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Présentée par
Benslimane Salim

Thème

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Soutenue le: 17/07/2021 **Devant le Jury composé de :**

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THESIS

IN ORDER TO GET A DOCTORAL DEGREE

Domain: Mathematics and Informatics   Tracks: Mathematics
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Presented by
Salim BENSLIMANE

Theme

On the fixed point theory on ordered Banach spaces and applications

Supported on: 07/17/2021   **In front of a jury composed of :**

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Contents

General Introduction

This thesis is devoted to the study of the existence, multiplicity, nonnegativity and localization of solutions for abstract equations of the form:

$$Tx + Fx = x, \quad x \in \Omega, \tag{1}$$

where Ω is a closed convex subset of a Banach space.

This work is motivated by the fact that many problems that arise from different areas of science (chemical reactors, neutron transport, population biology, infection diseases, epidemiology, economics, applied mechanics, fluid mechanics, ...) can be recast in the abstract formulation (1). In particular, ordinary, fractional, partial differential equations and integral equations can be formulated like abstract equations of the form (1). Note that the nonnegativity is a very important notion as the solutions may represent stuff that cannot be negative such as density, speed, volume, mass, voltage, distance, amperage, gravity, etc. The nonnegativity condition can, mathematically, be described by a closed convex subset $\mathcal{P}$ in a Banach space which satisfies $\lambda\mathcal{P} \subset \mathcal{P}$ for all nonnegative real number λ and $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. We are interested in solving Equation (1) in $\mathcal{P}$.

As a very important part of nonlinear analysis, fixed point theory plays a key role in the solvability of many complex problems from applied mathematics. The theory itself was developed in many directions starting from Brouwer's fixed point theorem (1910), Banach's contraction principle (1922), and Schauder's fixed point theorem for compact mappings (1930). Krasnosel'skii's fixed point theorem concerns the sum of a contraction and a compact mapping, and turns out to be an extension and a combination of these previous two results (see [24, 27, 73]). Among the very rich and recent literature on the development of the fixed point theory for the sums of operators, we quote, e.g., [20, 64, 79, 80].

Another fixed point result established by Krasnosel'skii in 1960 is the cone compression-expansion fixed point theorem; it is mostly used for proving existence, localization and multiplicity of nonnegative solutions for various nonlinear problems in some conical shells of a Banach space (see [46, 50, 51]). Recently, its extension have attracted many researchers (see [7, 54, 55, 65] and references therein).

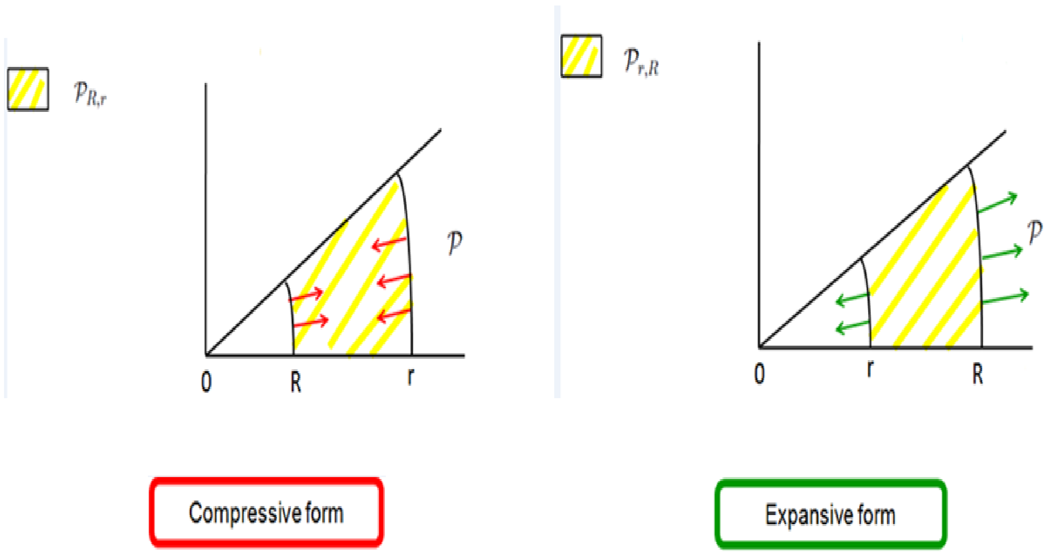
Let $\mathcal{P}$ be a cone of a Banach space X. Assume that there exist two positive constants r, R with $r \neq R$. The Krasnosel'skii-Guo compression-expansion of cone fixed point theorem guarantees that a completely continuous map $F : \mathcal{P}_{r,R} \to \mathcal{P}$ has a fixed point in the conical shell

$$\mathcal{P}_{r,R} := \{x \in P : r \leq \|x\| \leq R\}$$

under the following conditions:

$$\begin{aligned} \|Fu\| &\leq \|u\| \quad \text{for every } u \in \mathcal{P} \text{ with } \|u\| = r, \\ \|Fu\| &\geq \|u\| \quad \text{for every } u \in \mathcal{P} \text{ with } \|u\| = R. \end{aligned} \tag{2}$$

An illustration of this result in the special case where X is the two-dimensional plan $\mathbb{R}^2$ is depicted in the following figure



Note that the conditions (2) are imposed only on points of the two curved boundaries of $\mathcal{P}_{r,R}$. Interior points and points on the sides of the cone can be moved in any direction (as long as the image remains inside $\mathcal{P}$). Also it is not stipulated that any particular image point Fu must lie inside $\mathcal{P}_{r,R}$. The adjectives "compressive" and "expansive" in the names of the two forms of the theorem are conventional, and they are not meant to correctly describe the behaviour of F under all circumstances. Recently, many researchers have been interested in the extension of the above theorem in various directions (see [4, 7, 54, 55, 65, 69] and references therein). Our contribution is part of those generalizations leading to fixed point theory for sums of operators. More precisely, we derive several existence results for nonlinear equations of type (1).

In parallel to the development of the krasnosel'skii's type theorems and since 1979, there was the one of Leggett-Williams. While the Krasnosel'skii type compression-expansion fixed point theorems gives us fixed points localized in a conical shell of the form $\{x \in \mathcal{P} : a \leq \|x\| \leq b\}$, where $a, b \in (0, \infty)$, with the Leggett-Williams type they are localized in a conical shell of the form $\mathcal{P}(\alpha, \beta, a, b) := \{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \beta(x) \leq b\}$, where α is a concave nonnegative functional and β a convex nonnegative functional. The original Leggett-Williams fixed point theorem (see [56, Theorem 3.2]) discusses the existence of at least one fixed point in a conical shell of the form $\{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$, where $a, b \in (0, +\infty)$ and α is a concave nonngative functional. Noting that this result has been widely extended in many directions, (see, e.g., [4, 8]).

The fixed point theory has also been greatly influenced by the parallel progress of the research works made on the topological degree for different classes of mappings (see, e.g., [3, 54, 55]). In these regards, the pioneer works of Petryshyn [66, 67] have initiated important steps in establishing the relationship between the fixed point theory and the index fixed point theory. Our contribution ([10, 11, 12]) is a continuity in this direction. In [29], Djebali and Mebarki studied Equation (1) in the case where T is an expansive mapping with constant $h > 1$ and F a k-set contraction with $0 \leqslant k < h-1$. To do so, they developed a new fixed point index and then some fixed point theorems, including Krasnosel'skii type theorems, have been showed. The usefulness of the obtained fixed point theorems was showed in the same article and also in

[13, 14, 15, 16, 34, 35, 39, 36, 37]. In [10], we continue to extend the theory to the sum $T+F$ where T is an expansive mapping with constant $h>1$ and the perturbation $I-F$ is a k-set contraction with $0\leq k<h$. Our aim is to provide a new contribution to the fixed point index theory for this class of operators. First, we define and compute a topological index and then we prove several fixed point results in translates of cones. In [11], we used the fixed point index developed in [10] to establish an extension of a Leggett-Williams type expansion-compression fixed point theorem for a sum of two operators. It is also applied to prove the existence of nonnegative nontrivial solutions for two-point BVP and three-point BVP (see Section 4.2). In [12], we still use some results from [29] to get existence of multiple nonnegative solutions to a class of fourth-order boundary value problems with integral boundary conditions (see Chapter 5).

This thesis is organized as follows:

The chapter 1 gives a survey over some of the most important tools and results of nonlinear functional analysis in ordered Banach spaces. It provides the mathematical background needed to be applied in the rest of this work. We start in Section 1.1 with cones and partial ordering in Banach spaces which is required in this study since it is the tool that provides the ordering needed to describe the nonnegativity of the solution. Then in Section 1.2, we present some compactness criteria for functions defined on compact and noncompact intervals and we give a survey on the Kuratowski's measure of noncompactness. In Section 1.3, we will present different classes of operators. Then, we end this first chapter by a presentation of the topological degree theory.

In Chapter 2, we are concerned with the fixed point index theory for various classes of mapping: completely continuous mappings, strict-set contractions, condensing mappings and 1-set mappings.

In Chapter 3, we continue with the presentation of the fixed point index theory for the sum of two operators. In Sections 3.1, 3.2, and 3.3, we present the generalized fixed point index developed by Djebali and Mebarki for the sum $T+F$ where T is an expansive mapping with constant $h>1$ and F a k-set contraction with $0\leq k<h-1$ as well as we discuss the limit

case where F is a $(h-1)$-set contraction, and then the case where T is nonlinear expansive mapping. In each case, we give the definition and the computation of the fixed point index. In Section 3.4, we go on with the case where T is h-expansive with $h > 1$ and $I - F$ is a k-set contraction with $k < h$ and we present the definition of this index with respect to a translate of a cone $\mathcal{K}$ neither than to a cone.

In Chapter 4, we present fixed point theorems of several forms and some of their applications. In Section 4.1, three cone compression-expansion fixed point theorems of Krasnosel'skii type are established for sums $T + F$, where T is an expansive map with constant $h > 1$ and $I - F$ is a k-set contraction with $0 \leqslant k < h$. The proofs are based on the properties of the topological index i_* presented in Section 3.4. An extension of a cone expansion-compression fixed point theorem of Legget-Williams type for the same class of mappings is established in Section 4.2.

Chapter 5 is devoted to study a class of fourth-order boundary value problems with integral boundary conditions. The nonlinearity may have time-singularity and change sign. Moreover, it satisfies general polynomial growth conditions. A recent multiple fixed point theorem in cones is applied to prove the existence of at least two nonnegative classical solutions. Precisely, we investigate the existence of at least two nonnegative solutions to the fourth-order nonlinear boundary value problem

$$\begin{aligned} x^{(4)}(t) &= w(t)f(t, x(t), x''(t)), \quad t \in (0,1), \\ x(0) &= \int_0^1 h_1(s)x(s)ds, \quad x(1) = \int_0^1 k_1(s)x(s)ds, \\ x''(0) &= \int_0^1 h_2(s)x''(s)ds, \quad x''(1) = \int_0^1 k_2(s)x''(s)ds, \end{aligned} \tag{3}$$

where

(H1) $w \in L^1([0,1])$ is nonnegative and may be singular at $t = 0$ and (or) $t = 1$, $f \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R})$,

$$|f(t,u,v)| \leq a_1(t)|u|^{p_1} + a_2(t)|v|^{p_2} + a_3(t), \quad t \in [0,1], \quad u, v \in \mathbb{R},$$

$a_1, a_2, a_3 \in \mathcal{C}([0,1])$ are given nonnegative functions, p_1, p_2 are given nonnegative constants.

(H2) h_1, h_2, k_1, $k_2 \in L^1([0,1])$ with $m_1\nu_1 + n_1\mu_1 \neq 0$, $m_2\nu_2 + n_2\mu_2 \neq 0$, for

$$\begin{aligned}
m_1 &= \int_0^1 sh_1(s)ds, \quad m_2 = \int_0^1 sh_2(s)ds, \\
n_1 &= 1 - \int_0^1 sk_1(s)ds, \quad n_2 = 1 - \int_0^1 sk_2(s)ds, \\
\mu_1 &= 1 - \int_0^1 h_1(s)ds, \quad \mu_2 = 1 - \int_0^1 h_2(s)ds, \\
\nu_1 &= 1 - \int_0^1 k_1(s)ds, \quad \nu_2 = 1 - \int_0^1 k_2(s)ds.
\end{aligned}$$

List of publications

- S. Benslimane, S. Djebali and K. Mebarki, *On the fixed point index for sums of operators*, Fixed Point Theory, **Accepted, 2020**.
- S. Benslimane, S. G. Geogiev and K. Mebarki, *Expansion-Compression fixed point theorem of Leggett-Williams type for the sum of two operators and application in three-point BVPs*, Studia Universitatis Babeş-Bolyai Mathematica, **Accepted, 2020**.
- S. Benslimane, S. G. Geogiev and K. Mebarki, *Multiple nonnegative solutions for a class fourth-order BVPs*, Submitted.

Table of Notations

The most frequently used notations, symbols, and abbreviations are listed below.

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of all nonnegative real numbers.
$\mathbb{R}^n$	The n-dimensional Euclidean space.
$\inf(A)$	The infimum of the set A.
$\sup(A)$	The supremum of the set A.
$d(x,y)$	The distance between x and y.
$\operatorname{diam}(A)$	The diameter of the set A,where A is a subset of a metric space X.
$\operatorname{conv}(.)$	The convex hull.
$\overset{\circ}{\mathcal{P}}$	The set of interior points of $\mathcal{P}$.
$\mathcal{C}(G)$	The set of all real continuous functions from G into $\mathbb{R}$.
$\mathcal{C}^1([a,b])$	Space of all continuously differentiable and real valued functions defined on $[a,b]$.
$a.e.$	Almost everywhere.
BVPs	Boundary value problems.
u.s.c	Uniformly semicontinuous.
I	The identity application.
$f\|_V$	The restriction of f on V.
$i\,(f,U,D)$	Fixed point index of f on U with respect to D.
$Fix(f)$	The set of fixed points of f.
$mes(D)$	The Lebegues measure of the set D.

Chapitre 1

Preliminaries

Contents

In this chapter, we will present some basic tools that make the reading of this thesis easier. For more details on these tools, we refer the reader to the references [45] (for Section 1.1), [6, 17, 26, 32, 34, 41, 47, 49, 53, 70, 72, 82] (for Section 1.2), [18, 30, 33, 60, 79, 80] (for Section 1.3) and [19, 52, 61] (for Section 1.4).

1.1 Cones and partial ordering

Let E be a Banach space.

Definition 1.1.1 *A closed convex subset $\mathcal{P}$ of E is said to be a cone if it satisfies these two conditions:*

(1) $\lambda\mathcal{P} \subset \mathcal{P}$, $\forall\lambda \geq 0$,

(2) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

We denote by $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$ the punctured cone.

Every cone $\mathcal{P}$ defines a partial ordering $\leq$ in E defined by :

$$x \leq y \text{ if and only if } y - x \in \mathcal{P}.$$

Thus we make every Banach space E a partial ordered set. We say that

- $x < y \Leftrightarrow x \leq y$ and $x \neq y$.
- $x \ll y \Leftrightarrow y - x \in \mathring{\mathcal{P}}$ if $\mathring{\mathcal{P}} \neq \emptyset$.
- $x \nleq y \Leftrightarrow y - x \notin \mathcal{P}$.

Definition 1.1.2 *A segment of a cone $\mathcal{P}$ is defined by:*

$$[x, y] = \{z \in \mathcal{P} :\ x \leq z \leq y\}.$$

Example 1.1.3 1. *For $E = \mathbb{R}$, the set $\mathcal{P} = \{x \in \mathbb{R},\ x \geq 0\}$ is a cone in $\mathbb{R}$ and the order that it introduces is simply the usual one "$\leq$".*

2. *For $E = \mathbb{R}^2$, the set $\mathcal{P} = \{x = (x_1, x_2) \in \mathbb{R}^2, x_1 \geq 0$ and $x_2 \geq 0\}$ is a cone in E. Here the order introduced is not usual. For $x, y \in E$ saying that $x \leq y$ means $y_1 - x_1 \geq 0$ and $y_2 - x_2 \geq 0$.*

3. *Consider the Banach space* $E = \mathcal{C}([0,1],\mathbb{R})$ *with the sup-norm* $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. *The set* $\mathcal{P} = \{x \in E : \ x(t) \geq 0\}$ *is a cone in* E. *And saying that* $x \leq y$ *for* $x, y \in E$ *means that* $y(t) - x(t) \geq 0$, $\forall t \in [0,1]$.

Definition 1.1.4 1. *A cone* $\mathcal{P}$ *is said to be normal if there exists a positive constant* $N \neq 0$ *such that, for all* $x, y \in \mathcal{P}$, *we have* $x \leq y \Rightarrow \|x\| \leq N\|y\|$, *The least positive constant* N *is called the normal constant of* $\mathcal{P}$.

2. $\mathcal{P}$ *is called solid if his interior is not the empty set.*

Remark 1.1.5 *Geometrically, the normality of a cone means that the angle between any two positive unit vectors cannot exceed* π. *In other words, a normal cone cannot be too wide.*

Example 1.1.6 1. *Let* $E = \mathbb{R}^n$ *and* $\mathcal{P}_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n$: $x_i \geq 0, i = 1, \ldots, n\} = {\mathbb{R}_+}^n$.

(a) $\mathcal{P}_1$ *is a solid cone in* $\mathbb{R}^n$, *in fact* $\mathring{\mathcal{P}}_1 = (\mathbb{R}_+^*)^n \neq \emptyset$;

(b) *Furthermore, since all the norms of* $\mathbb{R}^n$ *are monotone we have*

$$\forall x, y \in \mathbb{R}^n,\ 0_{\mathbb{R}^n} \leq x \leq y \Rightarrow \|x\| \leq \|y\|.$$

Then, $\mathcal{P}_1$ *is normal with* $N = 1$.

2. *Let* $E = \mathcal{C}(G)$, *the space of continuous functions in a closed bounded space* $G \subset \mathbb{R}^n$, *provided with the norm* $\|x\|_{\mathcal{C}(G)} = \sup_{t\in G} |x(t)|$ *and* $\mathcal{P}_2 = \{x \in \mathcal{C}(G) : x(t) \geq 0,\ \forall t \in G\}$.

(a) $\mathcal{P}_2$ *is a solid cone in* $\mathcal{C}(G)$.

(b) $\mathcal{P}_2$ *is normal, since the norm* $\|.\|_{\mathcal{C}(G)}$ *is monotone in* $\mathcal{C}(G)$.

(c) *We define other cones in* $\mathcal{C}(G)$ *such that:*

$$\mathcal{P}_3 = \{x \in \mathcal{C}(G) : x(t) \geq 0,\ \text{and} \int_{G_0} x(t)dt \geq \varepsilon_0 \|x(t)\|_{\mathcal{C}(G)}\},$$

$$\mathcal{P}_4 = \{x \in \mathcal{C}(G) : x(t) \geq 0,\ \text{and} \min_{t\in G_1}(x(t)) \geq \varepsilon_1 \|x(t)\|_{\mathcal{C}(G)}\},$$

where G_0, G_1 *are closed subsets of* G, *and* ε_0 *and* ε_1 *are two constants such that :* $0 < \varepsilon_0 < mes(G_0)$ *and* $0 < \varepsilon_1 < 1$. *We have* $\mathcal{P}_3 \subset \mathcal{P}_2$ *et* $\mathcal{P}_4 \subset \mathcal{P}_2$ *and the two are solid and normal cones in* $\mathcal{C}(G)$.

3. *Let $E = L^p(\Omega)$, be the Lebesgue-integrable space on $\Omega \subset \mathbb{R}^n$ with $p \geq 1$ and $0 < mes(\Omega) < \infty$ provided with the norm $\|x\| = \left(\int_\Omega |x(t)|^p\, dt\right)^{\frac{1}{p}}$ and*

$$\mathcal{P}_5 = L^p_+(\Omega) = \{x \in L^p(\Omega) :\ x(t) \geq 0 \quad a.e \ in\ \Omega\}.$$

It is clear that $\mathcal{P}_5$ is a normal cone, since the norm of $L^p(\Omega)$ is increasing, but not solid, since $\overset{\circ}{\mathcal{P}}_5 = \emptyset$ except the cone $L^\infty_+(\Omega)$ which has an empty interior. Indeed, if $\overset{\circ}{\mathcal{P}}_5 \neq \emptyset$, then $\exists f \in \overset{\circ}{\mathcal{P}}_5$, i.e. $\exists \delta > 0$ such that $\mathcal{B}(f, \delta) \subset \mathcal{P}_5$.

We take $\Omega = [0, 1]$, and consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined by:

$$f_n(t) = \begin{cases} -f(t), & if\ t \in [0, \frac{1}{n}], \\ f(t), & if\ t \in]\frac{1}{n}, 1]. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 |f_n(t) - f(t)|^p dt &= \int_0^{\frac{1}{n}} |f_n(t) - f(t)|^p dt + \int_{\frac{1}{n}}^1 |f_n(t) - f(t)|^p dt \\ &= \int_0^{\frac{1}{n}} |-f(t) - f(t)|^p dt + \int_{\frac{1}{n}}^1 |f(t) - f(t)|^p dt \\ &= 2^p \int_0^{\frac{1}{n}} |f(t)|^p dt. \end{aligned}$$

Therefore, $\|f_n - f\| = \left(2 \int_0^{\frac{1}{n}} |f(t)|^p dt\right)^{\frac{1}{p}} \to 0$ when $n \to \infty$, since $f \in L^p_+(\Omega)$, so

$$\forall \delta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow \|f_n - f\| \leq \delta.$$

Hence

$$\forall \delta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow f_n \in \mathcal{B}(f, \delta),$$

which contradicts the fact that f_n is not in $\mathcal{P}_5$, since mes $\left([0, \frac{1}{n}]\right) \neq 0$.

Definition 1.1.7 *Let $\mathcal{P}$ be a cone in a Banach space E. For any $\theta \in \mathcal{P}$. The set $\mathcal{K} = \theta + \mathcal{P}$ is called a translate of the cone $\mathcal{P}$.*

Example 1.1.8 *Consider the Banach space $E = \mathcal{C}([0, 1], \mathbb{R})$ with the sup-norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. The set $\mathcal{K} = \{x \in E :\ x(t) \geq 1\}$ is a translate of a cone in E. In fact $\forall x \in \mathcal{K}$, we can find $y \in \mathcal{P}$ such that $x = \theta + y$ where $\theta \equiv 1$ and $\mathcal{P} = \{x \in E :\ x(t) \geq 0\}$.*

Noting that cones are a particular case of the translate of cones. Indeed, for $\theta = 0$ we have $\mathcal{K} = \mathcal{P}$, but translate of cones are not always cones, indeed for $\theta \neq 0$, $\mathcal{K}$ does not satisfy all the conditions of Definition 1.1.1.

1.2 Compactness and noncompactness

1.2.1 Some results about the compactnees

Compactness grew out of one of the most productive periods of mathematical activity. In mid to late nineteenth century, advanced mathematics began to take the form we know today. In the background was Cantor's work establishing the beginning of a systematic study of set theory and point-set topology. Also, many mathematicians, including Weierstrass, Hausdorff and Dedekind, were worried about the foundations of mathematics and began to make rigorous many of the ideas that had for centuries been taken for granted. We first recall two different characterizations of the compactness notion. One characterization, developed by Bolzano and Weierstrass among others, grew out of the study of sequence convergence. The second characterization, which grew out of work by Heine, Borel, and Lebesgue, was based on topological features, such as the covering of sets by open neighborhoods.

Definition 1.2.1 *Let (X, d) be a metric space. A subset $\mathcal{C}$ of X is compact if every sequence in $\mathcal{C}$ contains a convergent subsequence with a limit in $\mathcal{C}$. Equivalently, a subset $\mathcal{C}$ of X is called compact if every open cover of $\mathcal{C}$ has a finite subcover.*

Definition 1.2.2 *A subset $\mathcal{C}$ of X is said to be totally bounded if for each $\varepsilon > 0$, there exists a finite number of elements $\{x_1, x_2, \cdots, x_n\}$ such that $\mathcal{C} \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. The set $\{x_1, x_2, \cdots, x_n\}$ is called a finite ε-net.*

Remark 1.2.3

1. *Every subset of totally bounded set is totally bounded.*
2. *Every totally bounded set is bounded, but a bounded set dos not need to be totally bounded.*

Proposition 1.2.4 *A subset of a compact metric space is compact if and only if it is closed.*

Proposition 1.2.5 *Let X be a metric space. Then, the following assertions are equivalent:*

(a) X is compact.

(b) Every sequence in X has a convergent subsequence.

(c) X is complete and totally bounded.

Proposition 1.2.6 *Let C be a subset of a complete metric space X. Then we have :*

(a) C is compact if and only if C is closed and totally bounded.

(b) $\overline{C}$ is compact if and only if C is totally bounded.

Remark 1.2.7 *1. $X = (0, 1)$ with usual metric is totally bounded, but not compact.*

2. $X = \mathbb{R}$ with usual metric is complete. But it is not totally bounded and hence not compact.

Definition 1.2.8 *A subset $\mathcal{C}$ of a topological space is said to be relatively compact if its closure is compact, i.e., $\overline{\mathcal{C}}$ is compact. In particular, we have an interesting result:*

Proposition 1.2.9 *Let $\mathcal{C}$ be a closed subset of a complete metric space. Then $\mathcal{C}$ is compact if and only if it is relatively compact.*

We now state the following fundamental theorem concerning compactness.

Theorem 1.2.10 (The Heine-Borel property) *A subset $\mathcal{C}$ of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Definition 1.2.11 *A topological space is said to be locally compact if it is separable and if each of its points admits a compact neighborhood.*

Example 1.2.12 *1. A compact topological space E is locally compact because E is a neighborhood of each one of its points.*

2. $\mathbb{R}$ is locally compact because for all $x \in \mathbb{R}$ the interval $[x - 1, x + 1]$ is a compact neighborhood of x. $\mathbb{R}^n$ is locally compact because its closed unit ball is compact.

It is well known that infinite dimensional spaces like $\mathcal{C}([a,b],\mathbb{R})$ are not as well behaved as finite dimensional spaces like $\mathbb{R}^n$. For instance, closed, bounded subsets of continuous functions on $\mathbb{R}$ do not necessarily have the Heine-Borel property. The work in this area was done by Ascoli and in the last decades of the 1800s.

