of differential systems, in particular planar differential systems.

In the second chapter we have dealt with classes of planar differential systems having one limit cycle. This chapter is divided into two parts, in each part we have determined the exact expression of the first integral and the formula of the curves which are formed by the orbits of a class of planar differential systems. we used the Bernoulli equation.

In the third chapter we have treated two classes of planar differential systems having two limit cycles. This chapter is divided into two parts, in each part we have studied the coexistence of algebraic and non-algebraic limit cycles for a class of planar differential systems in which the expressions are given explicitly, we also determined the exact expression of the first integral and the formula of the curves which are formed by the orbits of a class of planar polynomial differential systems. We used the Ricati equation.

To our knowledge, it is a difficult problem to distinguish if a limit cycle is algebraic or not and it is rare to find, in the literature of differential systems, a differential system with a non-algebraic limit cycle given explicitly.

For the perspectives, given the techniques that we have used to find a class of systems with an algebraic and non-algebraic limit cycles, it is possible to hope to find a class of quadratic differential systems which admit a non-algebraic limit cycle and

given of a explicitly. Note that this issue is an open issue so far.
On the other hand, we have studied classes of planar systems from the point of view of coexistence of limit cycles. There remains the problem of existence of algebraic and non-algebraic limit cycle given explicitly for differential systems of a given degree n.

Our investment in the future is in this direction and this thesis serves as a powerful tool in the search for the first integral and the existence of limit cycle.

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الملخص

الهدف من هذه الأطروحة هي الدراسة النوعية لبعض الفئات من النظم التفاضلية كثيرات الحدود في المستوي. النتائج المتحصل عليها في هذه الدراسة متعلقة بتصرفات وسلوكيات الحلول في المستوي و وجود الحلول الدورية بصفة خاصة دورات الحد النهائية لبعض النظم التفاضلية ، بالإضافة إلى ذلك، تمكنا من تحديد الصيغة الرياضياتية لكل الكاملات الأولية ودورات الحد الجبرية أو الغير الجبرية

كلمات مفتاحية : مسألة هيلبرت 16، النظام التفاضلي، المنحى الصامد، الكاملات الأولية ، الحل الدوري ، الدورة الحد الجبري ، الدورة الحد الغير الجبري.

ABSTRACT

The objective of this thesis is the qualitative study of some classes of planar polynomial differential systems. The results obtained in this study concerns the integrability, the phase portraits and the existence of limit cycles of some classes of differential systems. In addition, we give explicitly an expression of the first integrals and limit cycles algebraic or non-algebraic found for all the classes studied.

Keywords: Hilbert 16th problem, differential system, invariant curve, first integral, periodic solution, algebraic limit cycle, non-algebraic limit cycle.

RÉSUMÉ

L'objectif de cette thèse est l'étude qualitative de quelques classes de systèmes différentiels polynômiaux planaires. Les résultats obtenus dans cette étude concernent l'intégrabilité, les portraits de phase et l'existence de cycles limites de quelques classes de systèmes différentiels. De plus nous donnons explicitement une expression des intégrales premières et des cycles limites algébriques ou non algébriques trouvées pour toutes les classes étudiées.

Mots clés: 16ème problème de Hilbert, système différentiel, courbe invariante, intégrale première, solution périodique, cycle limite algébrique, cycle limite non algébrique.