The following example illustrates that a closed, bounded subset of continuous functions on $\mathbb{R}$ is not compact.

Example 1.2.13 *Consider $B = \{f \in \mathcal{C}([0,1],\mathbb{R}) \; : \; \|f\| \leq 1\}$, where $\|\cdot\|$ is the sup norm. We will show that there is a sequence in B that does not have a convergent subsequence. Let $f_n(x) = x^n$, $n \in \mathbb{N}^*$. This sequence lies in B, but we cannot find a subsequence that converges uniformly to a function in $\mathcal{C}([0,1],\mathbb{R})$. Suppose to the contrary f is such a function. Then*

$$f(x) = \lim_{k\to\infty} f_{n_k}(x),$$

which would imply that

$$f(x) = \begin{cases} 0, & \text{if } x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Since f is a discontinuous function, it is not in $\mathcal{C}([0,1],\mathbb{R})$. Hence the sequence $(f_n)_{n\in\mathbb{N}}$ has no uniformly convergent subsequence.

The problem in this example comes from how functions converge. If convergence means pointwise convergence, then we get a behaviour different from that of sequences in closed unit balls of $\mathbb{R}^n$. In order to avoid this problem, Ascoli introduced the notion of equicontinuity.

Let (X,τ) be a topological space, (Y,d) a metric space, and $\mathcal{C}(X,Y)$ denotes the space of continuous functions from X to Y. Let $\mathcal{H} \subset \mathcal{C}(X,Y)$.

Definition 1.2.14 *$\mathcal{H}$ is said to be equicontinuous at a point $x_0 \in X$ if*

$$\forall \varepsilon > 0,\ \exists U_\varepsilon \in \mathcal{V}(x_0),\ \forall x \in X,$$

$$(x \in U_\varepsilon \Longrightarrow f(x) \in B(f(x_0),\varepsilon))\,,\ \forall f \in \mathcal{H}.$$

$\mathcal{H}$ is equicontinuous if it is equicontinuous at every point $x_0 \in X$. Noting that the prefix "equi" indicates uniformity with respect to the functions $f \in \mathcal{H}$.

Remark 1.2.15 *When (X, d) is a compact metric space, then $\mathcal{H}$ is equicontinuous if and only if (it is uniformly equicontinuous):*

$$\forall \varepsilon > 0, \exists \delta > 0 \, \forall x, y \in X,$$

$$(d(x, y) < \delta \Longrightarrow d(f(x), f(y)) < \varepsilon), \; \forall f \in \mathcal{H}.$$

Proof. Since uniform equicontinuity is a stronger condition, we only prove necessity. So, let $\mathcal{H}$ be an equicontinuous family of functions and let $\varepsilon > 0$. By assumption, for every $x \in X$, there exists $\delta = \delta(\varepsilon) > 0$ such that $d(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{H}$ and $d(x, y) < \delta$. Since X is compact, it can be covered by a finite number of balls $B(x_i, \delta_{x_i})$ $(1 \leq i \leq m)$. Let $\delta = \min\limits_{1 \leq i \leq m} \{\delta_{x_i}\}$ and let $x, y \in X$ be such that $d(x, y) < \delta$. Then there exists $i_0 \in \{1, \ldots, m\}$ such that $x \in B(x_{i_0}, \delta_{x_{i_0}}/2)$. Hence, $y \in B(x_{i_0}, \delta_{x_{i_0}}/2)$ and for all $f \in \mathcal{H}$,

$$d(f(x), f(y)) \leq d(f(x), f(x_{i_0})) + d(f(y), f(x_{i_0})) < \varepsilon.$$

■

Proposition 1.2.16 *Let $\mathcal{H}$ be equicontinuous and $T : \mathcal{H} \to T(\mathcal{H})$ a continuous mapping. Then, $T(\mathcal{H})$ is equicontinuous.*

Example 1.2.17 *Let X and Y be metric spaces. By definition we can see that any family $\mathcal{H}$ of a single function is equicontinuous. More generally, every finite subset of $\mathcal{C}(X, Y)$ is equicontinuous.*

Example 1.2.18 *If all the functions of $\mathcal{H}$ are k-lipschitzian, for a same constant k, then $\mathcal{H}$ is equicontinuous. More generally, it suffices that each point $x \in X$ has a neighborhood V_x that contains only k_x-Lipschitzian functions, where k_x is the same constant and only depends on x.*

Example 1.2.19 *If X and Y are normed vector spaces, and $\mathcal{H}$ is a bounded part of linear functions of $\mathcal{C}(X, Y)$, then $\mathcal{H}$, considered as a part of $\mathcal{C}(X, Y)$, is equicontinuous.*

Example 1.2.20 *Let $c > 0$ and $\mathcal{H} = \{f \in \mathcal{C}([0,1];\mathbb{R}) : \int_0^1 |f'(t)|^2 dt \leq c\}$. Notice that, if $f \in \mathcal{C}([0,1],\mathbb{R})$ and if $x < y$, we can write*

$$f(y) - f(x) = \int_x^y f'(t)dt.$$

Then, the Cauchy-Schwarz inequality leads

$$|f(y) - f(x)| \leq \left(\int_x^y |f'(t)|^2 dt\right)^{\frac{1}{2}} |y-x|^{\frac{1}{2}} \leq \sqrt{c}|y-x|^{\frac{1}{2}},$$

provided $f \in \mathcal{H}$. Taking $\delta = \frac{\varepsilon^2}{c}$, we can see that $\mathcal{H}$ is equicontinuous.

Example 1.2.21 *Let $f_n(x) = \sin nx$, $x \in [0, 2\pi]$ and $\mathcal{H} = \{f_n(.) : n \in \mathbb{N}\}$. Then $\mathcal{H}$ is bounded. However, it is not equicontinuous in $\mathcal{C}([0,2\pi],\mathbb{R})$ (for this, consider the sequence $x_n = \frac{\pi}{n}$, so $|f_n(x_n) - f_n(x_{2n})| = 1$). Hence, $\mathcal{H}$ is not relatively compact, i.e., we cannot extract a convergent subsequence.*

Example 1.2.22 *Let $X = [0,1]$, $Y = \mathbb{R}$, and for $n \in \mathbb{N}$, let $f_n(t) = t^n$. Then, the sequence $(f_n)_{n\in N}$ is not equicontinuous. Indeed, let $\varepsilon = \frac{1}{2}$, and suppose that there exists $\delta_1 > 0$, such that the condition of equicontinuity is satisfied. Define $\delta = \min(\delta_1, 1)$.*
Consider now $x = 1$, $y = 1 - \frac{\delta}{2}$. It is clear that

$$|x - y| = |1 - 1 + \frac{\delta}{2}| = \frac{\delta}{2} < \delta,$$

However, we have already seen that the sequence $(f_n)_{n\geq 1}$ is in the unit ball, and it converges to 0 for all $x \in [0,1)$ and to 1 for $x = 1$. Thus, $|f_n(1) - f_n(1-\frac{\delta}{2})|$ could be as close to 1 as wanted for all fixed $\delta > 0$.

Remark 1.2.23 *From Example 1.2.22, we conclude that the unit ball of $\mathcal{C}([0,1],\mathbb{R})$ is not equicontinuous, although it is bounded and closed.*

The Arzelà-Ascoli theorem then states the following.

Theorem 1.2.24 (Arzelà-Ascoli Theorem) *Any bounded equicontinuous sequence of functions in $\mathcal{C}([a,b],\mathbb{R})$ has a uniformly convergent subsequence.*

We can state a consequence of this theorem, analogous to the Heine-Borel Property.

Theorem 1.2.25 *[72, Corollary 2.14.31] A subset of $\mathcal{C}([a,b],\mathbb{R})$ is compact if and only if it is closed, bounded, and equicontinuous.*

Remark 1.2.26 *The Arzelà-Ascoli Theorem gives necessary and sufficient conditions for compactness in the space of continuous functions defined on a compact space X and taking values in $\mathbb{R}$ or, more generally, in any finite-dimensional Banach space.*

Corollary 1.2.27 *Let $M \subset \mathcal{C}^1([a,b],\mathbb{R})$ satisfy the following conditions:*

(a) there exists $L > 0$ such that for all $t \in [a,b]$ and $u \in M$,

$$|u(t)| \leq L \text{ and } |u'(t)| \leq L.$$

(b) For every positive $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\varepsilon)$ and for all $u \in M$,

$$|u(t_1) - u(t_2)| \leq \varepsilon \text{ and } |u'(t_1) - u'(t_2)| \leq \varepsilon.$$

Then, the set M is relatively compact in $\mathcal{C}^1([a,b],\mathbb{R})$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of $M \subset \mathcal{C}^1([a,b],\mathbb{R})$.
To prove that M is relatively compact in $\mathcal{C}^1([a,b],\mathbb{R})$, it is equivalent to prove that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence converging in $\mathcal{C}^1([a,b],\mathbb{R})$. Since $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of $M \subset \mathcal{C}^1([a,b],\mathbb{R})$, $\{u'_n\}_{n\in\mathbb{N}}$ (resp. $\{u_n\}_{n\in\mathbb{N}}$) is a sequence of $\mathcal{C}([a,b],\mathbb{R})$.
Arzelà-Ascoli Theorem and the assumptions (a)-(b) guarantee that the sequence of derivatives $\{u'_n\}_{n\in\mathbb{N}}$ (resp. $\{u_n\}_{n\in\mathbb{N}}$) is relatively compact in $\mathcal{C}([a,b],\mathbb{R})$.
As a consequence, there exists a subsequence, also denoted $\{u_n\}_{n\in\mathbb{N}}$ which converges in $\mathcal{C}([a,b],\mathbb{R})$ to a limit $u \in \mathcal{C}([a,b],\mathbb{R})$, and a subsequence of $\{u'_n\}_{n\in\mathbb{N}}$, also denoted $\{u'_n\}_{n\in\mathbb{N}}$, converging in $\mathcal{C}([a,b],\mathbb{R})$ to a limit $v \in \mathcal{C}([a,b],\mathbb{R})$.
Using the integral representation of u_n, we find that for all t, $t_0 \in [a,b]$,

$$u_n(t) = u(t_0) + \int_{t_0}^{t} u'_n(s)ds \to u(t_0) + \int_{t_0}^{t} v(s)ds,$$

as $n \to \infty$. Then for all $t \in [a,b]$, $\lim_{n\to\infty} u_n(t) = u(t)$ and the uniqueness of the limit yields that $u(t) = u(t_0) + \int_{t_0}^{t} v(s)ds$. Hence $u \in \mathcal{C}^1([a,b],\mathbb{R})$ and $u' = v$. ■

Corollary 1.2.28 *Every bounded sequence in $\mathcal{C}^1$ has a convergent subsequence in $\mathcal{C}$.*

Corollary 1.2.29 *For all $k \in \mathbb{N}$, the space $\mathcal{C}^{k+1}([a,b],\mathbb{R})$ is imbedded compactly in $\mathcal{C}^k([a,b],\mathbb{R})$.*

Proof. Let $k \geq 1$ and M bounded in $\mathcal{C}^{k+1}([a,b],\mathbb{R})$. Then, M is bounded in $\mathcal{C}([a,b],\mathbb{R})$ and there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in M such that $u_n \to u \in \mathcal{C}([a,b]\mathbb{R})$. Furthermore $\{u_n'\}_n$ is also bounded in $\mathcal{C}([a,b],\mathbb{R})$, therefore there exists a subsequence of $\{u_n\}_n$ such that $u_n' \to u' \in \mathcal{C}([a,b]\mathbb{R})$. We repeat the process until we get:

$$u_n^{(i)} \to u^{(i)}$$

for all i such that $0 \leq i \leq k$. ■

Example 1.2.30 *The set F of functions f on $[a,b]$ that is uniformly bounded and satisfies the Hölder condition of order $0 < \alpha \leq 1$ with a fixed constant K*

$$|f(x) - f(y)| \leq K|x-y|^{\alpha}, x, y \in [a,b],$$

is relatively compact in $\mathcal{C}([a,b],\mathbb{R})$.

Let (X,d) be a compact metric space and $(Y, \|\cdot\|)$ be a Banach space. The space $E = \mathcal{C}(X,Y)$ is endowed with the norm:

$$\|f\| = \sup_{x\in X} \|f(x)\|_Y.$$

Theorem 1.2.31 (Arzelà-Ascoli Theorem) *(see, e.g., [17, Corollary 1] or [26]) A subset $\mathcal{H} \subset \mathcal{C}(X,Y)$ is relatively compact if and only if*

(a) $\mathcal{H}$ is equicontinuous.

(b) $\forall\, x \in X$, the set $\mathcal{H}(x) = \{f(x),\ f \in \mathcal{H}\}$ is relatively compact in Y.

Next, let $J = [a,b]$. Then we have

Corollary 1.2.32 (Arzelà-Ascoli Theorem) *Let Y be a finite dimensional Banach space. If $\mathcal{H} \subset \mathcal{C}(J,Y)$ is bounded and equicontinuous, then $\mathcal{H}$ is relatively compact.*

Now, we consider the higher-order derivative spaces $E = \mathcal{C}^m(J,Y)$, which denotes the space of continuously differentiable functions defined on some interval $J \subset \mathbb{R}$ and taking values in a Banach space Y. For $\mathcal{H} \subset \mathcal{C}^m(J,Y)$ and $k = 1, 2, \ldots, m$, we denote by $\mathcal{H}^{(k)}$, the space of functions $\mathcal{H}^{(k)} = \{x^{(k)} : x \in \mathcal{H}\}$ and $\mathcal{H}^{(k)}(t) = \{x^{(k)}(t) : x \in \mathcal{H}\}$.

Now, we present a generalization of Arzelà-Ascoli Theorem to the space $\mathcal{C}^m(J,Y)$, when $J = [a,b]$ is compact. We have

Theorem 1.2.33 *[47, Theorem 1.2.7] $\mathcal{H} \subset \mathcal{C}^m(J,Y)$ is relatively compact if and only if*

(a) $\mathcal{H}^{(m)}$ is equicontinuous and, for any $t \in J$, $\mathcal{H}^{(m)}(t)$ is relatively compact in Y,

(b) for each $k \in \{0, 1, \ldots, m\}$, there exists $t_k \in J$ such that $\mathcal{H}^{(k)}(t_k)$ is relatively compact in Y.

Lemma 1.2.34 *[23, Page 62] Let $M \subseteq C_b(\mathbb{R}_+, \mathbb{R})$. Then M is relatively compact in $C_b(\mathbb{R}_+, \mathbb{R})$ if the following conditions hold:*

(a) M is uniformly bounded in $C_b(\mathbb{R}_+, \mathbb{R})$.

(b) The functions belonging to M are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact interval of $\mathbb{R}_+$.

(c) The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - l| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.

Concluding remarks

Compactness criteria in typical function spaces not only constitute important results describing properties of these spaces, but they also give a basic tool for investigating the existence of solutions to nonlinear equations of many kinds. The best known criterion is the Arzelà-Ascoli theorem that gives necessary and sufficient conditions for compactness in the space of continuous functions defined on a compact space X and taking values in $\mathbb{R}$ or, more generally, in any Banach space E. The natural topology in $\mathcal{C}(X,E)$ is the topology of uniform convergence given by the norm $\|f\| := \sup_{x \in X} \|f(x)\|_E$. If X is not a compact space but only a locally compact one, Arzelà-Ascoli theorem gives a compactness criterion in the space of continuous functions $\mathcal{C}(X,Y)$, where Y is a metric space, with the topology of compact convergence ([62], page 290). A sequence (f_n) tends to $f \in \mathcal{C}(X,Y)$, if $f_n|_K \to f|_K$ uniformly for each compact subset $K \subset X$. If one needs the boundedness of this limit f, then one should work in the space of bounded continuous functions $\mathcal{C}_b(X,Y)$ with its natural topology of uniform convergence. To deal with bounded closed subsets that are not compact, mathematicians introduced the concept of mesaures of non compactness.

1.2.2 Kuratowski's measure of noncompactness

In what follows, we consider a real Banach space $(E, \|.\|)$ and we let Ω_E be the class of all bounded subsets of E.

Definition 1.2.35 (Measure of noncompactness) *A fonction $\varphi : \Omega_E \to [0,+\infty[$ is called measure of noncompactness if it satisfies the following conditions:*

1. $\varphi(A) = 0 \Longleftrightarrow A$ *is relatively compact,* $\forall A \in \Omega_E$.
2. $\varphi(A) = \varphi\left(\overline{A}\right), \forall A \in \Omega_E$.
3. $\varphi(A_1 \cup A_2) = \max\{\varphi(A_1), \varphi(A_2)\}, \forall A_1, A_2 \in \Omega_E$.

There exist many measures of noncompactness, in the following, we shall present some of the most used in application. We will focus on Kuratowski measure of noncompactness since it is the one that we will use throughout this document.

Definition 1.2.36 *The Kuratowski measure of noncompactness (KMNC for short) $\alpha : \Omega_E \to [0,+\infty)$ is defined as*

$$\alpha(V) = \inf\left\{\delta > 0 \ : \ \exists\,(V_i)_{i=1}^n \subset E \ \text{ such that } \ V \subset \bigcup_{i=1}^n V_i \ \text{ and } \operatorname{diam}(\mathrm{V_i}) \leq \delta,\ \forall\, \mathrm{i} = 1,\ldots,\mathrm{n}\right\},$$

where $\operatorname{diam}(\mathrm{V_i}) = \sup\{\|\mathrm{x}-\mathrm{y}\|_{\mathrm{E}},\ \mathrm{x},\mathrm{y} \in \mathrm{V_i}\}$ *is the diameter of V_i.*

Proposition 1.2.37 (Monotonicity) *Let A and B be bounded subsets of E such that $A \subset B$. Then*

$$\alpha(A) \leq \alpha(B)$$

Proof. The proof comes directly from the Definition 1.2.36. ∎

Proposition 1.2.38 (Invariance under passage to the closure) *Let A be a bounded subset of E. Then*

$$\alpha(A) = \alpha(\overline{A})$$

Proof. Since $A \subset \overline{A}$, we get

$$\alpha(A) \le \alpha(\overline{A}). \tag{1.1}$$

Let now, $\varepsilon > 0$ be arbitrarily chosen and fixed. Then there exists a partition

$$A \subset \bigcup_{j=1}^{m} A_j$$

such that

$$A_j \subset E,$$

$$\operatorname{diam}(A_j) < \alpha(A) + \varepsilon, \ j \in \{1, \cdots, m\}.$$

Now, using that

$$\overline{A} \subset \bigcup_{j=1}^{m} \overline{A_j}$$

and

$$\begin{aligned} \operatorname{diam}(\overline{A_j}) &= \operatorname{diam}(A_j) \\ &< \alpha(A) + \varepsilon, \ j \in \{1, \cdots, m\}, \end{aligned}$$

we obtain

$$\alpha(\overline{A}) < \alpha(A) + \varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we obtain

$$\alpha(\overline{A}) \le \alpha(A). \tag{1.2}$$

From (1.1) and (1.2), we get

$$\alpha(A) = \alpha(\overline{A}).$$

This completes the proof. ∎

Proposition 1.2.39 (Subadditivity) *Let A and B be bounded subsets of E. Then*

$$\alpha(A \cup B) = \max\left(\alpha(A), \alpha(B)\right).$$

Proof. Let

$$\eta = \max(\alpha(A), \alpha(B)).$$

Then

$$\eta \leq \alpha(A \cup B). \tag{1.3}$$

Take $\varepsilon > 0$ arbitrarily. Then there are partitions

$$A \subset \bigcup_{j=1}^{m} A_j, B = \bigcup_{l=1}^{k} B_l,$$

such that

$$\begin{aligned} A_j &\subset E, \\ B_l &\subset E, \\ \mathrm{diam}(A_j) &\leq \alpha(A) + \varepsilon \\ &\leq \eta + \varepsilon, \ j \in \{1, \cdots, m\} \\ \mathrm{diam}(B_l) &\leq \alpha(B) + \varepsilon \\ &\leq \eta + \varepsilon, \ \ l \in \{1, \cdots, k\}. \end{aligned}$$

Because

$$A \cup B \subset \left(\bigcup_{j=1}^{m} A_j\right) \bigcup \left(\bigcup_{l=1}^{k} B_l\right),$$

we get

$$\alpha(A \cup B) \leq \eta + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we go to

$$\alpha(A \cup B) \leq \eta. \tag{1.4}$$

From (1.3) and (1.4), we arrive to

$$\alpha(A \cup B) = \eta.$$

This completes the proof. ■

Proposition 1.2.40 (Algebraic subadditivity) *Let A and B be bounded subsets of E. Then*

$$\alpha(A + B) \leq \alpha(A) + \alpha(B),$$

where

$$A + B = \{x + y : \ x \in A, \ y \in B\}.$$

Proof. Take $\varepsilon > 0$ arbitrarily. Then there are partitions

$$A \subset \bigcup_{j=1}^{m} A_j, \ B \subset \bigcup_{l=1}^{k} B_l,$$

such that

$$\begin{aligned} A_j &\subset E, \\ B_l &\subset E, \\ \mathrm{diam}(A_j) &\leq \alpha(A) + \varepsilon, \ j \in \{1, \cdots, m\} \\ \mathrm{diam}(B_l) &\leq \alpha(B) + \varepsilon, \ l \in \{1, \cdots, k\}. \end{aligned}$$

Denote

$$V_{jl} = \{x + y : x \in A_j, y \in B_j\}, \ \ j \in \{1, \cdots, m\}, \ l \in \{1, \cdots, k\}.$$

we have

$$\begin{aligned} A + B &\subset \bigcup_{j=1}^{m} \left(\bigcup_{l=1}^{k} V_{jl} \right) \\ &= \bigcup_{l=1}^{k} \left(\bigcup_{j=1}^{m} V_{jl} \right) \end{aligned}$$

and

$$\begin{aligned} \mathrm{diam}(V_{jl}) &\leq \mathrm{diam}(A_j) + \mathrm{diam}(B_l) \\ &\leq \alpha(A) + \varepsilon + \alpha(B) + \varepsilon \\ &= \alpha(A) + \alpha(B) + 2\varepsilon, \ j \in \{1, \cdots, m\}, \ l \in \{1, \cdots, k\}. \end{aligned}$$

Consequently

$$\alpha(A + B) \leq \alpha(A) + \alpha(B) + 2\varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we get

$$\alpha(A + B) \leq \alpha(A) + \alpha(B).$$

This completes the proof. ■

Proposition 1.2.41 (Invariance under shifting) *Let A be a bounded subset of E. Then*

$$\alpha(A + \{x\}) = \alpha(A).$$

Proof. Proposition 1.2.40 yields $\alpha(A + \{x\}) \leq \alpha(A) + \alpha(\{x\}) = \alpha(A)$. Note that

$$\alpha(\{x\}) \leq \mathrm{diam}(\{\mathrm{x}\}) = 0 \Rightarrow \alpha(\{\mathrm{x}\}) = 0.$$

Hence

$$\begin{aligned} A = A + \{x\} + \{-x\} \quad &\Rightarrow \quad \alpha(A) = \alpha(A + \{x\} + \{-x\}) \le \alpha(A + \{x\}) + \alpha(\{-x\}) \\ &\Rightarrow \quad \alpha(A) \le \alpha(A + \{x\}). \end{aligned}$$

Then $\alpha(A + \{x\}) = \alpha(A)$. ■

Proposition 1.2.42 (Semi-homogeneity) *Let A be a bounded subset of E and $\lambda \in \mathbb{R}$. Then*

$$\alpha(\lambda A) = |\lambda|\alpha(A),$$

where

$$\lambda A = \{\lambda x : \ x \in A\}$$

Proof.

1. Let $\lambda = 0$. Then

$$\lambda A = \{0\}.$$

   Hence,

$$0 = \alpha(\lambda A) = |\lambda|\alpha(A).$$

2. Let $\lambda \ne 0$. Take $\varepsilon > 0$ arbitrarily. Then there is a partition

$$A \subset \bigcup_{j=1}^{m} A_j,$$

   such that

$$\begin{aligned} A_j &\subset E, \\ \operatorname{diam}(A_j) &\le \alpha(A) + \varepsilon, \ j \in \{1, \cdots, m\}. \end{aligned}$$

   We have

$$\lambda A = \bigcup_{j=1}^{m} (\lambda A_j)$$

   and

$$\operatorname{diam}(\lambda A_j) \le |\lambda|(\alpha(A) + \varepsilon).$$

   Consequently,

$$\alpha(\lambda A) \le |\lambda|\alpha(A) + |\lambda|\varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, by the last inequality, we get

$$\alpha(\lambda A) \leq |\lambda|\alpha(A). \tag{1.5}$$

On the other hand, using that

$$A = \lambda^{-1}(\lambda A),$$

as in above, we obtain

$$\begin{aligned} \alpha(A) &= \alpha(\lambda^{-1}(\lambda A)) \\ &\leq |\lambda|^{-1}\alpha(\lambda A), \end{aligned}$$

whereupon

$$|\lambda|\alpha(A) \leq \alpha(\lambda A). \tag{1.6}$$

From (1.5) and (1.6), we get

$$\alpha(\lambda A) = |\lambda|\alpha(A).$$

This completes the proof. ■

Proposition 1.2.43 (Invariance under the convex hull) *([34], pp 8-11).*
Let A be a bounded subset of E. Then

$$\alpha(\mathrm{convA}) = \alpha(\mathrm{A}).$$

Proof. The proof follows from the following facts:

(a) $\mathrm{diam(A)} = \mathrm{diam(conv\ A)}$,

(b) $A \subset \mathrm{conv\,A} \Rightarrow \alpha(\mathrm{A}) \leq \alpha(\mathrm{conv\,A})$,

and uses the following Caratheodory's characterization of the convex hull:

$$\mathrm{conv\,A} = \left\{ \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}\, \mathrm{a_i},\ \ \mathrm{a_i} \in \mathrm{A},\ \ \mathrm{n} \in \mathbb{N}^*,\ \lambda_{\mathrm{i}} \geq 0,\ \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}} = 1 \right\}.$$

■

Proposition 1.2.44 (Lipschitzianity) *Let A and B be bounded subsets of E. Then*

$$|\alpha(A) - \alpha(B)| \leq 2\, d_h(A, B),$$

where $d_h(A,B)$ denotes the Hausdorff distance between the sets A and B, i.e.,

$$d_h(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(\cdot,\cdot)$ denotes the distance from an element of E to a subset of E.

Proof. Take $\varepsilon > 0$ arbitrarily. Then there exists a partition $A \subset \bigcup_{j=1}^{m} A_j$ such that

$$\begin{aligned} A_j &\subset E, \\ \operatorname{diam}(A_j) &< \alpha(A) + \varepsilon, j \in \{1, \cdots, m\}. \end{aligned}$$

Let

$$\mu = d_h(A,B) + \varepsilon$$

and define

$$B_j = \{y \in B : \exists x \in A_j : \|x - y\| < \mu\}, j \in \{1, \cdots, m\}.$$

Since $d_h(A,B) < \mu$, we have $B = \bigcup_{j=1}^{m} B_j$. Let $j \in \{1, \cdots, m\}$ and y_1, $y_2 \in B_j$ be arbitrarily chosen. Then there exist $x_1, x_2 \in A_j$ such that

$$\begin{aligned} \|x_1 - y_1\| &< \mu, \\ \|x_2 - y_2\| &< \mu. \end{aligned}$$

Hence,

$$\begin{aligned} \|y_1 - y_2\| &= \|y_1 - x_1 + x_1 - x_2 + x_2 - y_2\| \\ &\leq \|y_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - y_2\| \\ &< 2\mu + \operatorname{diam}(A_j). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{diam}(B_j) &\leq 2\mu + \operatorname{diam}(A_j) \\ &< 2d_h(A,B) + 2\varepsilon + \alpha(A) + \varepsilon \\ &= 2d_h(A,B) + \alpha(A) + 3\varepsilon \end{aligned}$$

and

$$\alpha(B) \leq 2d_h(A,B) + \alpha(A) + 3\varepsilon.$$

As in above, one can prove

$$\alpha(A) \leq 2d_h(A,B) + \alpha(B) + 3\varepsilon.$$

consequently

$$|\alpha(A) - \alpha(B)| \leq 2d_h(A, B) + 3\varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we obtain

$$|\alpha(A) - \alpha(B)| \leq 2d_h(A, B).$$

This completes the proof. ■

Remark 1.2.45 *It is not easy to calculate the measure of noncompactness for any bounded subset. Hence, we only know it from its different characteristics.*

Historically, KMNC given by (1.2.36) was the first measure of noncompactness introduced in nonlinear analysis in connection with metric spaces [53], 1930. Other measures of noncompactness have been defined since then. The most important ones are the measure of noncompactness of Hausdorff [41], 1957 and the measure of noncompactness of Istratescu [49], 1972. In what follows, we give the definition of these measures.

a) The Hausdorff measure of noncompactness (HMNC for short), also called ball measure of noncompactness, $\gamma : \Omega_E \to [0, +\infty)$ is defined by

$$\gamma(V) = \inf \{\varepsilon > 0 : \textit{ there exists a finite } \varepsilon - \textit{net for } V \textit{ in } E\},$$

where by ε-net, we mean a set $\{z_1, z_2, \cdots, z_m\} \subset E$ such that the balls $B(z_1, \varepsilon), B(z_2, \varepsilon), \cdots, B(z_m, \varepsilon)$ cover V.

b) The Istratescu measure of noncompactness (IMNC for short), also called lattice measure of noncompactness, $\chi : \Omega_E \to [0, +\infty)$ is defined by

$$\chi(V) = \sup \{\rho > 0\textit{: there exists a sequence } (x_n)_n \textit{ in } V \textit{ such that } ||x_m - x_n|| \geq \rho \textit{ for } m \neq n\}$$

Lemma 1.2.46 *Let (E, d) be a metric space. For any set $V \in \Omega_E$, we have*

$$\gamma(V) \leq \alpha(V) \leq 2\gamma(V).$$

Proof. Define the sets:

$$\begin{aligned} K(V) &= \{\delta > 0 : \exists\, n \in \mathbb{N},\ \exists\, (V_i)_{i=1}^{n} \subset E \text{ such that } V \subset \bigcup_{i=1}^{n} V_i \text{ with } \mathrm{diam(V_i) \leq \delta,\ \forall\, 1 \leq i \leq n}\}. \\ H(V) &= \{\varepsilon > 0 : \exists\, m \in \mathbb{N},\ \exists\, \{z_1, z_2, \cdots, z_m\} \subset E, \text{ such that } V \subset \bigcup_{i=1}^{m} B(z_i, \varepsilon)\}. \end{aligned}$$

The following inclusions are immediate.

$$2H(V) \subset K(V) \subset H(V).$$

Indeed,

(a) Given $\delta \in K(V)$, we have

$$\exists\, n \in \mathbb{N},\ \exists\, (V_i)_{i=1}^{n} \subset E \text{ such that } \mathrm{V \subset \bigcup_{i=1}^{n} V_i \text{ with } diam(V_i) \leq \delta,\ \forall\, 1 \leq i \leq n.}$$

Since V_i is bounded for all $1 \leq i \leq n$, then

$$V_i \subset B(z_i, \delta),\ \mathrm{with\ z_i \in V_i,\ \forall\, 1 \leq i \leq n.}$$

Hence,

$$\bigcup_{i=1}^{N} V_i \subset \bigcup_{i=1}^{n} B(z_i, \delta) \Rightarrow V \subset \bigcup_{i=1}^{n} B(z_i, \delta).$$

Therefore,

$$\delta \in H(V), \text{ and } K(V) \subset H(V).$$

(b) For $\varepsilon \in H(V)$, there exist $m \in \mathbb{N}$ and $\{z_1, z_2, \cdots, z_m\} \subset E$ such that

$$V \subset \bigcup_{i=1}^{m} B(z_i, \varepsilon) \text{ where } \mathrm{diam(B(z_i, \varepsilon)) \leq 2\varepsilon,\ \forall\, 1 \leq i \leq m.}$$

Therefore

$$2\varepsilon \in K(V) \Rightarrow 2H(V) \subset K(V).$$

Immediately, we have

(i) $2H(V) \subset K(V) \subset H(V)$. Hence

$$\inf(H(V)) \leq \inf(K(V)) \leq 2\inf(H(V)) \text{ and } \gamma(V) \leq \alpha(V) \leq 2\gamma(V).$$

(ii) Taking $n = 1$ and $\delta = \mathrm{diam(V)}$, we get $V \subset V$. Then

$$\begin{aligned} \mathrm{diam(V)} \in \mathrm{K(V)} \quad &\Rightarrow \quad \inf\left(K(V)\right) \leq \mathrm{diam(V)}, \\ &\Rightarrow \quad \alpha(V) \leq \mathrm{diam(V)}, \\ &\Rightarrow \quad \gamma(V) \leq \alpha(V) \leq \mathrm{diam(V)}. \end{aligned}$$

■

Proposition 1.2.47 *Let E be a metric space and $A \subset E$ be a bounded subset.*

$$\begin{aligned} \alpha(A) = 0 \quad &\Leftrightarrow \quad \gamma(A) = 0 \\ &\Leftrightarrow \quad A \ \textit{is totally bounded.} \end{aligned}$$

Proof. The first equivalence follows from Lemma 1.2.46. As for the second one, we have

$$\begin{aligned} \gamma(A) = 0 \quad &\Leftrightarrow \quad \inf\left\{\varepsilon > 0 : A \text{ has an } \varepsilon\textit{-net}\right\} = 0 \\ &\Leftrightarrow \quad A \text{ has an } \varepsilon\text{-net}, \ \forall\, \varepsilon > 0 \\ &\Leftrightarrow \quad A \text{ totally bounded.} \end{aligned}$$

Recall that a subset

1. A is totally bounded if and only if A has an ε-net, for all $\varepsilon > 0$.

2. $H(A) = \{\varepsilon > 0 : A \text{ has an } \varepsilon\text{-net}\}$.

■

Corollary 1.2.48 *Let E be a complete metric space and $A \subset E$ be a bounded subset. We have*

$$\alpha(A) = 0 \Leftrightarrow \gamma(A) = 0 \Leftrightarrow A \ \textit{is relatively compact.}$$

Proof. According to Proposition 1.2.47, if one MNC is zero, then $\overline{A}$ is totally bounded. Since $\overline{A}$ is a closed subset of the complete metric space E, then $\overline{A}$ is compact. The reverse implication is clear. ■

Remark 1.2.49 *Let $A \subset B \subset E$ be two bounded subsets of the metric space E. Then if B is relatively compact, then A is relatively compact. Moreover if A is relatively compact subset of E, then $0 = \alpha(A) \leq \alpha(B)$, that is the farther B is from A, the larger is its measure. This justifies why α and γ are called measures of noncompactness (MNCs).*

Recall the classical result from functional analysis.

Lemma 1.2.50 *(Riesz Lemma) [70] A normed linear space is finite-dimensional if and only if its closed unit ball is compact.*

Proposition 1.2.51 *Let $B = B(0,1)$ be the unit ball in a Banach space $(E, \|\cdot\|)$. Then*

$$\gamma(B) = \begin{cases} 0, & if \quad dim(E) < \infty, \\ 1, & if \quad dim(E) = \infty. \end{cases}$$

Proof. By Riesz Lemma, we have, since E is complete

$$\begin{aligned} dim(E) < \infty \quad &\Leftrightarrow \quad B \text{ is relatively compact} \\ &\Leftrightarrow \quad B \text{ is totally bounded} \\ &\Leftrightarrow \quad \gamma(B) = 0. \end{aligned}$$

Assume now that diam(E) $= \infty$. Then

$$B(0,1) \subset B(0,1) \Rightarrow 1 \in H(B) \Rightarrow \gamma(B) \leq 1.$$

To prove that $\gamma(B) = 1$, we proceed by contradiction and assume that $\gamma(B) < 1$ and let $0 < \varepsilon < 1 - \gamma(B)$. Then there exist $\varepsilon > 0$, $m \in \mathbb{N}, \{z_1, z_2, \cdots, z_m\} \subset E$ such that

$$B \subset \bigcup_{i=1}^{m} B(z_i, \varepsilon) \text{ and } \gamma(B) \leq \varepsilon < \gamma(B) + \varepsilon < 1.$$

Since $B \subset \bigcup_{i=1}^{m} B(z_i, \varepsilon)$, thus

$$\begin{aligned} \gamma(B) &\leq \max_{1 \leq i \leq m} \gamma(B(z_i, \varepsilon)) \\ &= \max_{1 \leq i \leq m} \gamma(\{z_i\} + \varepsilon B(0,1)) \\ &= \gamma(\varepsilon B) \\ &= \varepsilon \gamma(B). \end{aligned}$$

By Riesz Theorem, $\gamma(B) \neq 0$, which is a contradiction with $1 > \varepsilon$, so $\gamma(B) = 1$. ■

Corollary 1.2.52 *Let $(E, \|\cdot\|)$ be a Banach space and $B = B(x_0, r) \subset E$. Then*

$$\gamma(B) = \begin{cases} 0, & if \quad dim(E) < \infty, \\ r, & if \quad dim(E) = \infty. \end{cases}$$

Proof. Since $B(x_0, r) = \{x_0\} + rB(0, 1)$, then

$$\begin{aligned} \gamma(B(x_0, r)) &= \gamma(\{x_0\} + rB(0, 1)), \\ &= \gamma(rB(0, 1)), \\ &= r\gamma(B(0, 1)), \\ &= \begin{cases} 0, & if \quad dim(E) < \infty, \\ r, & if \quad dim(E) = \infty. \end{cases} \end{aligned}$$

■

Remark 1.2.53 *Let $(E, \|\cdot\|)$ be a normed space and S the unit sphere.*

Since $\mathrm{conv(S)} = \overline{\mathrm{B}}(0, 1)$, *from the properties of KMNC and HMNC, we deduce that*

$$\beta(S) = \beta(\mathrm{conv(S)}) = \beta(\overline{\mathrm{B}}(0, 1)) = \beta(\mathrm{B}(0, 1)), \quad where \;\; \beta = \alpha \; or \; \beta = \gamma.$$

However, in order to compute $\alpha(B(0, 1))$, we need the following lemma.

Lemma 1.2.54 *(Ljusternik-Schrinelman-Borsuk Theorem)[82]. Let S be the unit sphere in a normed space E with $dim(E) = n$. Then, for every covering of S by closed sets $(A_i)_{i=1}^n$, there exists at least one set A_{i_0} that contains two antipodal points of the sphere S.*

Recall that, two points on the sphere are antipodal if they are opposite through the center.

Proposition 1.2.55 *Let $(E, \|\cdot\|)$ be a normed space and $B = B(0, 1)$ be the unit ball in E. Then*

$$\alpha(B) = \begin{cases} 0, & if \quad dim(E) < \infty, \\ 2, & if \quad dim(E) = \infty. \end{cases}$$

Proof. By Riesz Lemma, we have

$$\begin{aligned} dim(E) < \infty \;\; &\Rightarrow \;\; B(0, 1) \text{ is relatively compact}, \\ &\Rightarrow \;\; \alpha(B) = 0. \end{aligned}$$

Assume that $dim(E) = \infty$. Then by Proposition 1.2.51

$$\gamma(B) \leq \alpha(B) \leq 2\gamma(B) \Rightarrow \alpha(B) \leq 2.$$

By contradiction, assume that $\alpha(S) = \alpha(B) < 2$ (by Remark 1.2.53). Then $\forall\, \varepsilon \in (0, 2-\alpha(S)),\ \exists\, n > 0,\ \exists\, (A_i)_{i=1}^n$ (chosen closed) such that

$$S \subset \bigcup_{i=1}^{n} A_i \ \text{ with } \ \mathrm{diam(A_i)} < \alpha(\mathrm{S}) + \varepsilon < 2,\ \forall\, \mathrm{i} \in [1, \mathrm{n}].$$

Let $L = \{x_1, x_2, \cdots, x_n\}$ be a linearly independent subset of E and $F = [L]$. Then $\mathrm{dim(F) = n}$. Let $S_n = \{x \in F : \|x\| = 1\}$. Then $S \cap S_n = S_n \subset \bigcup\limits_{i=1}^{n} (S_n \cap A_i)$ with $\mathrm{diam(S_n \cap A_i)} \leq \mathrm{diam(A_i)} < 2,\ \forall\, \mathrm{i} \in [1, \mathrm{n}]$. This is a contradiction with Lemma 1.2.54. Therefore

$$\alpha(B) = 2.$$

■ The Kuratowski measure is very important in application since it does not only give a new tool to deal with bounded sets in infinite dimension spaces but also helps to define new classes of operators that we will see in the next section.

1.3 Related classes of mappings

1.3.1 Compact and completely continuous maps

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two Banach spaces and $f : E \to F$ a map. Ω_E will denote the family of all bounded subsets of E.

We start by giving the definition of a bounded map, a compact map and a completely continuous map.

Definition 1.3.1 *Let $f \colon D \subset E \to F$ be a map. f is said to be:*

(1) bounded if it maps bounded sets into bounded sets;

(2) compact if the set $f(D)$ is relatively compact;

(3) completely continuous if it is continuous and it maps bounded sets into relatively compact sets.

Remark 1.3.2 *1. If f is a continuous map and D is a bounded set, then the definitions (2) and (3) coincide.*

2. For finite-dimensional spaces, continuous and completely continuous operators are the same. Indeed, if $M \subset D$ is bounded, then M is relatively compact, since $dim(E) < \infty$. Then $f(\overline{M})$ is compact, and hence $f(M)$ is relatively compact, since $\overline{f(M)} \subset f(\overline{M})$.

Example 1.3.3 *Let $G : [a, b] \times [a, b] \to \mathbb{R}$ be a continuous function and $T : \mathcal{C}([a, b], \mathbb{R}) \to \mathcal{C}([a, b], \mathbb{R})$ be the linear operator defined by*

$$Tx(t) = \int_a^b G(t, s)x(s)ds.$$

Then T is compact.

Example 1.3.4 *Let f be a p-integrable function on $[0, 1] (1 < p \leq \infty)$ and define F by*

$$F(x) = \int_0^x f(t)dt.$$

Let $\mathcal{H}$ be the set of functions F corresponding to functions f in the unit ball of the space $L^p([0, 1])$. If q is the Hölder conjugate of p, then Hölder's inequality implies that all functions in $\mathcal{H}$ satisfy the Hölder condition with $\alpha = \frac{1}{q}$ and constant $K = 1$. Hence, $\mathcal{H}$ is compact in $\mathcal{C}([0, 1])$, that is the correspondence $f \mapsto F$ is a linear compact operator from $L^p([0, 1])$ to $\mathcal{C}([0, 1])$. Composing with the injection of $\mathcal{C}([0, 1]$ into $L^p([0, 1])$, we find that F acts compactly from $L^p([0, 1])$ into itself.

Example 1.3.5 *Typical examples of compact operators on infinite-dimensional spaces are integral operators with sufficiently regular conditions. Set*

$$(Tx)(t) = \int_a^b K(t, s, x(s))ds,$$

$$(Sx)(t) = \int_a^t K(t, s, x(s))ds \text{ for all } t \in [a, b].$$

Suppose we have a continuous function

$$K : [a, b] \times [a, b] \times [-R, R] \to \mathbb{R},$$

where $a, b \in \mathbb{R}$. Set

$$M = \{x \in \mathcal{C}([a,b]), \mathbb{K}) : \|x\| \leq R\}.$$

Then the integral operators S and T map M into $\mathcal{C}([a,b], \mathbb{R})$ and are compact and continuous.

Proof. We will consider the operator S. The remaining cases are treated similarly.

(I) The set $A = [a,b] \times [a,b] \times [-R,R]$ is compact, whence K is bounded and uniformy continuous on A. Thus, there is a number δ such that $|K(t,s,x)| \leq \delta$, for all $(t,s,x) \in A$, and for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|K(t_1, s_1, x_1) - K(t_2, s_2, x_2)| < \varepsilon$$

for all (t_i, s_i, x_i) in A, $i = 1, 2$, satisfying $|t_1 - t_2| + |S_1 - S_2| + |x_1 - x_2| < \delta$.

(II) Let $z = Sx$ and $x \in M$. Then

$$|z(t)| \leq |\int_a^t K(t,s,x(s))ds| \leq (b-a)\delta, \ \textit{for all } t \in [a,b].$$

Furthermore, for $|t_1 - t_2| \leq \min(\delta, \varepsilon)$, we have the inequality

$$\begin{aligned} |z(t_1) - z(t_2)| &= |\int_a^{t_1} K(t_1, s, x(s)ds - \int_a^{t_2} K(t_2, s, x(s))ds| \\ &= |\int_a^{t_1} K(t_1, s, x(s))ds - \int_a^{t_1} K(t_2, s, x(s))ds \\ &- \int_{t_1}^{t_2} K(t_2, s, x(s))ds| \\ &\leq (b-a)\varepsilon + |t_1 - t_2|\delta \leq ((b-a) + \delta)\varepsilon. \end{aligned}$$

(III) The inequalities in (II) are uniformly true for all $z = Sx$ with arbitrary $x \in M$. By the Arzelà- Ascoli theorem, the set $S(M)$ is relatively compact.

(IV) The operator S is continuous on M. To see this, let (x_n) be a sequence in M with $\|x_n - x\| \to 0$ as $n \to \infty$, i.e, the functions $x_n(\cdot)$ converge uniformly on $[a,b]$ to $x(\cdot)$. Let $z_n = Sx_n$ and $z = Sx$. Then, Lebesgue's dominated convergence theorem leads

$$\begin{aligned} \|z - z_n\| &= \max_{a \leq t \leq b} |z(t) - z_n(t)| \\ &= \max_{a \leq t \leq b} |\int_a^t (K(t,s,x(s)) - K(t,s,x_n(s)))\, ds| \\ &\to 0 \ \textit{as } n \to \infty. \end{aligned}$$

Note the uniform continuity of K and the uniform convergence of the functions $x_n(\cdot)$ to $x(\cdot)$.

(III) and (IV) together imply that S is completely continuous. ■

1.3.2 k-set contraction maps

We consider two Banach spaces $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$and we let Ω_E be the class of all bounded subsets of E and $f : E \to F$.

Remark 1.3.6 *Lipschitz maps can be characterized by:*

$$f \ \ is \ \ k-Lipschitz \ \ \Leftrightarrow \forall\, A \in \Omega_E, \ \mathrm{diam(f(A))} \le \mathrm{k\,diam(A)}.$$

Indeed,

$$\begin{aligned} f \ \ is \ \ k-Lipschitz \ \ &\Rightarrow \ \ \exists\, k \ge 0 : \|f(x) - f(y)\|_F \le k\,\|x - y\|_E, \ \forall\, x, y \in A; \\ &\Rightarrow \ \ \|f(x) - f(y)\|_F \le k \sup_{x,y\in A} \|x - y\|_E = k\,\mathrm{diam(A)}, \ \forall\,\mathrm{x, y} \in \mathrm{A}; \\ &\Rightarrow \ \ \mathrm{diam(f(A))} \le \mathrm{k\,diam(A)}. \end{aligned}$$

Conversely, let $A = \{x, y\} \in \Omega_E$. Then

$$\begin{aligned} \mathrm{diam(f(A))} \le \mathrm{k\,diam(A)} \ \ &\Rightarrow \ \ \|f(x) - f(y)\|_F \le k\,\|x - y\|_E \\ &\Rightarrow \ \ f \ \ is \ \ k - Lipschitz. \end{aligned}$$

The observation in Remark 1.3.6 suggests to introduce k-set Lipschitz maps for the Kuratowski measure of noncompactness α:

Definition 1.3.7 *(a) f is called a k-set contraction, for some number $k \ge 0$, if it is continuous, bounded and*

$$\alpha(f(A)) \le k\alpha(A), \ \forall\, A \in \Omega_E.$$

(b) f is called a 1-set contraction, if $k = 1$.

(c) f is called a strict k-set contraction if $0 \le k < 1$.

(d) f is called a condensing, if $\forall\, A \in \Omega_E$ with $\alpha(A) > 0$, we have $\alpha(f(A)) < \alpha(A)$.

Example 1.3.8 *Let E be an infinite dimentional Banach space and let $T : E \to E$ be defined by:*

$$T(x) = \begin{cases} -x, & \text{if } \|x\| \leq 1; \\ -\frac{x}{\|x\|}, & \text{if } \|x\| \geq 1. \end{cases}$$

Then, T is a 1-set contraction. To see that, it suffices to show that for all subset K of E, we have $T(K) = \mathrm{conv}(-\mathrm{K} \cup \{0\})$. Indeed,

let $x \in \mathcal{K}$. If $\|x\| \leq 1$ then $T(x) = -x \in -K$. If $\|x\| \geq 1$, then

$$T(x) = \frac{1}{\|x\|}(-x) + (1 - \frac{1}{\|x\|})0 \in \mathrm{conv}(-\mathrm{K} \cup \{0\}).$$

Thus, By Propositions 1.2.42, 1.2.39, 1.2.38, 1.2.37 and the fact that $\alpha(\{0\}) = 0$ we have

$$\alpha(T(K)) \leq \alpha(\mathrm{conv}(-\mathrm{K} \cup \{0\})) = \alpha(\mathrm{K}).$$

which shows that T is a 1-set contraction.

Remark 1.3.9 *(a) f is completely continuous if and only if f is 0-set contraction. Indeed*

$$\begin{aligned} f \quad \text{is completely continuous} \quad &\Rightarrow \quad \overline{f(A)} \quad \text{is compact}, \forall A \in \Omega_E, \\ &\Rightarrow \quad \alpha(f(A)) = \alpha(\overline{f(A)}) = 0, \\ &\Rightarrow \quad f \quad \text{is } 0 - \text{set contraction}. \end{aligned}$$

Conversely,

$$\begin{aligned} f \quad \text{is } 0 - \text{set contraction} \quad &\Rightarrow \quad \alpha(\overline{f(A)}) = \alpha(f(A)) = 0, \ \forall A \in \Omega_E, \\ &\Rightarrow \quad \overline{f(A)} \quad \text{is compact, (since } E \text{ is complete)}, \\ &\Rightarrow \quad f \quad \text{is completely continuous}. \end{aligned}$$

(b) If f is a strict k-set contraction, then f is condensing. Indeed, let $A \in \Omega_E$ with $\alpha(A) > 0$. Then, since f is a strict k-set contraction, there exists $0 \leq k < 1$ such that $\alpha(f(A)) \leq k\,\alpha(A) < \alpha(A)$, that is f is condensing.

(c) If f is condensing, then f is 1-set contraction. Indeed, suppose that f is condensing. Then

(i) if $\alpha(A) > 0$, then $\alpha(f(A)) \leq \alpha(A) \Rightarrow f$ is 1-set contraction,

(ii) if $\alpha(A) = 0$, then $\overline{A}$ is compact and E is complete. Hence $f(\overline{A})$ is compact for f is continuous. As a consequence $\alpha(f(A)) = 0 \leq \alpha(A)$, since $\overline{f(A)} \subset f(\overline{A})$ and $\alpha(f(\overline{A})) = 0$).

(d) Let $f : (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ be a k-set contraction, and $g : (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ be a completely continuous mapping. Then $f + g$ is a k-set contraction. Indeed let $A \in \Omega_E$. We have

$$\begin{aligned} \alpha((f+g)(A)) &= \alpha(f(A) + g(A)), \\ &\leq \alpha(f(A)) + \alpha(g(A)) \\ &= \alpha(f(A)) + 0, \\ &\leq k\,\alpha(A). \end{aligned}$$

Hence $f + g$ is a k-set contraction.

Proposition 1.3.10 *Every k-Lipschitz map is a k-set contraction (with respect to the Kuratowski measure of noncompactness).*

Proof. Let $A \in \Omega_E$. Then

$$\forall\, \varepsilon > 0,\ \exists\, \delta_\varepsilon > 0,\ \exists\, n \in \mathbb{N},\ \exists\, \{A_1, A_2, \cdots, A_n\} \subset E : A \subset \bigcup_{i=1}^{n} A_i,$$

with $\mathrm{diam(A_i)} \leq \delta_\varepsilon,\ \forall\, \mathrm{i} \in \{1, ..., \mathrm{n}\}$ such that $\alpha(A) \leq \delta_\varepsilon < \alpha(A) + \varepsilon$. We have

$$f(A) \subset f\left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} f(A_i).$$

Then

$$\alpha(f(A)) \leq \alpha\left(\bigcup_{i=1}^{n} f(A_i)\right) \leq \max_{1\leq i\leq n} \alpha(f(A_i)) \leq \max_{1\leq i\leq n} \mathrm{diam(f(A_i))}.$$

By Remark 1.3.6, we have

$$\begin{aligned} \alpha(f(A)) \leq \max_{1\leq i\leq n} \mathrm{diam(f(A_i))} &\leq \max_{1\leq i\leq n} k\,\mathrm{diam(A_i)},\ \text{(since } f \text{ is Lipschitz)} \\ &\leq k\,\delta_\varepsilon < k\,(\alpha(A) + \varepsilon),\ \forall\, \varepsilon > 0. \end{aligned}$$

Hence $\alpha(f(A)) \leq k\,\alpha(A)$. ■

Remark 1.3.11 *In case of the Hausdorff MNC, we can show in a similar manner that every k-Lipschitz map is $2k$-set contraction. Thus, according to Propositions 1.2.51 and 1.2.55, we can say that every k-Lipschitz map is $\beta(B(0,1))k$-set contraction, where β is either α or γ and $B(0,1)$ the unit ball.*

Proposition 1.3.12 *Let $f : (E_1, \|\cdot\|_{E_1}) \to (E_2, \|\cdot\|_{E_2})$ and $g : (E_1, \|\cdot\|_{E_1}) \to (E_2, \|\cdot\|_{E_2})$ be k_1-set and k_2-set contraction, respectively. Then $f + g : (E_1, \|\cdot\|_{E_1}) \to (E_2, \|\cdot\|_{E_2})$ is a $(k_1 + k_2)$-set contraction.*

Proof. Given $A \in \Omega_E$, we have

$$\begin{aligned} \alpha(f(A) + g(A)) &\leq \alpha(f(A)) + \alpha(g(A)) \\ &\leq k_1\,\alpha(A) + k_2\,\alpha(A) \\ &= (k_1 + k_2)\,\alpha(A). \end{aligned}$$

■

Proposition 1.3.13 *Let $f : (E_1, \|\cdot\|_{E_1}) \to (E_2, \|\cdot\|_{E_2})$ and $g : (E_2, \|\cdot\|_{E_2}) \to (E_3, \|\cdot\|_{E_3})$ be k_1-set and k_2-set contraction, respectively. Then $g \circ f : (E_1, \|\cdot\|_{E_1}) \to (E_3, \|\cdot\|_{E_3})$ is a $k_1.k_2$-set contraction.*

Proof. Let $A \in \Omega_E$. Then

$$\begin{aligned} \alpha(g(f(A))) &\leq k_2\,\alpha(f(A)) \text{ (since } g \text{ is } k_2\text{-set contraction)} \\ &\leq k_2.k_1\,\alpha(A) \text{ (since } f \text{ is } k_1\text{-set contraction).} \end{aligned}$$

■

1.3.3 Expansive and nonexpansive maps

Definition 1.3.14 *Let A mapping $T : D \subset X \to X$, where (X, d) is a metric space.*

(1) T is called expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h\,d(x, y) \quad \textit{for all} \quad x, y \in D;$$

(2) T is called nonexpansive, if

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all} \quad x, y \in D;$$

(3) T is called a contraction, if there exists a constant $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq k\, d(x, y) \quad \text{for all} \quad x, y \in D;$$

(4) T is nonlinear expansive (or ψ-expansive) if

$$d(Tx, Ty) \geq \psi(d(x, y)),\ \forall\, x, y \in D,$$

where $\psi : [0, \infty) \to [0, \infty)$ satisfies $\psi(0) = 0$ and $\psi(t) > t$, for all $t \geq 0$.

(5) T is called a nonlinear contraction (or a ϕ-contraction) if

$$d(Tx, Ty) \leq \phi(d(x, y)),\ \forall\, x, y \in D,$$

where $\phi : [0, \infty) \to [0, \infty)$ satisfies $\phi(0) = 0$ and $\phi(t) < t$, for all $t \geq 0$.

Example 1.3.15 *(1) Let $T : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $Tx = x^3 + \lambda x$. Then T is expansive with constant $h = \lambda > 1$.*

(2) Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = e^x$. Then T is ψ-expansive with $\psi(t) = t + \frac{1}{2}t^2$. Indeed,

$$\forall\, x, y \in \mathbb{R},\ \ |e^x - e^y| = e^{\min(x,y)}(e^{|x-y|} - 1) \geq |x - y| + \frac{1}{2}|x - y|^2.$$

(3) In [33, Example 3.3], it is showed that, if r is the unit ball retraction of an infinite Banach space X, then $T = -r$ is a 1-set contraction and $I - T$ is h-expansive with constant $h > 1$.

Remark 1.3.16 *Noting that:*

(i) If we take $\psi(t) = ht$ with $h > 1$, the nonlinear expansive in (4) reduces to an expansion with constant h.

(ii) If we take $\phi(t) = kt$ with $0 < k < 1$, the nonlinear contraction in (5) reduces to a contraction with constant k.

(iii) The sum of a nonexpansive map and a completely continuous one is a 1-set contraction.

(vi) The sum of a contraction and a completely continuous map is a strict k-set contraction, hence a condensing mapping and then a 1-set contraction.

The following example demonstrates the usefulness of writing a map as the sum of two other ones to study it properties. Precisely, it is illustration of point (iii), Remark 1.3.16.

Example 1.3.17 *Let $l_1 = \{x = (x_1, x_2, x_3, x_4, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$ the set of sommable sequences provided with the norm $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$. Then the mapping $T : B_{l_1} \to B_{l_1}$ defined as follows :*

$$T(x) = (1 - \sum_{n=1}^{\infty} |x_n|, x_1, x_2, ..., x_n, ...),$$

is a 1-set contraction. It is obvious that with a direct reasoning we can only show that T is a 2-Lipschitz, it is then a 2-set contraction. Indeed, for all $x = (x_n), y = (y_n)$ in B_{l_1} we have

$$\|T(x) - T(y)\|_1 = \left|\sum_{n=1}^{\infty}(|y_n| - |x_n|)\right| + \sum_{n=1}^{\infty} |x_n - y_n| \leq 2\|x - y\|_1.$$

Now, to show that T is a 1-set contraction, we consider

$$T = S + R,$$

where $S, R : B_{l_1} \to B_{l_1}$ are tow continuous maps defined by

$$\begin{aligned} S(x) &= (1 - \sum_{n=1}^{\infty} |x_n|)e_1 = (1 - \sum_{n=1}^{\infty} |x_n|, 0, 0, 0, ...), \\ R(x) &= \sum_{n=1}^{\infty} x_n e_{n+1} = (0, x_2, x_3, x_4, ...). \end{aligned}$$

In what follows, we show that S is a compact map and R is a nonexpansive one.
Let $(x^{(m)})_{m\in\mathbb{N}}$ such that $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}, ...)$ a sequence of B_{l_1}, it is easy to show that the sequence $(y^{(m)})_{m\in\mathbb{N}}$ such that

$$y^{(m)} = (y_1^{(m)}, y_2^{(m)}, ..., y_n^{(m)}, ...) = S(x^{(m)}) = (1 - \sum_{n=1}^{\infty} |x_n^{(m)}|, 0, 0, 0, ...),$$

has a convergent subsequence in l_1. In fact, for all $m \in \mathbb{N}$, we have $\sum_{n=1}^{\infty} |x_n^{(m)}| \leq 1$, so $|y_n^{(m)}| \leq 1$ $(n = 1, 2, 3, ...)$. Hence, the sequence $(y_n^{(m)})_m$ is bounded in $\mathbb{R}$, which implies the existence of a convergent sub-sequence $(y_n^{(m_k)})$ such that $y_n^{(m_k)} \to \bar{y}_n$, when $m_k \to +\infty$.
Thus, $\bar{y} = (\bar{y}_1, \bar{y}_2, ..., \bar{y}_n, ...) \in l_1$ and $\|y^{(m_k)} - \bar{y}\|_1 = \sum_{n=1}^{\infty} |y_n^{(m_k)} - \bar{y}_n| \to 0$ when $m_k \to +\infty$.
Therefore, S is a compact, continuous map.
In the other hand, for all $x = (x_n), y = (y_n)$ in B_{l_1} we have

$$\|R(x) - R(y)\|_1 = 0 + \sum_{n=2}^{\infty} |x_n - y_n| \leq \|x - y\|_1.$$

Therefore, R is a nonexpansive map.

Now, we give some properties and results of expansive mappings that will be useful in the sequel.

Lemma 1.3.18 *Let $(E, \|.\|)$ be a linear normed space and $D \subset X$. Assume that the mapping $T : D \to E$ is expansive with constant $h > 1$. Then the inverse of $T : D \to T(D)$ exists and*

$$\|T^{-1}x - T^{-1}y\| \leq \frac{1}{h}\|x - y\|, \ \ \forall\, x, y \in T(D).$$

Proof. It is a direct consequence of the definition of expansive mapping. ■

Proposition 1.3.19 ([80, Lemma 2.1]) *Let $(E,\ \|.\|)$ be a normed linear space, $D \subset E$, and I be the identity map of E. If a mapping $T : D \to E$ is expansive with a constant $h > 1$, then the mapping $I - T : D \to (I - T)(D)$ is invertible and*

$$\|(I - T)^{-1}x - (I - T)^{-1}y\| \leq \frac{1}{h-1}\|x - y\| \ \ \textit{for all} \ \ x, y \in (I - T)(D).$$

Proof. For each $x,\ y \in D$, we have

$$\|(I - T)x - (I - T)y\| = \|(Tx - Ty) - (x - y)\| \geq (h - 1)\|x - y\|, \tag{1.7}$$

which shows that $(I - T)^{-1} : (I - T)(D) \to D$ exists. Hence, for $x, y \in (I - T)(D)$, we have $(I - T)^{-1}x,\ (I - T)^{-1}y \in D$. Thus, using $(I - T)^{-1}x,\ (I - T)^{-1}y$ substitute for x, y in (1.7), repesectively, we obtain

$$\|(I - T)^{-1}x - (I - T)^{-1}y\| \leq \frac{1}{h-1}\|x - y\|.$$

■

Proposition 1.3.20 *[79, Lemma 2.5] Let $(E,\ \|\cdot\|)$ be a linear normed space, $M \subset E$. Assume that the mapping $T : M \to E$ is a contraction with a constant $k < 1$, then the inverse of $I - T : M \to (I - T)(M)$ exists, and*

$$\|(I - T)^{-1}x - (I - T)^{-1}y\| \leq (1 - k)^{-1}\|x - y\| \quad \textit{for all} \ x, y \in (I - T)(M).$$

Proposition 1.3.21 *[30] Let $(E, \|.\|)$ a normed space. We have*

(a) If T is ϕ-contraction, then $(I-T)$ is ψ-expansive, invertible, and $(I-T)^{-1}$ is continuous mapping.

(b) If T is ψ-expansive, then $(I-T)$ is $(\psi-1)$-expansive, invertible, and $(I-T)^{-1}$ is continuous mapping.

Proof. (a)

$$\begin{aligned}\|(I-T)x-(I-T)y\| &\geq \|x-y\|-\|Tx-Ty\| \\ &\geq \|x-y\|-\phi(\|x-y\|)=\psi(\|x-y\|),\end{aligned}$$

where $\psi(s)=s-\phi(s)$, for $s>0$.

(b)

$$\begin{aligned}\|(I-T)x-(I-T)y\| &\geq \|Tx-Ty\|-\|x-y\| \\ &\geq \psi(\|x-y\|)-\|x-y\|=\widetilde{\psi}(\|x-y\|),\end{aligned}$$

where $\widetilde{\psi}(s)=\psi(s)-s$, for $s>0$. In particular, if $\psi(s)=hs$ with $h>1$, then $(I-T)^{-1}$ is $(h-1)^{-1}$-Lipschitz. ■

Remark 1.3.22 *Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $T(x,y)=(y,-x)$. Then $(I-T)$ is ψ-expansive with $\psi(t)=\sqrt{2}t$ but T is not a nonlinear contraction, showing that the converse in part (a) is not true.*

1.3.4 Some related fixed point theorems

Let us mention the following fixed point result for expansive mappings which accompanies the contraction fixed point theorem.

Proposition 1.3.23 ([80, Theorem 2.1]) *Let (X,d) be a complete metric space and D be a closed subset of X. Assume that the mapping $T:D\to X$ is expansive and $D\subset T(D)$, then there exists a unique point $x^*\in D$ such that $Tx^*=x^*$.*

Proof. Since T is expansive, there exists $h>1$ such that

$$d(Tx,Ty)\geq h\,d(x,y),\quad \forall\, x,y\in D.$$

So $T : D \to T(D)$ is injective. Hence $T^{-1} : T(D) \to D$ exists and it is $\frac{1}{h}$-contraction. Indeed, Let y_1, $y_2 \in T(D)$, then there exist x_1, $x_2 \in D$ such that

$$d(y_1, y_2) = d(Tx_1, Tx_2) \geq h\, d(x_1, x_2).$$

Therefore

$$d(T^{-1}y_1, T^{-1}y_2) \leq \frac{1}{h} d(y_1, y_2), \text{ for each } y_1, y_2 \in T(D).$$

Since, $D \subset T(D)$, from Banach's contraction principle, the equation $T^{-1}x = x$ has a unique solution on D which is the unique fixed point of T. ■

Corollary 1.3.24 *Assume that the mapping $T : E \to E$ is expansive and onto, then there exists a unique point $x^* \in E$ such that $Tx^* = x^*$.*

Corollary 1.3.25 *Let $T : E \to E$. Assume that there exists a positive integer n such that T^n is expansive and onto, then there exists a unique point $x^* \in E$ such that $Tx^* = x^*$.*

Proof. According to Corollary 1.3.24, there exists a unique point $x^* \in E$ such that $T^n x^* = x^*$, which implies that Tx^* is a fixed point of T^n. In view of uniqueness, we have $Tx^* = x^*$. And x^* is the unique fixed-point of T. This completes the proof. ■

Now, combining the Banach contraction mapping principle and Corollary 1.3.24, we obtain the following result

Corollary 1.3.26 *Let $T : E \to E$. If one of the following conditions holds*

(i) the mapping T is a contraction; or

(ii) the mapping T is expansive and onto.

Then there exists a unique point $x^* \in E$ such that $Tx^* = x^*$.

Example 1.3.27 *Let $x_0 \in \mathbb{R}$, k be a positive odd number, $h > 1$ and $T : \mathbb{R} \to \mathbb{R}$ defined by*

$$Tx = x^k + hx + x_0.$$

It is easy to check that the assumptions of Corollary 1.3.24 are satisfied, so there exists a unique point $x^* \in \mathbb{R}$ such that $Tx^* = x^*$. We cautiously note that the Banach contraction mapping principle cannot be directly applied in this case.

Remark 1.3.28 **(a)** *A nonexpansive mapping dosen't necessary have a fixed point, as shows the shift operator* $x \longmapsto x + v$ *for* $v \not\equiv 0$.

(b) *The identity operator shows that a fixed point for a nonexpansive mapping is not necessary unique.*

Remark 1.3.29 *Clearly if* $T : \Omega \longrightarrow X$ *is a* ψ*-expansive mapping, then* T *is injective and* T^{-1} *is uniformly continuous on the image set.*

Proposition 1.3.30 *[33, Lemma 3.1] Let* Ω *be a bounded closed convex subset of* X *and* $T : \Omega \longrightarrow X$ *such that*

(a) T *is continuous,*

(b) $(I - T)$ *is* ψ*-expansive,*

(c) T *has a sequence of approximate fixed points.*

Then T *has a unique fixed point.*

Corollary 1.3.31 *Assume that* Ω *is a nonempty closed convex subset and* $T : \Omega \longrightarrow \Omega$ *satisfies*

(a) T *is nonexpansive,*

(b) $I - T$ *is* ψ*-expansive,*

Then T *has a unique fixed point.*

Proof. For clarity, let $\Omega = B(0, R)$. Then $\left(1 - \frac{1}{n}\right) T$ is a contraction, hence admits a unique fixed point x_n, for each $n \in \mathbb{N}$. Hence

$$0 \leq \|T(x_n) - x_n\| = \|T(x_n) - \left(1 - \frac{1}{n}\right) T(x_n)\| \leq \frac{1}{n}\|T(x_n)\| \leq \frac{R}{n}.$$

Proposition 1.3.30 completes the proof. ∎

Remark 1.3.32 *(a) Boyd and Wong (1969, [18]) proved existence of a unique fixed point for a* ϕ*-contraction when* ϕ *is further u.s.c. from the right.*

(b)Matkowski ([60], 1975) replaced the condition $\phi(t) < t$, $\forall\, t > 0$ *by* $\lim_{n\to\infty} \phi^n(t) = 0$, *for* $t > 0$ *whenever* ϕ *is non-decreasing.*

1.4 Topological degree theory

In this section we present an introduction to the concept of topological degree from an analytic viewpoint. In particular, we summarize two of the most relevant constructions of the degree in literature: the Brouwer degree for continuous maps between Euclidean spaces of finite dimension and the Leray–Schauder degree for compact perturbations of the identity in real Banach spaces. We start by asking the following question : what is the topological degree? As a rough answer, the degree is a tool, precisely a number, which gives information about the solutions of equations of the form:

$$f(x) = y_0, \quad x \in \Omega \tag{1.8}$$

where

(i) $f : X \to Y$ is a given function, supposed at least continuous;

(ii) X and Y are finite or infinite dimensional Banach spaces;

(iii) y_0 is a fixed element of Y;

(iv) Ω is an open subset of X.

In the cases where a direct computation does not solve an equation as the equation above, neither give suitable approximations of the solutions, we can look for other methods to get information about the set of solutions. For example we can ask if the set of solutions is not empty. Is it finite or infinite? Where the solutions or some of them are? Are the solutions localized in Ω? Are they stable with respect to perturbations of f or y_0? And other even more complicated issues.

1.4.1 Brouwer's topological degree

After a pioneering work of Kronecker [52] in 1869, the first definition of degree for maps between Euclidean spaces is due to Brouwer [19] in 1912. In 1951, Nagumo [61] redefines the concept, today commonly known as Brouwer degree, by an analytical approach, which is different from the original Brouwer construction and uses Sard's Theorem [71]. We give in this section a short summary of the Brouwer degree with its most important properties.

Axiomatic definition of the degree

We consider the space $\mathcal{C}(\overline{\Omega})$ of continuous maps $f : \overline{\Omega} \to \mathbb{R}^n$ with the supremum norm

$$\|f\|_\infty = \sup_{x\in\overline{\Omega}} |f(x)|.$$

Let $y_0 \in \mathbb{R}^n$. We will be interested in the following subspace of $\mathcal{C}(\overline{\Omega})$:

$$\mathbb{K}(\overline{\Omega}) = \{f \in \mathcal{C}(\overline{\Omega}) : y_0 \notin f(\partial\Omega)\}.$$

That is $f \in \mathbb{K}(\overline{\Omega})$ if and only if $f \in \mathcal{C}(\overline{\Omega})$ and f(x) $\neq y_0$ for $x \in \partial\Omega$.

Now, we define a relation in the set $\mathbb{K}(\overline{\Omega})$, that will appear to be one of the most important tools that we will use.

Definition 1.4.1 *We call two maps f, $g \in \mathbb{K}(\overline{\Omega})$ homotopic if there exists a continuous map $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$, such that*

- *$H(t, \cdot) \in \mathbb{K}(\overline{\Omega})$, for $t \in [0, 1]$;*
- *$H(0, \cdot) = f$;*
- *$H(1, \cdot) = g$.*

We call the map H homotopy joining maps f and g.

Example 1.4.2 *Let $f, g : [-1, 1] \to \mathbb{R}$ be given by $f(x) = x^2$ and $g(x) = 2$. As we can see the map $H : [0, 1] \times [-1, 1] \to \mathbb{R}$ given by*

$$H(t, x) = (1 - t)f(x) + tg(x)$$

is a valid homotopy joining f and g.

Let us now proceed to the axiomatic definition of the topological degree.

Definition 1.4.3 *By topological degree we mean the family of maps* $deg(\cdot, \Omega, y_0) : \mathbb{K}(\overline{\Omega}) \to \mathbb{Z}$, *defined for open and bounded subset* $\Omega \subset \mathbb{R}^n$ *and satisfying the following axioms:*

(A1) (Normalization) If $y_0 \in \Omega$, *then* $deg(I, \Omega, y_0) = 1$, *where* I *is the indentity map in* $\mathbb{R}^n$;

(A2) (Additivity) Let $\Omega_1, \Omega_2 \subset \Omega$ *be such open subsets that* $\Omega 1 \cap \Omega_2 = \emptyset$ *and* $y_0 \notin f(\Omega \backslash (\Omega_1 \cup \Omega_2))$, *then*

$$deg(f, \Omega, y_0) = deg(f|_{\overline{\Omega 1}}, \Omega_1, y_0) + deg(f|_{\overline{\Omega_2}}, \Omega_2, y_0).$$

(A3) (Homotopy invariance) Let $f, g \in \mathbb{K}(\overline{\Omega})$ *be homotopic, then* $deg(f, \Omega, y_0) = deg(g, \Omega, y_0)$.

We call the integer value $deg(f, \Omega, y_0)$ *the topological degree of the map* f *on* y_0 *relative to* Ω.

Other Properties of the degree

We are going to present several simple properties that may be inferred from the set of axioms presented before.

Proposition 1.4.4 *(Invariance on the boundary) Assume* $f, g \in \mathbb{K}(\overline{\Omega})$ *are maps satisfying* $f(x) = g(x)$ *for* $x \in \partial\Omega$. *Then* $deg(f, \Omega, y_0) = deg(g, \Omega, y_0)$.

Proof. Let us define the homotopy $h : [0, 1] \times \overline{U} \to \mathbb{R}^n$ by

$$h(t, x) = (1 - t)f(x) + tg(x).$$

As we can see that $h(t, x) = f(x) = g(x)$ for all $(t, x) \in [0, 1] \times \partial\Omega$. But as $f \in \mathbb{K}(\overline{\Omega})$, we are sure that $f(x) \neq y_0$. This means that maps f and g are homotopic and hence by the homotopy axiom, we can see that $deg(f, \Omega, y_0) = deg(g, \Omega, y_0)$. ■

Proposition 1.4.5

$$deg(f, \emptyset, y_0) = 0.$$

Proof. Let us take $\Omega = \Omega_1 = \Omega_2 = \emptyset$. As we can see, we may apply the additivity axiom and conclude that

$$deg(f, \Omega, y_0) = deg(f, \Omega_1, y_0) + deg(f, \Omega_2, y_0);$$

$$deg(f,\emptyset,y_0) = deg(f,\emptyset,y_0) + deg(f,\emptyset,y_0) = 2\,deg(f,\emptyset,y_0).$$

Hence

$$deg(f,\emptyset,y_0) = 0.$$

■

Proposition 1.4.6 *(Excision property). Let $f \in \mathbb{K}(\overline{\Omega})$.*

a) If $V \subset \Omega$ is such open bounded set that $y_0 \notin f(\overline{\Omega} \setminus V)$, then

$$deg(f,\Omega,y_0) = deg(f,V,y_0).$$

b) If $W \subset \overline{\Omega}$ is closed and $y_0 \notin f(W) \cup f(\partial\Omega)$, then

$$deg(f,\Omega,y_0) = deg(f,\Omega \setminus W,y_0).$$

Proof. a) Let us take $\Omega_1 = V$ and $\Omega_2 = \emptyset$. We can see that applying additivity axiom to the sets Ω, Ω_1, and Ω_2 we arrive to

$$deg(f,\Omega,y_0) = deg(f|_{\overline{V}},V,y_0) + deg(f,\emptyset,y_0).$$

b) Let us take $\Omega_1 = \Omega \setminus W$ and $\Omega_2 = \emptyset$. We can see that applying additivity axiom to the sets Ω, Ω_1, and Ω_2 we arrive to $deg(f,\Omega,y_0) = deg(f|_{\overline{\Omega \setminus W}},\Omega \setminus W,y_0) + deg(f,\emptyset,y_0)$. ■

Proposition 1.4.7 *Let $f \in \mathbb{K}(\overline{\Omega})$ be such that $y_0 \notin f(\overline{\Omega})$. Then $deg(f,\Omega,y_0) = 0$.*

Proof. By using the excision property given above for $V = \emptyset$ and the Proposion 1.4.5, we get that $deg(f,\Omega,y_0) = deg(f,\emptyset,y_0) = 0$. ■

Proposition 1.4.8 *(Existence property) Assume $deg(f,\Omega,y_0) \neq 0$. Then there exists $x_0 \in \Omega$, such that $f(x_0) = y_0$.*

Proof. This is just the logical transposition of the Property given in Proposition 1.4.7. ■

Proposition 1.4.9 *(Translation property) Let $f \in \mathbb{K}(\overline{\Omega})$. For all $z \in \mathbb{R}^n$,*

$$deg(f,\Omega,y_0) = deg(f-z,\Omega,y_0-z).$$

Proof. We consider the natural homotopy between (f, y_0) and $(f - z, y_0 - z)$, that is

$$h(t, x) = (1 - t)f(x) + t(f(x) - z) = f(x) - tz \text{ and } y(t) = (1 - t)y_0 + t(y_0 - z) = y_0 - tz.$$

For $t \in [0, 1]$, if there exists some x_0 in $\partial\Omega$ such that $h(t, x_0) = y(t)$, that is $f(x_0) - tz = y_0 - tz$. So $f(x_0) = y_0$, Contradicting the assumption. The result then follows from the homotopy invariance axiom in Definition 1.4.3. ■

Proposition 1.4.10 *(Continuity property: Continuity with respect to the function and y_0) Let $f \in \mathbb{K}(\overline{\Omega})$ and $r = dist(y_0, f(\partial\Omega)) > 0$. If $g : \overline{\Omega} \to \mathbb{R}^n$ continuous and $z \in \mathbb{R}^n$ are such that $\sup\limits_{\partial\Omega}(|g - f|) + |y_0 - z| < r$, then $deg(f, \Omega, y_0) = deg(g, \Omega, z)$.*

Proof. Let us note first that r is indeed strictly positive because Ω being bounded leads to $\partial\Omega$ is compact, so that $f(\partial\Omega)$ is closed (in fact compact). Then all point which does not belong to $f(\partial\Omega)$ is at a strict positive distance from this set (i.e., $y_0 \notin f(\partial\Omega)$ means $r > 0$). Let $h(t, x) = tg(x) + (1 - t)f(x)$ and $y(t) = tz + (1 - t)y_0$. If there exist $t_1 \in [0, 1]$ and $x_1 \in \partial\Omega$ such that $t_1 g(x_1) + (1 - t_1)f(x_1) = t_1 z + (1 - t_1)y_0$, then
$|y_0 - f(x_1)| \leq t_1|g(x_1) - f(x_1)| + t_1|y_0 - z| < r$ (since $|g(x) - f(x)| + |y_0 - z| < r$), which contradicts the definition of r. The result then follows from the homotopy invariance axiom in Definition 1.4.3.

■

Proposition 1.4.11 *(Invariance on the connected components of $R^n \setminus f(\partial\Omega)$). Let $f \in \mathbb{K}(\overline{\Omega})$. Then, $deg(f, \Omega, \cdot)$ is constant on the connected components of $\mathbb{R}^n \setminus f(\partial\Omega)$.*

Proof. The application $y_0 \mapsto deg(f, \Omega, y_0)$ is defined on $\mathbb{R}^n \setminus f(\partial\Omega)$ and, by the Proposition 1.4.10, is locally constant, we deduce that it is constant on the connected components of $\mathbb{R}^n \setminus f(\partial\Omega)$, which ends the proof. ■

Remark 1.4.12 *The Existence property shows the main power of the topological degree as the tool for solving different problems. By showing that the degree has the nonzero value in the given open set Ω, we may conclude that there must exists zero of the map $f - y_0$ somewhere in the open set Ω. Although we don't know how the value of the degree may be computed yet, we can* *feel that if this technical issue is overcome, we can have quite nice tool of showing that a solution to our problem exists. When* $deg(f, \Omega, y_0) \neq 0$, *then not only* $f(x) = y_0$ *admits a solution in* Ω, *but this equation is still soluble in* Ω *for every second members in a small neighborhood of* y_0.

Determinant formula of the degree on $\mathbb{K}(\overline{\Omega}) \cap \mathcal{C}^1(\Omega, \mathbb{R}^n)$

The process of constructing Brouwer's degree $deg(f, \Omega, y_0)$, in this case, is done in two steps. In the following, we say that $y_0 \in f(\Omega)$ is a regular value of f if $J_f(x_0) \neq 0$ for all $x_0 \in f^{-1}(y_0)$. The points $x_0 \in \Omega$ for which $J_f(x_0) = 0$ are called critical points of f. The set of critical points of f is denoted by K_f. We start by defining the degree map $(f, \Omega, y_0) \mapsto deg(f, \Omega, y_0)$ in the "generic" case, that is, for $f \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ and $y_0 \in \mathbb{R}^n \setminus f(K_f \cup \partial\Omega)$. In this case,

$$f^{-1}(y_0) \neq \emptyset \textit{ if and only if } y_0 \textit{ is a regular value,}$$

in which case $f^{-1}(y_0)$ is compact and discrete (by virtue of the inverse function theorem, since $J_f(x_0) \neq 0$ for all $x \in f^{-1}(y_0)$); hence $f^{-1}(y_0)$ is finite. Therefore, the following definition makes sense.

Definition 1.4.13 *If* $\Omega \subset \mathbb{R}^n$ *is nonempty, bounded, and open,* $f \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$, *and* $y_0 \in \mathbb{R}^n \setminus f(K_f \cup \partial\Omega)$, *then*

$$deg(f, \Omega, y_0) = \begin{cases} \displaystyle\sum_{x \in \Omega \cap f^{-1}(y_0)} sgn J_f(x), & \textit{if} \quad \Omega \cap f^{-1}(y_0) \neq \emptyset, \\ 0, & \textit{if} \quad \Omega \cap f^{-1}(y_0) = \emptyset, \end{cases}$$

where $sign J_f(x_0)$ *is the sign of the determinant of the Jacobian matrix* $Df(x_0)$.

Then, as a second step in the construction, we remove the assumption that y_0 is regular, f still being $\mathcal{C}^1$. This step in the construction will be based on the Sard's Theorem. One fundamental property of the degree in Definition 1.4.13 is that

$$deg(f, \Omega, y_1) = deg(f, \Omega, y_2)$$

whenever $y_1, y_2 \in \mathbb{R}^n \setminus f(K_f \cup \partial\Omega)$ belong to the same connected component of $\mathbb{R}^n \setminus f(\partial\Omega)$. This property makes it possible to extend the degree to the case of $y \in f(K_f) \setminus f(\partial\Omega)$, i.e., the next definition makes sense.

Definition 1.4.14 *Let $\Omega \subset \mathbb{R}^n$ be nonempty, bounded and open subset $f \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$, and $y_0 \in f(K_f) \setminus f(\partial\Omega)$. We define*

$$deg(f, \Omega, y_0) = deg(f, \Omega, y_1)$$

whenever $y_1 \in \mathbb{R}^n \setminus f(K_f \cup \partial\Omega)$ is such that $\|y_1 - y_0\| < d(y_0, f(\partial\Omega))$, (where $d(y_0, f(\partial\Omega))$ stands for the distance from y_0 to $f(\partial\Omega)$).

Remark 1.4.15 *In the preceding definition, the existence of y_1 is guaranteed by Sard's theorem, which assures that the set of critical values of f is Lebesgue-null in $\mathbb{R}^n$.*

The degree defined in Definitions 1.4.13 and 1.4.14 for a function $f \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ and $y_0 \in \mathbb{R}^n \setminus f(\partial\Omega)$ satisfies the following continuity property: if $g \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ satisfies $\|f - g\|_\infty < d(y_0, f(\partial\Omega))$, thus $y_0 \notin g(\partial\Omega)$), then

$$deg(f, \Omega, y_0) = deg(g, \Omega, y_0).$$

This leads to the following definition.

Definition 1.4.16 *Let $\Omega \subset \mathbb{R}^n$ be nonempty, bounded and open subset, $f \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$, and $y_0 \notin f(\partial\Omega)$. Then Brouwer's degree $deg(f, \Omega, y_0)$ is defined by*

$$deg(f, \Omega, y_0) = deg(g, \Omega, y_0)$$

whenever $g \in \mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ satisfies $\|f - g\|_\infty < d(y_0, f(\partial\Omega))$.

In Definition 1.4.16, the existence of g is guaranteed by the density of $\mathcal{C}^1(\Omega, \mathbb{R}^n) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ in $\mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$.

Example 1.4.17 *Let the problem*

$$(\mathcal{P}) \qquad \text{Find } x \in \Omega \text{ such that } f(x) = y_0,$$

where $n = 1$ et $\Omega =]0, 1[$ and $f : \overline{\Omega} \to \mathbb{R}$ be a function of a class $\mathcal{C}^1$ that verifies:

$$\text{for all solution } x \text{ of the problem } (\mathcal{P}), \quad f'(x) \neq 0. \tag{1.9}$$

We introduce then the integer

$$deg(f, \Omega, y_0) = \begin{cases} \sum\limits_{i \in I} sgn(f'(x_i)), & \text{if } \{x_i, \quad i \in I\} \text{ is the solutions set of } (\mathcal{P}), \\ 0, & \text{if } (\mathcal{P}) \text{has no solution..} \end{cases} \tag{1.10}$$

1.4.2 Leray–Schauder's topological degree

We now wish to construct a degree with the same purpose as Brouwer's degree, but in infinite domension spaces, which means a tool that makes it possible to ensure that an equation of the form $f(x) = y_0$, where f is continuous from a Banach space X in itself, has at least one solution x.

An obstruction in infinite dimension

However, we quickly realize, on one example, that there is no hope to such a tool to happen in the infinite dimension. Indeed, let $X = \{(x_n)_{n\geq 1} \subset E \textit{ such that } (x_n) \text{ is bounded } \forall n \geq 1 \}$, and $S : X \to X$ the right shift, that is

$$S(x) = (0, x_1, x_2, \cdots).$$

Let $H(t, x) = tx + (1-t)S(x) = (tx_1, tx_2 + (1-t)x_1, tx_3 + (1-t)x_2, \cdots)$ the natural homotopy between I and S.

We see that, for all $t \in [0, 1]$, the only solution of $H(t, x) = 0$ is the null sequence. If the Brouwer degree was defined for all continuous fucntions on X, from homotopy invariance property, we would get

$$deg(S, B(0, 1), 0) = deg(I, B(0, 1), 0) = 1.$$

Using homotopy invariance, since $dist(0, S(\partial B(0, 1))) = 1 > 0$, we still get

$deg(S, B(0, 1), z) = 1$ for all $z \in X$ close to 0; but for $z = (\epsilon, 0, 0, \cdots)$ has no precedent by S as soon as $\epsilon \neq 0$.

The Brouwer degree in infinite dimension cannot therefore be defined for all continuous applications of a Banach space X in itself. We must then restrict the functions we are considering. There exist several degrees in infinite dimension, whose main difference is precisely the classe of functions to which each applies; the degreee we are going to study here, called the Leray-Schauder degree, is built on applications that differ from identity by compact application.

Definition of the Leray-Schauder degree

The Leray–Schauder degree theory follows from Brouwer's degree theory. The key step is provided by the next lemma.

Lemma 1.4.18 *Let X be a Banach space, $\Omega \subset X$ nonempty, bounded, and open subset, and $K : \overline{\Omega} \to X$ a completely continuous map with $0 \notin (I-K)(\partial\Omega)$, so that $\rho := d(0,(I-K)(\partial\Omega)) > 0$. If $K_1, K_2 : \overline{\Omega} \to X$ are finite rank maps such that*

$$\|K_i - K\|_\infty < \rho \ \text{ and } \ K_i(\overline{\Omega}) \in Z \ \text{ for } i \in \{1,2\},$$

where $Z \in X$ is a finite-dimensional vector subspace intersecting Ω, then

$$deg((I-K_1)\mid_{\overline{\Omega\cap Z}}, \Omega\cap Z, 0) = deg((I-K_2)\mid_{\overline{\Omega\cap Z}}, \Omega\cap Z, 0),$$

where $deg((I-K_i)\mid_{\overline{\Omega\cap Z}}, \Omega\cap Z, 0)$ stands for the Brouwer degree of the map $(I-K_1)\mid_{\overline{\Omega\cap Z}} \in \mathcal{C}(\overline{\Omega\cap Z}, Z)$, $i \in \{1,2\}$.

Now we give the definition of the Leray–Schauder degree.

Definition 1.4.19 *Let X be a Banach space, $\Omega \subset X$ a nonempty, bounded, and open subset, and $f : \overline{\Omega} \to X$ a compact perturbation of the identity, that is, $f = I - K$, where $K : \overline{\Omega} \to X$ is a completely continuous map.*

(a) If $0 \notin f(\partial\Omega)$, then the Leray-Schauder degree of the triple $(f,\Omega,0)$ is defined by

$$deg_{LS}(f,\Omega,0) = deg((I-\tilde{K})\mid_{\overline{\Omega\cap Z}}, \Omega\cap Z, 0), \tag{1.11}$$

where $Z \subset X$ is a finite-dimensional vector subspace intersecting Ω and $\tilde{K} : \overline{\Omega} \to X$ is a finite-rank map such that $\|K-\tilde{K}\|_\infty < d(0, f(\partial\Omega))$ and $\tilde{K}(\overline{\Omega}) \in Z$.

(b) If $y_0 \in X \setminus f(\partial\Omega)$, $y_0 \neq 0$, then the Leray-Schauder degree of the triple (f,Ω,y_0) is defined by

$$deg_{LS}(f,\Omega,y_0) = deg_{LS}(f-y_0,\Omega,0).$$

Remark 1.4.20 *If $X = \mathbb{R}^n$ $(n \geq 1)$ and $f \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$, then f is a compact perturbation of the identity and $deg_{LS}(f,\Omega,y_0) = deg(f,\Omega,y_0)$ for all $y_0 \in \mathbb{R}^n \setminus f(\partial\Omega)$.*

By virtue of (1.11), the main properties of the Leray-Schauder degree follow from the corresponding properties of Brouwer's degree. We consider triples (f, Ω, y_0) such that $\Omega \subset X$ nonempty, bounded, and open subset, $f = I - K$, with $K : \overline{\Omega} \to X$ is completely continuous map, and $y_0 \notin f(\partial\Omega)$.

Theorem 1.4.21 *The Leray-Schauder degree map $(f, \Omega, y_0) \mapsto deg_{LS}(f, \Omega, y_0)$, defined on triples (f, Ω, y_0) as previously, introduced in Definition 1.4.19, is the unique integer-valued map satisfying the following properties:*

(a) *Normalization property:*

$$deg_{LS}(I, \Omega, y_0) = \begin{cases} 1, & \text{if } y_0 \in \Omega, \\ 0, & \text{if } y_0 \notin \Omega. \end{cases}$$

(b) *Additivity property: if $\Omega_1, \Omega_2 \subset \Omega$ are disjoint, nonempty, open subsets and $y_0 \notin (I - K)(\partial\Omega_1) \cup (I - K)(\partial\Omega_2)$, then*

$$deg_{LS}(I - K, \Omega_1 \cup \Omega_2, y_0) = deg_{LS}(I - K, \Omega_1, y_0) + deg_{LS}(I - K, \Omega_2, y_0);$$

(c) *Homotopy invariance property: if $h : [0, 1] \times \overline{\Omega} \to X$ is completely continuous and for all $t \in [0, 1]$, letting $f_t = I - h(t, .)$, we have $y_0 \notin f_t(\partial\Omega)$, then $deg_{LS}(f_t, \Omega, y_0)$ does not depend on $t \in [0, 1]$;*

(d) *Existence property: if $deg_{LS}(I - K, \Omega, y_0) \neq 0$, then there exists $x \in \Omega$ such that $I - K(x) = y_0$;*

(e) *Excision property: if $V \subset \overline{\Omega}$ is closed and $y_0 \notin I - K(V) \cup I - K(\partial\Omega)$, then*

$$deg_{LS}(I - K, \Omega, y_0) = deg_{LS}(I - K, \Omega \setminus V, y_0);$$

(f) *Continuity with respect to the function property: if $K, G : \overline{\Omega} \to X$ are completely continuous maps, $y_0 \notin (I - K)(\partial\Omega)$, and $\|K - G\|_\infty < d(y_0, (I - K)(\partial\Omega))$, then $y_0 \notin (I - G)(\partial\Omega)$ and*

$$deg_{LS}(I - K, \Omega, y_0) = deg_{LS}(I - G, \Omega, y_0).$$

Moreover, $deg_{LS}(I - K, \Omega, .)$ is constant on each connected component of $X \setminus (I - K)(\partial\Omega)$;

(g) *Boundary invariance property: if* $K, G : \overline{\Omega} \to X$ *are compact maps,* $K\mid_{\partial\Omega} = G\mid_{\partial\Omega}$*, and* $y_0 \notin (I-K)(\partial\Omega)$*, then* $deg_{LS}(I-K,\Omega,y_0) = deg_{LS}(I-G,\Omega,y_0)$*;*

(h) *Translation property:* $deg_{LS}(I-K,\Omega,y_0) = deg_{LS}(I-K-u,\Omega,y_0-u)$ *for all* $u \in X$*.*

1.4.3 Applications of the topological degree

Study of the existence of solutions for nonlinear equations

The topological degree has been constructed with in head the solving of equations of the form (1.8). It is therefore natural that many of its applications revolve around this problem.

Fixed point theorems

Theorem 1.4.22 (Brouwer fixed point) *Let* $\overline{B}$ *the closed unit ball of* $\mathbb{R}^n$ *and let* $f : \overline{B} \to \overline{B}$ *be continuous. Then,* f *has a fixed point:* $x \in \overline{B}$*.*

Theorem 1.4.23 (Schauder fixed point) *Let* $\overline{B}$ *the closed unit ball of a Banach space* E *and let* $f : \overline{B} \to \overline{B}$ *be completely continuous. Then,* f *has a fixed point: there exists* $x \in \overline{B}$ *such that* $f(x) = x$*.*

The Brouwer and Leray-Schauder theorems in question are, however, similar (in fact, the theorem of Brouwer is a special case of Schauder's theorem, since any continuous map is completely continuous in finite dimension), and it would be natural to have similar proofs for each of them. Thanks to the topological degrees, we can give a quick proof and totally common to the theorems of Brouwer and Schauder.

Proof of the two theorems. If there is a fixed point on ∂B, then we are done. Otherwise $f(x) \neq x$ for all $x \in \partial B$. On this case $deg(I-f,B,0)$ is well-defined; we will show that $deg(I-f,B,0) = 1$, which will prove that $I-f$ has at least one zero in B, and therefore f has at least one fixed point in this set.
Let $H(t,x) = tf(x)$, a continuous function over $[0,1] \times \overline{B}$ (and completely continuous in the framework of Schauder's theorem). If, for some $t \in [0,1]$ and $x \in \partial B$, we have $x - H(t,x) = 0$,

then $tf(x) = x$; like $|x| = 1$ and $|f(x)| \leq 1$, this imposes $t = 1$, which leads to a contradiction. From the invariance homotopy and normalization properties of the degree, we deduce

$$deg(I - f, B, 0) = deg(I, B, 0) = 1,$$

which ends proof of the two theorems. ■

The topological degree is a much more powerful tool, more general and often even easier to use than some fixed point theorems. In what follows we give a simple example which confirms this observation.

Example 1.4.24 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function which, for some α, $\beta \in \mathbb{R}$, satisfies $\varphi(\alpha) > \alpha$ and $\varphi(\beta) < \beta$. Then, as an immediate consequence of the intermediate value theorem, φ has at least one fixed point in (α, β). Suppose now, we can find a second interval $(\gamma, \delta) \subset \mathbb{R}$ with $\beta < \gamma$ such that $\varphi(\gamma) > \gamma$ and $\varphi(\delta) < \delta$ then, by the same argument, there exists at least one fixed point in the interior of each of the intervals $I_1 \equiv [\alpha, \beta]$, $I_2 \equiv [\beta, \gamma]$, $I_3 = [\gamma, \delta]$.*
Suppose now in addition that φ, is nondecreasing. Then, we find a fundamental difference in the behavior of φ, on the intervals $I_i, i = 1, 3$, compared with its behavior on I_2. In fact, φ maps each of $I_i, i = 1, 3$ into itself, but this is not true for the middle interval I_2.
Hence we can deduce the existence of fixed points in each of $I_i, i = 1, 3$ also by Brouwer's fixed-point theorem. This method generalizes to nonlinear operational equations in infinite-dimensional spaces, but by this method one does not obtain the "middle" fixed point.
Consider now the equivalent problem of finding zeros of the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = x - \varphi(x)$ and suppose for convenience that ψ is differentiable and has only simple zeros. Then it is obvious that on each interval $I_j, j = 1, 2, 3$, ψ must have an odd number of zeros. Moreover, if we take an algebraic count of the number of the zeros ξ has the value $+1$ if $\psi'(\xi) > 0$ and the value -1 otherwise, then, denoting by $i(I)$ the algebraic number of zeros in the interval I, obviously $i(I_1) = i(I_3) = +1$ and $i(I_2) = -1$. Since on the boundary of the large interval $I \equiv I_1 \cup I_2 \cup I_3$ the function φ has the same behavior as on the boundary of I_1, we have $i(I) = 1$. Hence we see that we can compute $i(I_2)$ also indirectly by means of the behavior formula

$$i(I_2) = i(l) - i(I_1) - i(I_3) = -1. \tag{1.12}$$

But the algebraic number $i(I)$ of zeros of ψ in l is nothing else than the one-dimensional version of the Leray-Schauder degree or, more generally, of the fixed-point index for nonlinear mappings in Banach.

Chapitre 2

Fixed point index

Contents

2.1 Introduction

We have seen that the Leray-Schauder degree is an important tool, in nonlinear analysis, allowing to show the existence of fixed points for a mapping defined on an open bounded subset of a Banach space to this space. But there are many interesting problems for which we cannot use the entire Banach space, but instead their formulation leads us to a map of a closed convex subset of a Banach space that is not a vector subspace, as the non solid cones. There is a generalization of the Leray–Schauder degree, called the fixed point index, that is, designed to find fixed points of such a map. Our goal in this chapter is to define this index and list its properties for the class of completely continuous mappings then extend it to larger classes like the class of strict-set contractions and that of 1-set contraction mappings.

Recall that a subset $D \neq \emptyset$ of a metric (more generally : topological) space Y is called a retract of Y if there exists a continuous map $r : Y \to D$, called a retraction, such that $r(x) = x$, $\forall x \in D$. The key to define the fixed point index is the following result of Dugundji [75]:

If X is a nonempty closed, convex subset of a Banach space E, then X is a retract of E. In particular, every cone $\mathcal{P} \subset E$ is a retract of E.

2.2 Fixed point index for completely continuous maps

Now, we present the definition and the most important properties of the fixed point index in the class of completely continuous mappings. For more details, see [2, 25, 46].

Theorem 2.2.1 *Let X be a retract of E. For every open bounded subset $U \subset X$ and every compact mapping $f : \overline{U} \to X$ without fixed point on the boundary ∂U, there exists a unique integer $i(f, U, X)$ satisfying the following conditions:*

(i) (Normalization property). The index $i(f, U, X) = 1$ whenever f is constant on $\overline{U}$.

(ii) (Additivity property). Let U_1, U_2 be two disjoint open subsets of U such that f has no fixed point on $\overline{U} \setminus (U_1 \cup U_2)$, then

$$i(f, U, X) = i(f, U_1, X) + i(f, U_2, X)$$

where $i(f, U_k, X) = i(f|_{\overline{U_k}}, U_k, X)$, $k = 1, 2$.

(iii) (Homotopy Invariance property). The index $i(h(\cdot, t), U, X)$ does not depend on the parameter $t \in [0, 1]$, where $h : \overline{U} \times [0, 1] \to X$ is a compact mapping and $h(x, t) \neq x$ for every $x \in \partial U$ and $0 \leq t \leq 1$.

(iv) (Permanence property). If Y is a retract of X and $f(\overline{U}) \subset Y$, then

$$i(f, U, X) = i(f, U \cap Y, Y)$$

where $i(f, U \cap Y, Y) := i(f|_{\overline{U \cap Y}}, U \cap Y, Y)$.

The integer $i(f, U, X)$ is called fixed point index of f on U with respect to X.

Sketch of the proof.

Let $i(f, U, X)$ satisfying the conditions (i)-(v). With $X = E$, conditions (i)-(v) are just the main properties of the Leray-Schauder degree with

$$i(f, U, E) = deg(I - f, U, 0). \tag{2.1}$$

Let $r : E \to X$ be a retraction. The fixed point index $i(f, U, X)$ is defined by:

$$i(f, U, X) = i(f \circ r, r^{-1}(U), E) = deg_{LS}(I - f \circ r, r^{-1}(U), 0). \tag{2.2}$$

Indeed, $r : E \to X$ is a retraction, thus continuous. Then,

- $U \subset X$ is open, thus $r^{-1}(U) \subset E$ is also open.
- $f \circ r : E \to X$ is a compact mapping, has no fixed point on the boundary $\partial r^{-1}(U)$ and satisfies $[f \circ r](\overline{r^{-1}(U)} \subset X$ for

$$\begin{aligned} f \circ r \textit{ is continuous } \Rightarrow [f \circ r](\overline{r^{-1}(U)}) &\subset \overline{[f \circ r](r^{-1}(U))} \\ &\subset \overline{f(U)} \subset \overline{X} = X. \end{aligned}$$

By the permanence property of the degree, we have

$$\begin{aligned} i(f \circ r, r^{-1}(U), E) &= i(f \circ r, r^{-1}(U) \cap X, X) \\ &= i(f \circ r, U, X) = i(f \circ r|_{\overline{U}}, U, X) \\ &= i(f, U, X). \end{aligned}$$

Since (2.1) implies that

$$i(f \circ r, r^{-1}(U), E) = deg_{LS}(I - f \circ r, r^{-1}(U), 0),$$

we deduce (2.2). We may further check that (2.2) does not depend on the retraction r. Indeed, ler $r_1 : E \to X$ be another retraction. Let $V := r^{-1}(U) \cap r_1^{-1}(U)$ and $r_0 := r$. By the excision property of the Leray-Schauder degree, we find that:

$$deg((I - f \circ r_j, r_j^{-1}(U), 0) = deg_{LS}(I - f \circ r_j, V, 0), \; j = 0, 1.$$

Define a compact mapping $h : [0,1] \times V \to X$ by

$$h(\lambda, x) = r_0[(1-\lambda) f(r_0(x)) + \lambda f(r_1(x))].$$

Notice that $h(\lambda, x) \neq x$, $\forall (\lambda, x) \in [0,1] \times \partial V$. The Leray-Schauder degree $deg(I - h(\lambda, \cdot), V, 0)$ is well defined $\forall \lambda \in [0,1]$. The invariance under homotopy of the degree impliers that:

$$deg_{LS}(I - f \circ r_0, V, 0) = deg_{LS}(I - f \circ r_1, V, 0),$$

and our claim follows. ■

Corollary 2.2.2 *The fixed point index satisfies:*

(v) Excision property. Let $V \subset U$ an open subset such that f has no fixed point in $\overline{U} \setminus V$; then

$$i(f, U, X) = i(f, V, X).$$

(vi) Existence property. If $i\,(f, U, X) \neq 0$, then f has a fixed point in U.

Proof.

(v) Given $U_1 = U, U_2 = \emptyset$, the additivity property of the fixed point index yields

$$i\,(f, U, X) = i\,(f, U_1, X) + i\,(f, U_2, X) = i\,(f, U, X) + i\,(f, \emptyset, X).$$

This implies that $i(f, \emptyset, X) = 0$. Taking $U_1 = V$ and $U_2 = \emptyset$, we get

$$i(f, U, X) = i(f, V, X) + i(f, \emptyset, X) = i\,(f, V, X).$$

(vi) Assume f has no fixed point in U. So f has no fixed point in $U \cup \partial U = \overline{U}$.
Let $V = \emptyset$ in the excision property, i.e $\overline{U} \setminus V = \overline{U}$. Then $i\,(f, U, X) = i\,(f, \emptyset, X) = 0$, which contradicts the fact that $i\,(f, U, X) \neq 0$.

■

The fixed point index has been extended to wider classes of maps. We describe three of these extensions: to the classes of strict-set contractions, condensing and 1-set contraction maps.

2.3 Fixed point index for strict-set contraction maps

We now describe the definition of the fixed point index for strict-set contractions. For more details, see [3, 24, 45, 46]. The key result here is the following:

Lemma 2.3.1 *Let $\mathcal{C}$ be a closed convex subset of a Banach space E. Let U be a bounded open subset of $\mathcal{C}$ and $f : \overline{U} \to \mathcal{C}$ be a strict k-set contraction. Then:*

(i) *there exists a compact convex $\mathcal{C}_\infty \subset \mathcal{C}$ such that $f(\overline{U} \cap \mathcal{C}_\infty) \subset \mathcal{C}_\infty$.*

(ii) *there exists a completely continuous map $F : \overline{U} \to \mathcal{C}_\infty$ extending the restriction $f_{|\overline{U}} : \overline{U} \cap \mathcal{C}_\infty \to \mathcal{C}_\infty$ and satisfying $Fix(F) = Fix(f)$; any two such extensions are homotopic via a completely continuous homotopy $H_t : \overline{U} \to \mathcal{C}_\infty$ such that $Fix(H_t) = \emptyset$ for each $t \in [0, 1]$.*

Proof. Define inductively a descending sequence $\mathcal{C}_1 \supset \mathcal{C}_2 \supset ...$ of closed convex sets by setting

$$\mathcal{C}_1 = \mathrm{conv(f(\overline{U}))}, \quad \mathcal{C}_\mathrm{n} = \mathrm{conv(f(\overline{U} \cap \mathcal{C}_{n-1}))}.$$

Letting $\mathcal{C}_\infty = \cap_{n=1}^{\infty} \mathcal{C}_n$ and using $f(\overline{U} \cap \mathcal{C}_{n-1}) \subset \mathcal{C}_n$ gives $f(\overline{U} \cap \mathcal{C}_\infty) \subset \mathcal{C}_\infty$.
Because f is strict k-set contraction,

$$\alpha(\mathcal{C}_n) \leq k\alpha(\mathcal{C}_{n-1}) \leq k^{n-1}\alpha(\mathcal{C}_1),$$

and as $k \in (0, 1)$, the generalized Cantor's intersection theorem shows that $\mathcal{C}_\infty$, is compact.

Now, from [46, Theorem A.5.1], $f : \overline{U} \cap \mathcal{C}_\infty \to \mathcal{C}_\infty$ extends to $F : \overline{U} \to \mathcal{C}_\infty$, and since $\mathcal{C}_\infty$ is compact it follows that F is completely continuous.
If $x = F(x)$ for some $x \in \overline{U}$? then $x \in \overline{U} \cap \mathcal{C}_\infty$? so $x = F(x) = f(x)$, and thus $Fix(F) = Fix(f)$.
Now assume that $G : \overline{U} \to \mathcal{C}_\infty$ were another such extension and define a completely continuous homotopy $H_t : \overline{U} \to \mathcal{C}_\infty$ by

$$H_t(x) = (I - t)F(x) + tG(x).$$

If $x = H_t(x)$ for some $x \in \overline{U}$ and some $t \in I$ then $x \in \overline{U} \cap \mathcal{C}_\infty$, and therefore $x = F(x) = f(x)$, i.e., $Fix(H_t) = Fix(f)$ for all $t \in [0, 1]$. ■

The fixed point index $i(F, U, \mathcal{C})$, $i(G, U, \mathcal{C})$ are the same, and we define

$$i(f, U, \mathcal{C}) = i(F, U, \mathcal{C}).$$

It can be shown that $i(f, U, \mathcal{C})$ satisfies the main axioms for the index on the class of strict-set contraction maps. In particular, it is also unique.

The basic properties of fixed point index for strict k-set contractions are collected in the following lemma. For the proof, we refer the reader to [45, Theorem 1.3.5] or [3, 46].

Lemma 2.3.2 *Let X be a retract of a Banach space E. For every open bounded subset $U \subset X$ and every strict k-set contraction $f : \overline{U} \to X$ without fixed point on the boundary ∂U, there exists uniquely one integer $i(f, U, X)$ satisfying the following conditions:*

(a) *(Normalization property). If $f : \overline{U} \to U$ is a constant map (that is, $f(x) = y_0$ for all $x \in \overline{U}$), then $i(f, U, X) = 1$.*

(b) *(Additivity property). For any pair of disjoint open subsets U_1, U_2 in U such that f has no fixed point on $\overline{U} \backslash (U_1 \cup U_2)$, we have*

$$i(f, U, X) = i(f, U_1, X) + i(f, U_2, X),$$

where $i(f, U_j, X) := i(f|_{\overline{U_j}}, U_j, X)$, $j = 1, 2$.

(c) *(Homotopy Invariance property). The index $i(h(t, \cdot), U, X)$ does not depend on the parameter $t \in [0, 1]$, where*

(i) *$h : [0, 1] \times \overline{U} \to X$ is continuous and $h(t, x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,*

(ii) *$h(t, .) : \overline{U} \to X$ is a strict k-set contraction, where k does not depend on $t \in [0, 1]$,*

(iii) *$h(t, x) \neq x$, for every $t \in [0, 1]$ and $x \in \partial U$.*

(d) *(Permanence property). If Y is a retract of X and $f(\overline{U}) \subset Y$, then*

$$i(f, U, X) = i(f, U \cap Y, Y),$$

where $i(f, U \cap Y, Y) := i(f|_{\overline{U \cap Y}}, U, Y)$.

(e) *(Excision property). Let $V \subset U$ an open subset such that f has no fixed point in $\overline{U} \backslash V$. Then*

$$i\,(f, U, X) = i\,(f, V, X).$$

(f) *(Existence property). If $i\,(f, U, X) \neq 0$, then f has a fixed point in U.*

In the following, we compute the fixed point index for the class of mappings under consideration. Note that these computations follow directly from the properties of this index.

Proposition 2.3.3 *Let X be a closed convex of a Banach space E and $U \subset X$ an open bounded subset with $0 \in U$. Assume that $A : \overline{U} \to X$ is a strict k-set contraction that satisfies the Leray-Schauder boundary condition:*

$$Ax \neq \lambda x, \quad \forall\, x \in \partial U,\ \forall\, \lambda \geq 1.$$

Then $i\,(A, U, X) = 1$.

Corollary 2.3.4 *Let $\mathcal{P}$ be a cone of a Banach space E and $U \subset \mathcal{P}$ an open bounded subset with $0 \in U$. Assume that $A : \overline{U} \to \mathcal{P}$ is a strict k-set contraction satisfying*

$$\|Ax\| \leq \|x\| \;\; \text{and} \;\; Ax \neq x \;\; \text{for all} \;\; x \in \partial U.$$

Then $i\,(A, U, \mathcal{P}) = 1$.

Proposition 2.3.5 *[45, Corollary 1.3.1] Let X be a closed convex of a Banach space E and $U \subset X$ a nonempty open bounded convex subset of X. Assume that $A : \overline{U} \to X$ is a strict set contraction such that $A(\overline{U}) \subset U$. Then $i\,(A, U, X) = 1$.*

Proposition 2.3.6 *[45, Theorem 1.3.8] Let X be a closed convex of a Banach space E and $U \subset X$ be an open bounded subset. Assume that $A : \overline{U} \to X$ is a strict k-set contraction. If there exists $u_0 \in X, u_0 \neq 0$, such that $\lambda u_0 \in X$, $\forall\, \lambda \geq 0$ and*

$$x - Ax \neq \lambda u_0,\ \forall\, x \in \partial U,\ \forall\, \lambda \geq 0,$$

then $i\,(A, U, X) = 0$.

2.4 Fixed point index for condensing maps

In this section, we explain how to extend the definition of the fixed point index given in section 2.3 to the class of condensing mappings. For more details, (see [6, 46, 67]). In what follows, let X be a nonempty closed convex subset of a Banach space E, U a relative bounded open set with respect to X, and $\overline{U}$ and ∂U the closure and boundary of U in X respectively.

Let $F : \overline{U} \to C$ be a condensing map with $Fix(F) \subset U$. Take

$$0 < \delta < \delta_0 = \inf\{\|x - F(x)\| \ : \ x \in \partial U\}.$$

Select any $x_0 \in U$ and let

$$K_t(x) = (1-t)x_0 + tF(x) \ \text{ for } \ x \in \overline{U} \ \text{ and } t \in [0,1].$$

Now choosing t_0 sufficiently close to 1 so that $\|K_{t_0}(x) - F(x)\| < \delta$ for all $x \in \partial U$.
Noting that $K_{t_0} : \overline{U} \to X$ is a strict-set contraction with $Fix(K_{t_0}) \subset U$. Thus, we define

$$i\,(F, U, X) = i\,(K_{t_0}, U, X).$$

It can be verified that the definition does not depend on t_0, x_0 chosen and that $i\,(F, U, X)$ satisfies the main axioms for the index on the class of condensing maps that are fixed point free on ∂U. In particular, it is also unique.

2.5 Fixed point index for 1-set contraction maps

Now, we extend the concept of fixed point index to the class of 1-set contraction mappings. For the proofs and more details, we refer the reader to the reference [48]. Suppose that $F : \overline{U} \to X$ is a 1-set contraction mapping and $0 \notin \overline{(I-T)\partial U}$, so there exists $\delta > 0$ such that

$$\inf_{x \in \partial U} \|x - Fx\| \geq \delta. \tag{2.3}$$

We set $F_k = kF$, where $k \in (1 - \frac{\delta}{M}, 1)$, $M = \sup\limits_{x \in \overline{U}} \|Fx\| + \delta$.
Obviously F_k is a strict-set contraction mapping. Thus the fixed point index $i\,(F_k, U, U)$ is well

defined. Then we put

$$i\,(T,U,X) = i\,(T_k,U,X). \tag{2.4}$$

Note that the index $i\,(F,U,X)$ is independent of T_k. In fact, suppose that $W_i : \overline{U} \to X$ is a k_i-set contraction mapping $(0 < k_i < 1)$ with

$$\|W_i x - Fx\| < \delta,\ x \in \partial U i,\ i = 1,2. \tag{2.5}$$

We make a homotopic mapping on $\overline{U}$ as follows

$$H(t,x) = tW_1x + (1-t)W_2x \quad x \in \overline{U},\ t \in [0,1].$$

$H_t : \overline{U} \to X$ is a l-set contraction mapping, where $l = \max\{k_1,k_2\}$. For every $x \in \partial U$, we have

$$\begin{aligned}
\|x - H(t,x)\| &= \|x - tW_1x - (1-t)W_2x\| \\
&\geq \|x - Tx\| - t\|Tx - W_1x\| - (1-t)\|Tx - W_2x\| \\
&> \delta - t\delta - (1-t)\delta = 0.
\end{aligned}$$

By homotopy invariance property of the index (Theorem 2.2.1 (iii)), we get

$$i(W_1,U,X) = i(W_2,U,X).$$

This equality proves our claim.

The fixed point index $i\,(F,U,X)$ defined in (2.4)) for 1-set contractions has the following properties:

(i) (Normalization property). If $F : \overline{U} \to U$ is a constant mapping (that is, $Fx \equiv y_0$ for all $x \in \overline{U}$), then $i(T,U,X) = 1$.

(ii) (Additivity property). If U_1 and U_2 are disjoint open subsets of U such that F has no fixed point in $\overline{U} \setminus (U_1 \cup U_2)$, then

$$i\,(F,U,X) = i(F,U_1,X) + i\,(F,U_2,X).$$

(iii) (Homotopy Invariance property). If we assume that

(a) $H : [0,1] \times \overline{U} \to X$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$ and $0 \notin (I - H(t,x))([0,1] \times \partial U)$;

(b) $H(t, \cdot) : \overline{U} \to X$ is a 1-set contraction;

then $i(H(t, \cdot), U, X) =$ constant for all $t \in [0, 1]$.

Chapitre 3

Generalized fixed point index for maps of the form $T + F$

Contents

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space E, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$. For some constant $r > 0$, we will denote $\mathcal{P}_r = \mathcal{P} \cap \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in E : \|x\| < r\}$ is the open ball centered at the origin with radius r.

3.1 The case where T is h-expansive mapping and F is a k-set contraction with $0 \leq k < h-1$

In [29], Djebali and Mebarki have developped a generalized fixed point index theory for the sum of a k-set contraction and an expansive mapping with constant $h > 1$ when $0 \leq k < h-1$. In what follows, we present the definition of this index as well as some of its properties.

3.1.1 Definition of the index

Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$ and $F : \overline{U} \to E$ is a k-set contraction. Suppose that

$$0 \leqslant k < h-1,$$

$$F(\overline{U}) \subset (I-T)(\Omega), \tag{3.1}$$

and

$$x \neq Tx + Fx, \quad \text{for all } \; x \in \partial U \cap \Omega. \tag{3.2}$$

Then $x \neq (I-T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict $\frac{k}{h-1}$-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I-T)^{-1}F)(B)) \leqslant \frac{1}{h-1}\,\alpha(F(B)) \leqslant \frac{k}{h-1}\alpha(B).$$

By Lemma 2.3.2, the fixed point index $i\left((I-T)^{-1}F, U, \mathcal{P}\right)$ is well defined. Thus we put

$$i_*\left(T+F, U\cap\Omega, \mathcal{P}\right) = \begin{cases} i\left((I-T)^{-1}F, U, \mathcal{P}\right) & \text{if } \; U\cap\Omega \neq \emptyset \\ 0, & \text{if } \; U\cap\Omega = \emptyset. \end{cases} \tag{3.3}$$

This integer is called *the generalized fixed point index* of the sum $T+F$ on $U \cap \Omega$ with respect to the cone $\mathcal{P}$.

Using the main properties of the fixed point index for strict set contractions, Djebali and Mebarki in [29] have discussed the properties of the generalized fixed point index i_*.

Theorem 3.1.1 *The fixed point index defined in (3.3) satisfies the following properties:*

(a) *(Normalization property). If $U = \mathcal{P}_r,\ 0 \in \Omega,$ and $Fx = z_0 \in \mathcal{B}(-T0,(h-1)r) \cap \mathcal{P}$ for all $x \in \overline{\mathcal{P}_r}$, then*

$$i_*\left(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}\right) = 1.$$

(b) *(Additivity property). For any pair of disjoint open subsets U_1, U_2 in U such that $T+F$ has no fixed point on $(\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega$, we have*

$$i_*\left(T+F, U \cap \Omega, \mathcal{P}\right) = i_*\left(T+F, U_1 \cap \Omega, \mathcal{P}\right) + i_*\left(T+F, U_2 \cap \Omega, \mathcal{P}\right),$$

where $i_*\left(T+F, U_j \cap \Omega, X\right) := i_*\left(T+F|_{\overline{U_j}}, U_j \cap \Omega, \mathcal{P}\right), \quad j = 1,2.$

(c) *(Homotopy Invariance property). The fixed point index $i_*\left(T+H(t,.), U \cap \Omega, \mathcal{P}\right)$ does not depend on the parameter $t \in [0,1]$ whenever*

(i) *$H : [0,1] \times \overline{U} \to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,*

(ii) *$H([0,1] \times \overline{U}) \subset (I-T)(\Omega)$,*

(iii) *$H(t,.) : \overline{U} \to E$ is a l-set contraction with $0 \leqslant l < h-1$ and l does not depend on $t \in [0,1]$,*

(iv) *$Tx + H(t,x) \neq x$, for all $t \in [0,1]$ and $x \in \partial U \cap \Omega$.*

(d) *(Existence property). If $i_*\left(T+F, U \cap \Omega, \mathcal{P}\right) \neq 0,$ then $T+F$ has a fixed point in $U \cap \Omega$.*

Proof. Properties (b), (c) and (d) follow directly from the (3.3) and the corresponding properties of the fixed point index for strict-set contractions (see Lemma 2.3.2). We only check that if $U = \mathcal{P}_r$, then

$$i\left((I-T)^{-1}z_0, U, \mathcal{P}\right) = 1.$$

For this, we show that $y_0 := (I-T)^{-1}z_0 \in \mathcal{P}_r \cap \Omega$. We have $F(\overline{\mathcal{P}_r}) = \{z_0\} \subset (I-T)(\Omega)$, which gives $y_0 \in \Omega$ and since T is an expansive operator with $h > 1$ and $F(\overline{\mathcal{P}_r}) \subset \mathcal{B}(-T0, (h-1)r) \cap \mathcal{P}$, Lemma 1.3.18 guarantees that

$$\|(I-T)y_0 + T0\| = \|(I-T)y_0 - (I-T)0\| \geq (h-1)\|y_0\|.$$

Hence

$$(h-1)\|y_0\| \leq \|(I-T)y_0 + T0\| = \|z_0 - (-T0)\| < (h-1)r,$$

that is $y_0 = (I-T)^{-1}z_0 \in \mathcal{P}_r$. By property (a) in Lemma 2.3.2, we deduce that

$$i\left((I-T)^{-1}z_0, \mathcal{P}_r, \mathcal{P}\right) = 1.$$

Therefore $i_*\left(T + z_0, \mathcal{P}_r \cap \Omega, \mathcal{P}\right) = 1$, which completes the proof. ■

Remark 3.1.2 *Theorem 3.1.1 still holds if instead of the cone $\mathcal{P}$, we consider a retract X of E. In this case, the set $\mathcal{P}_r$ is replaced by $X \cap \mathcal{B}_r$.*

3.1.2 Computation of the index

The following results give the computation of the generalized fixed point index i_*. For Proofs and more details see [29].

Proposition 3.1.3 *Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leqslant k < h-1$, and $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $0 \in \Omega$, $\|T0\| < (h-1)r$, and*

$$Fx \neq \lambda(x - Tx) \ \text{ for all } \ x \in \partial\mathcal{P}_r \cap \Omega \ \text{ and } \ \lambda \geqslant 1, \tag{3.4}$$

then $i_*\left(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}\right) = 1$.

As a consequence of Proposition 3.1.3, we have the following result.

Proposition 3.1.4 *Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leqslant k < h-1$, $F(\partial\mathcal{P}_r \cap \Omega) \subset \mathcal{P}$, and $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $0 \in \Omega$, $\|T0\| < (h-1)r$, and*

$$Fx \ngeq x - Tx \ \text{ for all } \ x \in \partial\mathcal{P}_r \cap \Omega,$$

then $i_*\left(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}\right)=1$.

Proposition 3.1.5 *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U \cap \Omega$. Assume that $T: \Omega \subset \mathcal{P} \rightarrow E$ is an expansive mapping with constant $h>1$, $F: \overline{U} \rightarrow E$ is a k-set contraction with $0 \leqslant k<h-1$, and $F(\overline{U}) \subset(I-T)(\Omega)$. If*

$$\|F x+T 0\| \leqslant(h-1)\|x\| \quad \text{and} \quad T x+F x \neq x, \quad \text{for all } x \in \partial U \cap \Omega, \tag{3.5}$$

then $i_*(T+F, U \cap \Omega, \mathcal{P})=1$.

Proposition 3.1.6 *Let U be an open bounded subset of $\mathcal{P}$. Assume that $T: \Omega \subset \mathcal{P} \rightarrow E$ is an expansive mapping with constant $h>1$, $F: \overline{U} \rightarrow E$ is a k-set contraction with $0 \leqslant k<h-1$, and $F(\overline{U}) \subset(I-T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$F x \neq(I-T)(x-\lambda u_0), \quad \text{for all } \lambda \geqslant 0 \text{ and } x \in \partial U \cap(\Omega+\lambda u_0), \tag{3.6}$$

then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P})=0$.

As a consequence of Proposition 3.1.6, we have the following result.

Proposition 3.1.7 *Assume that $T: \Omega \subset \mathcal{P} \rightarrow E$ is an expansive mapping with constant $h>1$, $F: \overline{\mathcal{P}_r} \rightarrow E$ a k-set contraction with $0 \leqslant k<h-1$, and $F(\overline{\mathcal{P}_r}) \subset(I-T)(\Omega)$. Assume in addition that there exists $w_0 \in \mathcal{P}^*$ such that $T(x-\lambda w_0) \in \mathcal{P}$, for all $\lambda \geqslant 0$ and $x \in \partial \mathcal{P}_r \cap(\Omega+\lambda w_0)$, and*

$$F x \not\leq x-\lambda w_0 \quad \text{for all} \quad x \in \partial \mathcal{P}_r \quad \text{and} \quad \lambda \geqslant 0. \tag{3.7}$$

Then $i_*\left(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}\right)=0$.

Proposition 3.1.8 *Let U be an open bounded subset of $\mathcal{P}$. Assume that $T: \Omega \subset \mathcal{P} \rightarrow E$ is an expansive mapping with constant $h>1$, $F: \overline{U} \rightarrow E$ a k-set contraction with $0 \leqslant k<h-1$, and $F(\overline{U}) \subset(I-T)(\Omega)$. Suppose further that there exists $u_0 \in \mathcal{P}^*$ such that $T(x-\lambda u_0) \in \mathcal{P}$, for all $\lambda \geqslant 0$ and $x \in \partial U \cap(\Omega+\lambda u_0)$, and one of the following conditions holds:*

(a) $F x \not\leq x-\lambda u_0, \ \forall x \in \partial U, \forall \lambda \geqslant 0$.

(b) $\|F x\|>\|x-\lambda u_0\|, \ \forall x \in \partial U, \forall \lambda \geqslant 0$ *and the cone $\mathcal{P}$ is normal with constant $N=1$.*

Then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P})=0$.

3.2 The case where T is h-expansive mapping and F is an $(h-1)$-set contraction

The results of this section are obtained by Djebali and Mebarki. For Proofs and more details we refer the reader to [29].

3.2.1 Definition of the index

Suppose that $T : \Omega \to E$ is h-expansive and $F : \overline{U} \to E$ is an $(h-1)$-set contraction. Since $(I-T)^{-1}$ is $\frac{1}{h-1}$-Lipschtzian, then $(I-T)^{-1}F : \overline{U} \to \mathcal{P}$ is a 1-set contraction. Assume that

$$tF(\overline{U}) \subset (I-T)(\Omega),\ \forall\, t \in [0,1] \tag{3.8}$$

and

$$0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}. \tag{3.9}$$

Then there exists $\gamma > 0$ such that

$$\inf_{x \in \partial U \cap \Omega} \|x - Tx - Fx\| \geq \gamma.$$

Thus

$$0 \notin (I-T-kF)(\partial U \cap \Omega),\ \forall\, k \in (1-\gamma/M, 1),$$

where $M = \gamma + \sup\limits_{x \in \overline{U}} \|Fx\|$. In fact, for all $x \in \partial U \cap \Omega$, we have

$$\begin{aligned} \|0 - (x - Tx - kFx)\| &\geq \|x - Tx - Fx\| - (1-k)\|Fx\| \\ &\geq \gamma - (1-k)M > 0. \end{aligned}$$

In other words, $x \neq (I-T)^{-1}kFx$, for all $x \in \partial U$ and $k \in (1-\frac{\gamma}{M}, 1)$. Clearly, $(I-T)^{-1}kF$ is a strict k-set contraction mapping. As a consequence, by (3.3) and Lemma 2.3.2, the fixed point index $i_*(T+kF, U\cap\Omega, \mathcal{P})$ is well defined. Thus we set

$$\begin{aligned} i_*(T+F, U\cap\Omega, \mathcal{P}) &= i_*(T+kF, U\cap\Omega, \mathcal{P}) \\ &= i((I-T)^{-1}kF, U, \mathcal{P}),\ k \in (1-\tfrac{\gamma}{M}, 1). \end{aligned} \tag{3.10}$$

However we must check that $i_*\,(T+F, U\cap\Omega, \mathcal{P})$ does not depend on the parameter $k \in (1-\frac{\gamma}{M}, 1)$. For this, let $G_j = k_jF : \overline{U} \to E$ be $k_j(h-1)$-set contractions with $k_j \in (1-\frac{\gamma}{M}, 1)$ $(j = 1, 2)$. Then $\|G_jx - Fx\| = (1-k_j)\|Fx\| \leq (1-k_j)M < \gamma$, $\forall\, x \in \partial U$. Define the convex deformation $H : [0, 1] \times \overline{U} \to E$:

$$H(t, x) = tG_1x + (1-t)G_2x.$$

The operator H is continuous, uniformly continuous in t for each x, and $H([0, 1] \times \overline{U}) \subset (I-T)(\Omega)$. In addition $H(t, .)$ is a $\overline{k}(h-1)$-set contraction for each t, where $\overline{k} = \max(k_1, k_2)$ and $T + H(t, .)$ has no fixed point on $\partial U \cap \Omega$. In fact, for all $x \in \partial U \cap \Omega$, we have

$$\begin{aligned} \|x - Tx - H(t, x)\| &= \|x - Tx - tG_1x - (1-t)G_2x\| \\ &\geq \|x - Tx - Fx\| - t\|Fx - G_1x\| \\ &\quad -(1-t)\|Fx - G_2x\| \\ &> \gamma - t\gamma - (1-t)\gamma = 0. \end{aligned}$$

From the invariance property by homotopy of the index in Theorem 3.1.1, we deduce that

$$i_*\,(T + G_1, U \cap \Omega, \mathcal{P}) = i_*\,(T + G_2, U \cap \Omega, \mathcal{P}),$$

which shows that the index $i_*\,(T + F, U \cap \Omega, \mathcal{P})$ does not depend on k.

The integer defined in (3.10) satisfies some properties grouped in the following:

Theorem 3.2.1

(a) (Normalization property). If $U = \mathcal{P}_r = \mathcal{P} \cap \mathcal{B}_r$ is a conical shell and $Fx = z_0 \in \mathcal{B}(-T0, (h-1)r) \cap \mathcal{P}$, for all $x \in \overline{\mathcal{P}_r}$, then $i_*\,(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

(b) (Additivity property). For any pair of disjoint open subsets U_1, U_2 in U such that $0 \notin \overline{(I - T - F)((\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega)}$, we have

$$i_*\,(T + F, U \cap \Omega, \mathcal{P}) = i_*\,(T + F, U_1 \cap \Omega, \mathcal{P}) + i_*\,(T + F, U_2 \cap \Omega, \mathcal{P}),$$

where $i_*\,(T + F, U_j \cap \Omega, X) := i_*\,(f|_{\overline{U_j}}, U_j \cap \Omega, \mathcal{P})$, $j = 1, 2$.

(c) (Homotopy Invariance property). The fixed point index $i_*\,(T + H(t, .), U \cap \Omega, \mathcal{P})$ does not depend on the parameter $t \in [0, 1]$, where

(i) $H : [0,1] \times \overline{U} \to E$ *is continuous and* $H(t,x)$ *is uniformly continuous in* t *with respect to* $x \in \overline{U}$,

(ii) $H(t,.) : \overline{U} \to E$ *is an* $(h-1)$*-set contraction,*

(iii) $tH([0,1] \times \overline{U}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$,

(iv) $0 \notin \overline{(I-T-H(t,.))(\partial U \cap \Omega)}$, *for all* $t \in [0,1]$,

(d) (Existence property). If $i_*(T+F, U \cap \Omega, \mathcal{P}) \neq 0$, *then* $0 \in \overline{(I-T-F)(U \cap \Omega)}$.

Proof.

(a) Since F is a constant mapping, it is a 0-set contraction (completely continuous), which implies that $(I-T)^{-1}F$ is a 0-set contraction. As in the proof of Theorem 3.1.1, part (a), $y_0 = (I-T)^{-1}z_0 \in \mathcal{P}_r$. By the normalization property in Lemma 2.3.2, we deduce that

$$i\left((I-T)^{-1}z_0, \mathcal{P}_r, \mathcal{P}\right) = 1.$$

Therefore $i_*(T+z_0, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$, proving our claim.

(b) Let

$$\gamma = \inf_{(\overline{U}\setminus(U_1 \cup U_2)) \cap \Omega} \|x - Tx - Fx\| > 0.$$

Suppose that $G = kF : \overline{U} \to E$ is a $k(h-1)$-set contraction and

$$\|Gx - Fx\| < \gamma, \ \forall\, x \in \overline{U}\setminus(U_1 \cup U_2) \cap \Omega. \tag{3.11}$$

From (3.10), we have

$$i_*(T+F, U \cap \Omega, \mathcal{P}) = i_*(T+G, U \cap \Omega, \mathcal{P})$$

and

$$i_*(T+F, U_j \cap \Omega, \mathcal{P}) = i_*(T+G, U_j \cap \Omega, \mathcal{P}), \quad j = 1,2.$$

Hence $T+G$ has no fixed point in $\overline{U}\setminus(U_1 \cup U_2) \cap \Omega$. In fact, if there exists $x_0 \in \overline{U}\setminus(U_1 \cup U_2) \cap \Omega$ such that $x_0 = Tx_0 + Gx_0$, then

$$\gamma \leq \|x_0 - Tx_0 - Fx_0\| = \|x_0 - Tx_0 - Gx_0 + Gx_0 - Fx_0\| = \|Gx_0 - Fx_0\|,$$

which contradicts (3.11). The claim follows from (3.10) and property (b) of the fixed point index in Theorem 3.1.1.

(c) By the property of the function H, there exist $\gamma > 0$ and $N > 0$ such that

$$\|x - Tx - H(t,x)\| \geq \gamma, \quad \text{for all } x \in \partial U \cap \Omega \text{ and } t \in [0,1],$$

as well as $\|H(t,x)\| \leq N$, for all $x \in \overline{U}$ and $t \in [0,1]$. Let $K(t,x) = kH(t,x)$, where $k \in (1 - \frac{\gamma}{2N}, 1)$. Then for all $x \in \partial U \cap \Omega$ and $t \in [0,1]$, we have

$$\begin{aligned} \|x - Tx - K(t,x)\| &= \|x - Tx - H(t,x)\| + \|H(t,x) - K(t,x)\| \\ &\geq \gamma - (k-1)N > \gamma - \tfrac{\gamma}{2} > 0. \end{aligned}$$

Obviously, $K(t,.) : \overline{U} \to E$ is a $k(h-1)$-set contraction, where k does not depend on $t \in [0,1]$ and $K([0,1] \times \overline{U}) \subset (I - T)(\Omega)$.

Then our claim follows from (3.10) and property (c) of the fixed point index in Theorem 3.1.1.

(d) Consider a sequence $(k_n)_n \subset (0,1)$ such that $k_n \to 1$, as $n \to \infty$ and define the function $G_n = k_n F$ $n = 1, 2, \dots$ Then $G_n : \overline{U} \to E$ is a $k(h-1)$-set contraction. Since $\|Fx\| < \infty$, $\forall x \in \overline{U}$, we obtain that

$$\|Fx - G_n x\| = \|Fx - k_n Fx\| = (1 - k_n)\|F_n x\| \to 0, \quad \text{as } n \to +\infty.$$

Hence there exists $n_0 > 0$ such that for every $n \geq n_0$

$$\|Fx - G_n x\| < \gamma, \quad \text{where } 0 < \gamma < \inf_{x \in \partial U \cap \Omega} \|x - Tx - Fx\|.$$

By assumption and Definition 3.10,

$$i_* (T + F, U \cap \Omega, \cap P) = i_* (T + G_n, U \cap \Omega, \cap P) \neq 0.$$

Thus, property (d) in Theorem 3.1.1 guarantees that for all $n = 1, 2, \dots$, the mapping $T + G_n$ has a fixed point x_n in $U \cap \Omega$. Consequently,

$$\begin{aligned} \|x_n - Tx_n - Fx_n\| &= \|x_n - Tx_n - G_n x_n + G_n x_n - Fx_n\| \\ &= \|G_n x_n - Fx_n\| \to 0, \quad \text{as } n \to +\infty. \end{aligned}$$

Then $x_n - Tx_n - Fx_n \to 0,$ as $n \to +\infty$, that is $0 \in \overline{(I-T-F)(U \cap \Omega)}$. ∎

Remark 3.2.2 *As for the additivity property in Theorem 3.2.1, we cannot replace the condition* $0 \notin \overline{(I-T-F)((\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega)}$ *by the weaker one that* $T+F$ *has no fixed point on* $(\overline{U} \setminus (U_1 \cup U_2)) \cap \Omega$. *In fact, let us consider the Banach space* c_0 *of real sequences converging to zero with the sup-norm and the cone* $\mathcal{P}$ *of sequences* (a_n) *with only positive entries* a_n. *Let* $r : \overline{\mathcal{P}}_5 \to \overline{\mathcal{P}}_1$ *be the radial retraction,* $s : \overline{\mathcal{P}}_1 \ni (a_1, a_2, \dots) \mapsto (1, a_1, a_2, \dots) \in \overline{\mathcal{P}}_1$ *the well-known shift map, and let* $\hat{F} := s \circ r$. *For* $T = 2I, F = -\hat{F}$, *and* $U = \Omega = \mathcal{P}_5, U_1 = \mathcal{P}_3 \backslash \overline{\mathcal{P}}_2, U_2 = \mathcal{P}_5 \backslash \overline{\mathcal{P}}_4$, *we get*

$$i_* (T+F, \mathcal{P}_5, \mathcal{P}) = 1 \neq 0 + 0 = i_* (T+F, U_1, \mathcal{P}) + i_* (T+F, U_2, \mathcal{P}).$$

Remark 3.2.3 *Notice that a sufficient condition for (3.9) to hold is:*

$$\exists\, \delta > 0,\ \forall\, x \in \partial U \cup \Omega,\ \ \|x - Tx - Fx\| \geq \delta.$$

3.2.2 Computation of the index

The proofs are omitted. For more details, we refer the reader to [30].

Proposition 3.2.4 *Assume that* $T : \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F : \overline{\mathcal{P}_r} \to E$ *is an* $(h-1)$*-set contraction with* $F(\partial \mathcal{P}_r \cap \Omega) \subset \mathcal{P}$ *and* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial \mathcal{P}_r \cap \Omega)}$. *If* $0 \in \Omega$, $\|T0\| < (h-1)r$, *and*

$$Fx \not\geq x - Tx,\ \ \forall\, x \in \partial \mathcal{P}_r \cap \Omega,$$

then the fixed point index $i_* (T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.2.5 *Assume that* $T : \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F : \overline{\mathcal{P}_r} \to E$ *is a* $(h-1)$*-set contraction with* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial \mathcal{P}_r \cap \Omega)}$. *If* $0 \in \Omega$, $\|T0\| < (h-1)r$, *and*

$$Fx \neq \lambda(x - Tx)\ \ \text{for all } x \in \partial \mathcal{P}_r \cap \Omega\ \ \text{and}\ \ \lambda > 1,$$

then the fixed point index $i_* (T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.2.6 *Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ and $F : \overline{\mathcal{P}_r} \to E$ is an $(h-1)$-set contraction with $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$ and $0 \notin \overline{(I-T-F)(\partial\mathcal{P}_r \cap \Omega)}$. If $0 \in \Omega$, $\|T0\| < (h-1)r$, and*

$$\|Fx\| \leq \|x - Tx\| \quad \textit{for all } x \in \partial\mathcal{P}_r \cap \Omega,$$

then the fixed point index $i_* (T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.2.7 *Let U be a bounded open subset of $\mathcal{P}$ such that $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ and $F : \overline{U} \to E$ is an $(h-1)$-set contraction with $F(\overline{U}) \subset (I-T)(\Omega)$ and $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. If*

$$Fx \neq (I-T)(\lambda x), \quad \textit{for all } x \in \partial U \cap \Omega \;\; \textit{and} \;\; \lambda > 1,$$

then the fixed point index $i_* (T + F, U \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.2.8 *Let U be a bounded open subset of $\mathcal{P}$ such that $0 \in U \cap \Omega$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ and $F : \overline{U} \to E$ is an $(h-1)$-set contraction with $F(\overline{U}) \subset (I-T)(\Omega)$ and $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. If*

$$\|Fx + T0\| \leq (h-1)\|x\| \quad \textit{for all } x \in \partial U \cap \Omega, \tag{3.12}$$

then the fixed point index $i_* (T + F, U \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.2.9 *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ such that $F : \overline{U} \to E$ is an $(h-1)$-set contraction with $tF(\overline{U}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$ and $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$\gamma Fx \neq (I-T)(x - \lambda u_0), \; \textit{for all} \;\; \lambda \geq 0, \; x \in \partial U \cap (\Omega + \lambda u_0), \;\; \textit{and} \; \gamma \in (0,1), \tag{3.13}$$

then the fixed point index $i_* (T + F, U \cap \Omega, \mathcal{P}) = 0$.

Proposition 3.2.10 *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ such that $F : \overline{U} \to E$ is an $(h-1)$-set contraction* *with* $tF(\overline{U}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. *Suppose that there exists* $u_0 \in \mathcal{P}^*$ *such that* $T(x-\lambda u_0) \in \mathcal{P}$, *for all* $\lambda \geq 0$ *and* $x \in \partial U \cap (\Omega + \lambda u_0)$, *and one of the following conditions is satisfied:*

(a) $\gamma Fx \not\leq x - \lambda u_0$, *for all* $x \in \partial U$, $\lambda \geq 0$, *and* $\gamma \in (0,1)$.

(b) $Fx \in \mathcal{P}$, $\gamma \|Fx\| > \|x - \lambda u_0\|$, *for all* $x \in \partial U$, $\lambda \geq 0$, $\gamma \in (0,1)$, *and the cone* $\mathcal{P}$ *is normal with constant* $N = 1$.

Then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 0$.

Proposition 3.2.11 *Let* U *be a bounded open subset of* $\mathcal{P}$. *Assume that* $T : \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, *and* $F : \overline{U} \to E$ *is an* $(h-1)$*-set contraction with* $F(\partial U) \subset \mathcal{P}$ *and* $tF(\overline{U}) \subset (I-T)(\Omega), \forall\, t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. *Suppose further that there exists* $u_0 \in \mathcal{P}^*$ *such that*

$$c_0 Fx \not\leq x - T(x - \lambda u_0), \quad \textit{for all } \lambda \geq 0,\ x \in \partial U \cap (\Omega + \lambda u_0) \textit{ and } c_0 \in (0,1). \tag{3.14}$$

Then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 0$.

3.3 The case where T is nonlinear expansive mapping and F is a k-set contraction

The results given in this section are obtained by Djebali and Mebarki. For Proofs and more details we refer the reader to [29].

Let (X,d) be a metric space. Following [77], we put

Definition 3.3.1 *The mapping* $T : X \to X$ *is said to be nonlinear expansive, if there exists a function* $\phi : [0,+\infty) \to [0,+\infty)$ *such that*

$$d(Tx,Ty) \geq \phi(d(x,y)),\ \forall\, x,y \in X,$$

with $\phi(t) > t$, $\forall\, t > 0$ *and*

$$\exists\, r > 0, \quad \omega = \inf_{t \in (0,2r]} \frac{\phi(t) - t}{t} > 0, \tag{3.15}$$

that is T *is* $(\omega+1)$*-expansive. We will denote by* $D = B(0,r)$.

Lemma 3.3.2 *Let $(X, \|.\|)$ be a linear normed space and $T : D \to X$ a nonlinear expansive mapping. Then the inverse of $A := I - T : D \to (I - T)(D)$ exists and is $\frac{1}{\omega}$-Lipschtzian.*

Proof. For each $x, y \in D$, $x \neq y$, and $0 < \|x - y\| \leq 2r$, we have

$$\|Ax - Ay\| = \|(Tx - Ty) - (x - y)\| \geq \phi(\|x - y\|) - \|x - y\| \geq \omega\|x - y\|, \tag{3.16}$$

showing that A is injective. Thus $A^{-1} : D \to A(D)$ exists. Taking $x, y \in A(D)$ and using (3.16), we get

$$\|A^{-1}x - A^{-1}y\| \leq \frac{1}{\omega}\|x - y\|, \quad \text{for all } x, y \in A(D).$$

■

3.3.1 Definition of the index

In this section, $\mathcal{P}$ will refer to a cone in a Banach space E. Let $\Omega = \mathcal{P} \cap B_r$ and U be a bounded open subset of $\mathcal{P}$ such that $\overline{U} \cap \Omega \neq \emptyset$. Assume that $T : \Omega \to X$ is a nonlinear expansive mapping and $F : \overline{U} \to E$ is a k-set contraction. By Lemma 3.3.2, the operator $(I - T)^{-1}$ is $\frac{1}{\omega}$-Lipschtzian on $(I - T)(\Omega)$.

Suppose that $0 \leq k < \omega$, $F(\overline{U}) \subset (I - T)(\Omega)$, and $x \neq Tx + Fx$, for all $x \in \partial U \cap \Omega$. Then $x \neq (I - T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict $\frac{k}{\omega}$-set contraction. Indeed, for any bounded set B in U, we have

$$\alpha(((I - T)^{-1}F)(B)) \leq \frac{1}{\omega}\alpha(F(B)) \leq \frac{k}{\omega}\alpha(B).$$

By Lemma 2.3.2, the fixed point index $i\left((I - T)^{-1}F, U, \mathcal{P}\right)$ is well defined. Thus we put

$$i_*\left(T + F, U \cap \Omega, \mathcal{P}\right) = i\left((I - T)^{-1}F, U, \mathcal{P}\right). \tag{3.17}$$

3.3.2 Computation of the index

In what follows, we compute the fixed point index for the class of mappings under consideration by appealing to some results of Section 2.3.

Proposition 3.3.3 *Assume that $T : \Omega \to E$ is a nonlinear expansive mapping. Let $\rho > 0$ and $F : \overline{\mathcal{P}_\rho} \to E$ be a k-set contraction with $0 \leq k < \omega$, $tF(\overline{\mathcal{P}_\rho}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$ and $F(\partial\mathcal{P}_\rho \cap \Omega) \subset \mathcal{P}$. If $0 \in \Omega$, $\|T0\| < \omega\rho$ and*

$$Fx \ngeq x - Tx, \quad \text{for all } x \in \partial\mathcal{P}_\rho \cap \Omega,$$

then the fixed point index $i\,(T+F, \mathcal{P}_\rho \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.3.4 *Let U be an open bounded subset of $\mathcal{P}$ with $0 \in U \cap \Omega$. Assume that $T : \Omega \to E$ is a nonlinear expansive mapping, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \omega$, and $F(\overline{U}) \subset (I-T)(\Omega)$. If*

$$\|Fx + T0\| \leq (\omega - 1)\|x\| \quad \text{and} \quad Tx + Fx \neq x, \quad \text{for all } x \in \partial U \cap \Omega, \tag{3.18}$$

then the fixed point index $i\,(T+F, U \cap \Omega, \mathcal{P}) = 1$.

Corollary 3.3.5 *Assume that $T : \Omega \to E$ is a nonlinear expansive mapping, $F : \overline{\mathcal{P}_\rho} \to E$ is a k-set contraction with $0 \leq k < \omega$, and $F(\overline{\mathcal{P}_\rho}) \subset (I-T)(\Omega)$. If $0 \in \Omega$ and*

$$\|Fx + T0\| < (\omega - 1)\rho, \quad \text{for all } x \in \overline{\mathcal{P}_\rho}, \tag{3.19}$$

then $i\,(T+F, \mathcal{P}_\rho \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.3.6 *Let U be an open bounded subset of $\mathcal{P}$. Assume that $T : \Omega \to E$ is a nonlinear expansive mapping, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \omega$ and $F(\overline{U}) \subset (I-T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$T(x - \lambda u_0) \in \mathcal{P} \text{ and } Fx \nleq x - \lambda u_0 \text{ for all } (x, \lambda) \in \partial U \times [0,1],$$

then the fixed point index $i\,(T+F, U \cap \Omega, \mathcal{P}) = 0$.

3.4 The case where T is h-expansive mapping and $I-F$ is a k-set contraction with $0 \leq k < h$

In [10], Benslimane, Djebali and Mebarki developed a fixed point index for the sum $T+F$ where T is an expansive mapping with constant $h > 1$ and $I-F$ a k-set contraction with

$k < h$. In this section, we present the definition of this index with respect to a translate of a cone $\mathcal{K}$ neither than to a cone.

3.4.1 Definition of the index

Given a real Banach space $(E, \|.\|)$, let $Y \subset E$ be a closed convex subset. Let Ω be any subset of Y and U be a bounded open subset of Y. Consider an expansive mapping $T : \Omega \to E$ with constant $h > 1$ and let $I - F : \overline{U} \to E$ be a k-set contraction with $0 \leq k < h$ and supppose that

$$(I-F)(\overline{U}) \subset T(\Omega).$$

If $x \neq Tx + Fx$, for all $x \in \partial U \cap \Omega$, then $x \neq T^{-1}(I-F)x$, for all $x \in \partial U$.

As in [29], a fixed point index of the sum $T + F$ on $U \cap \Omega$ with respect to the closed convex set Y can be defined by

$$i_*\left(T+F, U\cap\Omega, Y\right) = \begin{cases} i\left(T^{-1}(I-F), U, Y\right), & \text{if } U\cap\Omega \neq emptyset \\ 0, & \text{if } U\cap\Omega = \emptyset. \end{cases} \tag{3.20}$$

Theorem 3.4.1 *The fixed point index $i_*(T+F, U\cap\Omega, Y)$ defined in (3.20) has the following properties:*

(i) (Normalization property). *If $U = Y \cap \mathcal{B}(\omega, r)$, $\omega \in \Omega$, and $(I-F)x = z_0$ for all $x \in \overline{U}$, where $z_0 \in Y$ and $\|z_0 - T\omega\| < hr$, then*

$$i_*\left(T+F, U\cap\Omega, Y\right) = 1.$$

(ii) (Additivity property). *For any pair of disjoint open subsets $U_1, U_2 \subset U$ such that $T+F$ has no fixed point on $(\overline{U}\setminus(U_1\cup U_2))\cap\Omega$, we have*

$$i_*(T+F, U\cap\Omega, Y) = i_*(T+F, U_1\cap\Omega, Y) + i_*(T+F, U_2\cap\Omega, Y).$$

(iii) (Homotopy invariance property). *The generalized fixed point index $i_*(T+H(.,t), U\cap\Omega, Y)$ does not depend on the parameter $t \in [0,1]$, where*

(a) $(I-H) : [0,1]\times\overline{U} \to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,

(b) $(I-H)([0,1]\times \overline{U}) \subset T(\Omega)$,

(c) $(I-H(t,.)) : \overline{U} \to E$ *is a* ℓ*-set contraction with* $0 \leq \ell < h$, *for all* $t \in [0,1]$,

(d) $Tx + H(t,x) \neq x$ *for all* $t \in [0,1]$ *and* $x \in \partial U \cap \Omega$.

(iv) (Existence property). *If* $i_*(T+F, U\cap\Omega, Y) \neq 0$, *then* $T+F$ *has a fixed point in* $U \cap \Omega$.

Proof. We argue as in [29, Theorem 2.1]. Properties (ii), (iii), and (iv) are consequences of (3.20) and of the properties of the fixed point index for strict set contractions (see [45, Theorem 1.3.5]). It remains to check the normalization property. If $U = Y \cap \mathcal{B}(w,r)$, then

$$i\,(T^{-1}(I-F), U, Y) = i\,(T^{-1}z_0, U, Y) = 1.$$

For this purpose, we show that $y_0 := T^{-1}z_0 \in \mathcal{B}(\omega, r)\cap\Omega$. $(I-F)(\overline{U}) = \{z_0\} \subset T(\Omega)$ implies that $y_0 \in \Omega$ and since T is an expansive operator with $h > 1$, then

$$\|Ty_0 - T\omega\| \geq h\|y_0 - \omega\|.$$

Then

$$h\|y_0 - \omega\| \leq \|Ty_0 - T\omega\| = \|z_0 - T\omega\| < hr,$$

and thus $y_0 = T^{-1}z_0 \in U$. Using the normalization property of the index [45, Theorem 1.3.5], we find that

$$i\,(T^{-1}z_0, U, Y) = 1.$$

Finally $i_*\,(T+F, U\cap\Omega, Y) = 1$, as claimed. ■

Remark 3.4.2 *Let* $\mathcal{P} \subset E$ *be a cone,* $0 \in \Omega$, *and* $U = \mathcal{P} \cap \{x \in E : \psi(x) < R\}$, *where* ψ *is a nonnegative continuous functional on* $\mathcal{P}$ *satisfying* $\psi(x) \leq \|x\|$ *for all* $x \in \mathcal{P}$. *If* $(I-F)x = z_0$, *for all* $x \in \overline{U}$, *where* $z_0 \in \mathcal{P}$ *and* $\|z_0 - T0\| < hR$, *then we can prove as for the normalization property that*

$$i_*\,(T+F, U\cap\Omega, \mathcal{P}) = 1.$$

3.4.2 Computation of the index

In this section, we show that the fixed point index can be computed in case of a translate of a cone, rather than in a cone, and in some cases even in an arbitrary closed convex subset.

A fixed point index in translates of cones of Banach spaces is defined in [28] for completely continuous mappings and can be extended to the case of a strict set contractions. Let $\mathcal{P} \neq \{0\}$ be a cone in E and $\mathcal{K} = \mathcal{P} + \theta$ ($\theta \in E$) a θ-translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ be a subset and $U \subset \mathcal{K}$ be a bounded open subset such that $\Omega \cap U \neq \emptyset$. Since $\mathcal{K}$ is a closed convex of E, the fixed point index $i_* (T + F, U \cap \Omega, \mathcal{K})$ is well defined whenever $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, $(I-F)(\overline{U}) \subset T(\Omega)$, and $x \neq Tx + Fx$ for all $x \in \partial U \cap \Omega$, where $\overline{U}$ and ∂U denote the closure and the boundary of U in $\mathcal{K}$, respectively. For two real numbers $0 < r < R$, define the sets:

$$\begin{aligned} \mathcal{K}_r &= \{x \in \mathcal{K} : \|x - \theta\| < r\} \\ \partial\mathcal{K}_r &= \{x \in \mathcal{K} : \|x - \theta\| = r\} \\ \mathcal{K}_{r,R} &= \{x \in \mathcal{K} : r < \|x - \theta\| < R\}. \end{aligned}$$

Proposition 3.4.3 *Let $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$ and $I - F : \overline{\mathcal{K}_r} \to E$ be a k-set contraction with $0 \leq k < h$ such that $t(I-F)(\overline{\mathcal{K}_r}) + (1-t)\theta \subset T(\Omega)$, for all $t \in [0,1]$. Assume that $\theta \in \Omega$, $\|T\theta - \theta\| < hr$, and*

$$Tx \neq \lambda(x - Fx) + (1-\lambda)\theta, \quad \text{for all } x \in \partial\mathcal{K}_r \cap \Omega \text{ and } 0 \leq \lambda \leq 1. \tag{3.21}$$

Then $i_* (T + F, \mathcal{K}_r \cap \Omega, \mathcal{P}) = 1$.

Proof. Define the line homotopy $H : [0,1] \times \overline{\mathcal{K}_r} \to E$ by

$$H(t,x) = tFx + (1-t)(x - \theta).$$

Then, the operator $(I - H)$ is continuous and uniformly continuous in t for each x. Moreover the mapping $(I - H(t,.))$ is a k-set contraction for each t. Actually, for any bounded set B in $\mathcal{K}_r$, we have

$$\alpha((I - H(t,.))(B)) = \alpha(t(I-F)(B) + (1-t)\theta) \leq k\alpha(B).$$

In addition, $T + H(t,.)$ has no fixed point on $\partial\mathcal{K}_r \cap \Omega$. If not, there exist some $x_0 \in \partial\mathcal{K}_r \cap \Omega$ and $t_0 \in [0,1]$ such that

$$Tx_0 + t_0 Fx_0 + (1-t_0)(x_0 - \theta) = x_0.$$

Then $Tx_0 = t_0(x_0 - Fx_0) + (1-t_0)\theta$, leading to a contradiction with the hypothesis. By properties (i) and (iii) of the fixed point index in Theorem 3.4.1, we get

$$i_*\left(T+F, \mathcal{K}_r \cap \Omega, \mathcal{K}\right) = i_*\left(T+I-\theta, \mathcal{K}_r \cap \Omega, \mathcal{K}\right) = 1.$$

■

From Proposition 3.4.3, we capture the following two results.

Corollary 3.4.4 *Assume that $T : \Omega \subset \mathcal{K} \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{\mathcal{K}_r} \to E$ is a k-set contraction with $0 \leq k < h$, and $\big(t(I-F)(\overline{\mathcal{K}_r}) + (1-t)\theta\big) \subset T(\Omega)$, for all $t \in [0,1]$. If $\theta \in \Omega$, $\|T\theta - \theta\| < hr$, and*

$$\|Tx - \theta\| \geq \|x - Fx - \theta\| \text{ and } Tx + Fx \neq x, \text{ for all } x \in \partial\mathcal{K}_r \cap \Omega.$$

Then $i_*\left(T+F, \mathcal{K}_r \cap \Omega, \mathcal{K}\right) = 1$.

Proof. It is sufficient to prove that Assumption (3.21) holds. By contradiction, let $x_0 \in \mathcal{K}_r \cap \Omega$ and let $0 \leq \lambda_0 \leq 1$ satisfy $Tx_0 = \lambda_0(x_0 - Fx_0) + (1-\lambda_0)\theta$. If $\lambda_0 = 1$, then $x_0 - Fx_0 = Tx_0$ which is impossible. If $0 \leq \lambda_0 < 1$, then $\|Tx_0 - \theta\| = \lambda_0\|x_0 - Tx_0 - \theta\| < \|x_0 - Tx_0 - \theta\|$, which is a contradiction. ■

Corollary 3.4.5 *Let $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$ and let $I - F : \overline{\mathcal{K}_r} \to E$ be a k-set contraction with $0 \leq k < h$ such that $\big((I-F)(\overline{\mathcal{K}_r}) + (1-t)\theta\big) \subset T(\Omega)$, for all $t \in [0,1]$. Assume further that $\theta \in \Omega$, $\|T\theta - \theta\| < hr$,*

$$x - Fx \in \mathcal{K} \text{ for all } x \in \partial\mathcal{K}_r \cap \Omega,$$

and

$$Tx \nleq x - Fx \text{ for all } x \in \partial\mathcal{K}_r \cap \Omega.$$

Then $i_*\left(T+F, \mathcal{K}_r \cap \Omega, \mathcal{K}\right) = 1$.

Proof. Assumption (3.21) is readily checked for otherwise there would exist some $x_0 \in \mathcal{K}_r \cap \Omega$ and $0 \leq \lambda_0 \leq 1$ such that $Tx_0 = \lambda_0(x_0 - Fx_0) + (1-\lambda_0)\theta$. Hence $Tx_0 - \theta = \lambda_0(x_0 - Fx_0 - \theta)$. Since $x_0 - Fx_0 - \theta \in \mathcal{P}$, then $\lambda_0(x_0 - Fx_0 - \theta) \leq x_0 - Fx_0 - \theta$, which is a contradiction to our assumption. ■

Proposition 3.4.6 *Let $U \subset \mathcal{K}$ be a bounded open subset with $\theta \in U_1$ and $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, and $(I-F)(\overline{U}) \subset T(\Omega)$. Assume further that*

$$x - Fx \neq T(\lambda x + (1-\lambda)\theta), \quad \textit{for all } x \in \partial U,\ \lambda \geq 1 \textit{ and } \lambda x + (1-\lambda)\theta \in \Omega.$$

Then $i_*\left(T + F, U \cap \Omega, \mathcal{K}\right) = 1$.

Proof. The mapping $T^{-1}(I-F) : \overline{U} \to \mathcal{K}$ is a strict set contraction and it is clear that

$$T^{-1}(I-F)x - \theta \neq \lambda(x - \theta), \quad \text{for all } x \in \partial U \ \text{ and } \ \lambda \geq 1. \tag{3.22}$$

Owing to [28, Proposition 2.2], $i\left(T^{-1}(I-F), U, \mathcal{K}\right) = 1$. Then Equality (3.20) ends this proof. ■

Proposition 3.4.7 *Let $U \subset \mathcal{K}$ be a bounded open subset, $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, and $(I-F)(\overline{U}) \subset T(\Omega)$. Assume that $\theta \in \Omega$,*

$$\|x - Fx - T\theta\| \leq h\|x - \theta\|, \quad \textit{and} \ \ Tx + Fx \neq x, \quad \textit{for all } x \in \partial U \cap \Omega. \tag{3.23}$$

Then $i_*\left(T + F, U \cap \Omega, \mathcal{P}\right) = 1$.

Proof. According to Lemma 1.3.18, $T^{-1}(I-F) : \overline{U} \to \mathcal{K}$ is a strict set contraction. From the inclusion $(I-F)(\overline{U}) \subset T(\Omega)$, for all $x \in \overline{U}$, we can find some $y \in \Omega$ such that $x - Fx = Ty$. For all $x \in \overline{U}$, we have $T^{-1}(x - Fx) \in \Omega$ and

$$T((T^{-1}(x - Fx)) = x - Fx,$$

which implies that

$$\|T(T^{-1}(x - Fx)) - T\theta\| = \|x - Fx - T\theta\|.$$

Since T is expansive with constant h, we have

$$\|T(T^{-1}(x - Fx)) - T\theta\| \geq h\|T^{-1}(x - Fx) - \theta\|.$$

Hence

$$h\|T^{-1}(I-F)x-\theta\| \le \|x-Fx-T\theta\|. \tag{3.24}$$

From (3.24) and Assumption (3.23), we get

$$\|T^{-1}(I-F)x-\theta\| \le \frac{1}{h}\|x-Fx-T\theta\| \le \|x-\theta\|, \ \forall x \in \partial U.$$

Therefore for all $x \in \partial U \cap \Omega$

$$\|T^{-1}(I-F)x-\theta\| \le \|x-\theta\| \text{ and } T^{-1}(I-F)x \ne x.$$

Due to [28, Corollary 2.2], $i\,(T^{-1}(I-F), U, \mathcal{K}) = 1$. Equality (3.20) completes the proof. ∎

In case of a cone, i.e., $\theta = 0$, Proposition 3.4.6 and Proposition 3.4.7 become

Corollary 3.4.8 *Let $U \subset \mathcal{K}$ be a bounded open subset and $T : \Omega \subset \mathcal{P} \to E$ be an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ be a k-set contraction with $0 \le k < h$, and $(I-F)(\overline{U}) \subset T(\Omega)$. Assume further that*

$$x - Fx \ne T(\lambda x), \quad \text{for all } x \in \partial U \cap \Omega, \ \lambda \ge 1, \quad \text{and } \lambda x \in \Omega.$$

Then $i_*\,(T+F, U\cap\Omega, \mathcal{P}) = 1$.

Corollary 3.4.9 *Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ is a k-set contraction with $0 \le k < h$, and $(I-F)(\overline{U}) \subset T(\Omega)$. Let $0 \in \Omega$,*

$$\|x - Fx - T0\| \le h\|x\|, \quad \text{and} \quad Tx + Fx \ne x, \quad \text{for all } x \in \partial U \cap \Omega. \tag{3.25}$$

Then $i_*\,(T+F, U\cap\Omega, \mathcal{P}) = 1$.

The following result can be directly proved by replacing the operator A in [45, Corollary 1.3.1] by $T^{-1}(I-F)$.

Proposition 3.4.10 *Assume that $T : \Omega \subset \mathcal{K} \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{\mathcal{K}_r} \to E$ is a k-set contraction with $0 \le k < h$, and $(I-F)(\overline{\mathcal{K}_r}) \subset T(\Omega)$. In addition, if $T^{-1}(I-F)(\overline{\mathcal{K}_r}) \subset \mathcal{K}_r$, then $i_*\,(T+F, \mathcal{K}_r \cap \Omega, \mathcal{K}) = 1$.*

In particular, we have

Corollary 3.4.11 *Assume that $T : \Omega \subset \mathcal{K} \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{\mathcal{K}}_r \to E$ is a k-set contraction with $0 \leq k < h$, and $(I - F)(\overline{\mathcal{K}}_r) \subset T(\Omega)$. If $\theta \in \Omega$, and*

$$\|x - Fx - T\theta\| < hr, \quad \textit{for all } x \in \overline{\mathcal{K}}_r. \tag{3.26}$$

Then $i_* (T + F, \mathcal{K}_r \cap \Omega, \mathcal{K}) = 1$.

Proof. From (3.24) and Assumption (3.26), for all $x \in \overline{\mathcal{K}}_r$, we conclude that

$$\|T^{-1}(I - F)x - \theta\| \leq \frac{1}{h}\|x - Fx - T\theta\| < r.$$

Hence $T^{-1}(I - F)(\overline{\mathcal{K}}_r) \subset \mathcal{K}_r$. ■

A special situation of Corollary 3.4.11 is

Corollary 3.4.12 *Assume that $T : \Omega \subset \mathcal{K} \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{\mathcal{K}}_r \to E$ is a k-set contraction with $0 \leq k < h$, r is sufficiently large, and $(I - F)(\overline{\mathcal{K}}_r) \subset T(\Omega)$. If further $\theta \in \Omega$ and*

$$\|x - Fx\| \leq \|x - \theta\|, \quad \textit{for all } x \in \overline{\mathcal{K}}_r, \tag{3.27}$$

then $T + F$ has at least one fixed point in $\mathcal{K}_r \cap \Omega$.

Proof. Notice that

$$\begin{aligned} \|x - Fx - T\theta\| &\leq \|x - Fx\| + \|T\theta\| \\ &\leq \|x - \theta\| + \|T\theta\| \\ &\leq r + \|T\theta\| \\ &\leq hr, \end{aligned}$$

for all $r > \frac{\|T\theta\|}{h-1}$. By Corollary 3.4.11, $i_* (T + F, \mathcal{K}_r \cap \Omega, \mathcal{P}) = 1$. As a consequence, $T + F$ has a fixed point in $\mathcal{K}_r \cap \Omega$. ■

Before giving results for zero index i_*, we need an auxiliary lemma on index fixed point of strict set contractions.

Lemma 3.4.13 *Let $\mathcal{K}$ be a translate of a cone $\mathcal{P} \neq \emptyset$ and U be a bounded open subset of $\mathcal{K}$. Assume that $A : \overline{U} \to \mathcal{K}$ is a strict set contraction and there is $w_0 \in \mathcal{P}^*$ such that*

$$x - Ax \neq \lambda w_0, \quad \textit{for all } x \in \partial U, \lambda \geq 0. \tag{3.28}$$

Then $i(A, U, \mathcal{K}) = 0$.

Proof. Define the homotopy $H : [0, 1] \times \overline{U} \to \mathcal{K}$ by

$$H(t, x) = Ax + t\lambda_0 w_0,$$

for some

$$\lambda_0 > \sup_{x \in \overline{U}} (\|w_0\|^{-1}(\|x\| + \|Ax\|)). \tag{3.29}$$

Such a choice is possible since U is a bounded subset and so is $A(\overline{U})$. The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t, .)$ is a strict set contraction for each $t \in [0, 1]$. In addition, $H(t, .)$ has no fixed point on ∂U. On the contrary, there would exist some $x_0 \in \partial U$ and $t_0 \in [0, 1]$ such that

$$x_0 = Ax_0 + t_0 \lambda_0 w_0,$$

contradicting the hypothesis. By Lemma 2.3.2,, we get

$$i(A, U, \mathcal{K}) = i(H(0, .), U, \mathcal{K}) = i(H(1, .), U, \mathcal{K}) = 0. \tag{3.30}$$

Indeed, suppose that $i(H(1, .), U, \mathcal{K}) \neq 0$. Then there would exist $x_0 \in U$ such that $Ax_0 + \lambda_0 w_0 = x_0$, which implies that $\lambda_0 \leq \|w_0\|^{-1}(\|x_0\| + \|Ax_0\|)$, contradicting (3.29). ■

Proposition 3.4.14 *Let $U \subset \mathcal{K}$ be a bounded open subset, $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, and $(I - F)(\overline{U}) \subset T(\Omega)$. Let $u_0 \in \mathcal{P}^*$ be such that*

$$x - Fx \neq T(x - \lambda u_0), \ \textit{for all} \ x \in \partial U \cap (\Omega + \lambda u_0) \ \textit{and} \ \lambda \geq 0. \tag{3.31}$$

Then $i_* (T + F, U \cap \Omega, \mathcal{K}) = 0$.

Proof. The mapping $T^{-1}(I - F) : \overline{U} \to \mathcal{K}$ is a strict set contraction and in view of (3.31), we have

$$x - T^{-1}(I - F)x \neq \lambda u_0 \ \text{ for all } \ x \in \partial U \ \text{ and } \ \lambda \geq 0.$$

By (3.20) and Lemma 3.4.13, we deduce that

$$i_* (T + F, U \cap \Omega, \mathcal{P}) = i \,(T^{-1}(I - F), U, \mathcal{P}) = 0.$$

■

The following two propositions are direct consequences of Proposition 3.4.14; the proofs are omitted.

Proposition 3.4.15 *$U \subset \mathcal{K}$ be a bounded open subset and $T : \Omega \subset \mathcal{K} \to E$ be an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, and $(I - F)(\overline{U}) \subset T(\Omega)$. Suppose further that there exists $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$, for all $x \in \partial U \cap (\Omega + \lambda u_0)$ and*

$$Fx \not\leq x, \quad \textit{for all } x \in \partial U \ \textit{ and } \lambda \geq 0.$$

Then $i_* (T + F, U \cap \Omega, \mathcal{K}) = 0$.

Proposition 3.4.16 *Let $U \subset \mathcal{K}$ be a bounded open subset. Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$, and $(I - F)(\overline{U}) \subset T(\Omega)$. Let $u_0 \in \mathcal{P}^*$ satisfy $T(x - \lambda u_0) \in \mathcal{P}$, for all $x \in \partial U \cap (\Omega + \lambda u_0)$ and $\lambda \geq 0$. Suppose that the following conditions hold:*

$$Fx \in \mathcal{K}, \ \textit{and } \|Fx - \theta\| > N\|x - \theta\|, \quad \textit{for all } x \in \partial U.$$

Then $i_* (T + F, U \cap \Omega, \mathcal{K}) = 0$.

Remark 3.4.17 *(1) Letting $\theta = 0$, we obtain computations of the index in case of a cone.*
(2) Proposition 3.4.3 and Corollary 3.4.4 remain valid in the more general setting of $Y \cap \mathcal{B}(\theta, R)$, where $Y \subset E$ is an arbitrary closed convex subset and $\mathcal{B}(\theta, R) = \{x \in E : \|x - \theta\| < R\}$.
(3) Proposition 3.4.6 holds in the framework of any closed convex subset Y of E containing θ.

3.5 Concluding remarks

In this section, we will compare between the generalized fixed point index developed by Djebali and Mebarki in [29] and the one developed by Benslimane, Djebali and Mebarki in [10].

(1) Since T and $I - T$ have the same properties in terms of invertibility and since $I - F$ is an ℓ-set contraction with $\ell < h$, one could think that the fixed point index developed in [10] is

a generalization of the one developed in [29]. Unfortunately the implication

$$F(\overline{U}) \subset (I-T)(\Omega) \Rightarrow (I-F)(\overline{U}) \subset T(\Omega)$$

does not in general hold. For example:

(a) Let $T:[0,1]\to\mathbb{R}$ be such that $Tx=-\frac{5}{2}e^{x}$ and $F:[0,4]\to\mathbb{R}$ is $Fx=e^{-x}+3$. Then, the conditions of the fixed point index developed in [29] are satisfied. Indeed, T is a $\frac{5}{2}$-expansive mapping and F is a 1-set contraction. In addition $F([0,4])=[e^{-4}+3,4]\subset(I-T)([0,1])=[\frac{5}{2},1+\frac{5e}{2}]$ but $(I-F)([0,4])=[-4,1-\frac{1}{e}]\not\subset T([0,1])=[-\frac{5e}{2},-\frac{5}{2}]$.

(b) Let $T:[0,1]\to\mathbb{R}$ be such that $Tx=2x$ and $F:[0,5]\to\mathbb{R}$ is $Fx=-\frac{1}{10}x+g(x)$, where $g:[0,5]\to[-\frac{1}{2},0]$ is a $\frac{4}{5}$-set contraction such that the equation $g(x)+\frac{9}{10}x=0$ has a solution in $(0,1]$. Then the conditions of the fixed point index developed in [29] are satisfied. Indeed, T is a 2-expansive mapping and F is a $\frac{9}{10}$-set contraction. In addition $F([0,5])\subset[-1,0]=(I-T)([0,1])$ but $(I-F)([0,5])\not\subset T([0,1])=[0,2]$.

(2) Conversely, define two mappings $T,F:[0,1]\to\mathbb{R}$ by $Tx=\frac{3}{2}e^{x}$ and $Fx=-2e^{-x}$. Then T is a $\frac{3}{2}$-expansive mapping, $(I-F)x=x+2e^{-x}$ is a 1-set contraction, and $(I-F)([0,1])=[\frac{2+e}{e},2]\subset T([0,1])=[\frac{3}{2},\frac{3}{2}e]$. It is clear that the conditions of the fixed point index developed in [10] are satisfied, while that of the index defined in [29] are not (F is a 2-set contraction). Moreover, the equation $Fx+Tx=x$ cannot be rewritten in the abstract form $\tilde{T}x+\tilde{F}x=x$, where $\tilde{T}$ is $\tilde{h}$-expansive and $\tilde{F}\not\equiv 0$ is $\tilde{k}$-set contraction with $\tilde{k}<\tilde{h}-1$.

(3) These two examples show that the fixed point index presented in [10] and the one developed in [29] do not coincide and are not easily comparable.

Chapitre 4

New fixed point theorems for the sum of two operators

Contents

After computing the new index i_*, several fixed point theorems and recent results are derived. In [10, 11], we have obtained some new Krasonesl'skii type and Leggett-Williams type fixed point theorems for the sum of two operators $T+F$, where T is expansive with constant $h>1$ and $I-F$ is a k-set contraction with $0 \leq k < h$. These are extensions of a Krasnosel'skii type as well as of a Leggett-Williams type expansion-compression fixed point theorem on cones for a sum of two operators. Each section ends with applications to nonlinear integral or differential equations illustrating the abstract results obtained in our works.

Throughout this chapter, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$.

4.1 Expansion-Compression fixed point theorem of Krasnosel'skii type for the sum of two operators

The results given in this section are obtained by Benslimane-Djebali-Mebarki in [10].

4.1.1 Main results

Some results from the section 3.4 are combined to establish three fixed point theorems of cone compression and expansion type. The proofs are based on the properties of the topological index i_*. We omit the details.

Theorem 4.1.1 *(Homotopy version). Let E be a Banach space, $\mathcal{P} \subset E$ a cone, and $\mathcal{K} = \mathcal{P} + \theta$ a translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ with $\theta \in \Omega$. Let U_1 and U_2 be two open bounded subsets of $\mathcal{K}$ such that $\theta \in \overline{U_1} \subset U_2$. Let $T : \Omega \to E$ be an expansive mapping with constant $h > 1$ and $I - F : \overline{U_2} \to E$ a k-set contraction with $0 \leq k < h$ such that $(I - F)(\overline{U_2}) \subset T(\Omega)$. Assume that $(\overline{U_2} \setminus U_1) \cap \Omega \neq \emptyset$ and there exists $u_0 \in \mathcal{P}^*$ such that either one of the following conditions holds:*

(i) $x - Fx \neq T(x - \lambda u_0)$, for all $x \in \partial U_1 \cap (\Omega + \lambda u_0)$, and $\lambda > 0$

$x - Fx \neq T(\lambda x + (1-\lambda)\theta)$, for all $x \in \partial U_2$, $\lambda > 1$ and $\lambda x + (1-\lambda)\theta \in \Omega$.

(ii) $x - Fx \neq T(x - \lambda u_0)$ for all $x \in \partial U_2 \cap (\Omega + \lambda u_0)$, and

$x - Fx \neq T(\lambda x + (1 - \lambda)\theta)$, *for all* $x \in \partial U_1$, $\lambda > 1$ *and* $\lambda x + (1 - \lambda)\theta \in \Omega$.

Then $T + F$ *has a fixed point* $x \in (\overline{U_2} \setminus U_1) \cap \Omega$.

Proof. Without loss of generality, suppose that $Tx + Fx \neq x$ on $\partial U_1 \cap \Omega$ and on $\partial U_2 \cap \Omega$, otherwise we are finished. If condition (i) holds, by Propositions 3.4.6 and 3.4.14, we have

$$i_* (T + F, U_1 \cap \Omega, \mathcal{K}) = 1 \text{ and } i_* (T + F, U_2 \cap \Omega, \mathcal{K}) = 0.$$

The additivity property of the index yields

$$i_* (T + F, (\overline{U}_2 \setminus U_2 \cap \Omega, \mathcal{K}) = -1.$$

By the existence property of the index, the sum $T + F$ has at least one fixed point in the closed set $(\overline{U_2} \setminus U_1) \cap \Omega$. The proof is similar in case (ii). ■

Theorem 4.1.2 *(Norm version). Let E be a Banach space, $\mathcal{P} \subset E$ a normal cone with constant N, and $\mathcal{K} = \mathcal{P} + \theta$ a translate of $\mathcal{P}$. Let $\theta \in \Omega \subset \mathcal{K}$ and U_1, U_2 be two bounded open subsets of $\mathcal{K}$ such that $\theta \in \overline{U_1} \subset U_2$. Let $T : \Omega \to E$ be an expansive mapping with constant $h > 1$ and $I - F : \overline{U_2} \to E$ a k-set contraction with $0 \leq k < h$ such that $(I - F)(\overline{U_2}) \subset T(\Omega)$. Assume that $(\overline{U_2} \setminus U_1) \cap \Omega \neq \emptyset$ and there are $u_0 \in \mathcal{P}^*$ with $T(x - \lambda u_0) \in \mathcal{P}$, for all $\lambda > 0$ and $x \in \partial U_1 \cap \partial U_2 \cap (\Omega + \lambda u_0)$. Let one of the following conditions holds:*

(i) $\|x - Fx - T\theta\| < h\|x - \theta\|$, *for all* $x \in \partial U_1 \cap \Omega$ *and* $Fx \in \mathcal{K}$, $\|Fx - \theta\| > N\|x - \theta\|$, *for all* $x \in \partial U_2$,

(ii) $\|x - Fx - T\theta\| < h\|x - \theta\|$ *for all* $x \in \partial U_2 \cap \Omega$ *and* $Fx \in \mathcal{K}$, $\|Fx - \theta\| > N\|x - \theta\|$, *for all* $x \in \partial U_1$.

Then $T + F$ *has a fixed point* $x \in (\overline{U_2} \setminus U_1) \cap \Omega$.

Proof. The proof uses Propositions 3.4.7 and 3.4.16. ■

Theorem 4.1.3 *(Order version). Let E be a Banach space, $\mathcal{P} \subset E$ a cone, and $\mathcal{K} = \mathcal{P} + \theta$ a translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ with $\theta \in \Omega$, $\gamma, \beta > 0$, $\gamma \neq \beta$, $r = \min\{\gamma, \beta\}$, and $R = \max\{\gamma, \beta\}$. Let $T : \Omega \to E$ be an expansive mapping with constant $h > 1$ such that $\|T\theta - \theta\| < h\gamma$, and $I - F : \overline{\mathcal{K}}_R \to E$ be a k-set contraction with $0 \leq k < h$. Assume that $\mathcal{K}_{r,R} \cap \Omega \neq \emptyset$,*

$$(I - F)(\partial \mathcal{K}_\gamma \cap \Omega) \subset \mathcal{K},$$

and there is

$$u_0 \in \mathcal{P}^* \text{ with } T(x - \lambda u_0) \in \mathcal{P}, \text{ for all } \lambda > 0, x \in \partial\mathcal{K}_\beta \cap (\Omega + \lambda u_0).$$

If further

$$\begin{cases} Tx & \nleq & x - Fx, \text{ for all } x \in \partial\mathcal{K}_\gamma \cap \Omega, \\ Fx & \nleq & x, \text{ for all } x \in \partial\mathcal{K}_\beta, \end{cases}$$

then $T + F$ *has a fixed point* $x \in \overline{\mathcal{K}}_{r,R} \cap \Omega$.

Proof. The proof uses Corollary 3.4.5 and Proposition 3.4.15. ■

Clearly, the following result on a cone is a particular case of Theorem 4.1.1.

Corollary 4.1.4 *Let* E *be a Banach space,* $\mathcal{P} \subset E$ *a cone, and* $\Omega \subset \mathcal{P}$ *with* $0 \in \Omega$. *Let* U_1 *and* U_2 *be two open subsets of* $\mathcal{P}$ *such that* $0 \in \overline{U}_1 \subset U_2$. *Let* $T : \Omega \to E$ *be an expansive mapping with constant* $h > 1$ *and* $I - F : \overline{U}_2 \to E$ *a* k*-set contraction with* $0 \leq k < h$. *Assume that* $(\overline{U}_2 \setminus U_1) \cap \Omega \neq \emptyset$ *and*

$$(I - F)(\overline{U}_2) \subset T(\Omega).$$

Assume that there exists $u_0 \in \mathcal{P}^*$ *such that either one of the following conditions holds:*

(i) $x - Fx \neq T(\lambda x)$, *for all* $x \in \partial U_1 \cap \lambda > 1$ *and* $\lambda x \in \Omega$, *and*

$(I - F)x \neq T(x - \lambda u_0)$, *for all* $x \in \partial U_2 \cap (\Omega + \lambda u_0), \lambda > 0$,

(ii) $x - Fx \neq T(\lambda x)$, *for all* $x \in \partial U_2 \cap \Omega$ *and* $\lambda > 1$, *and*

$(I - F)x \neq T(x - \lambda u_0)$, *for all* $x \in \partial U_1 \cap (\Omega + \lambda u_0), \lambda \geq 0$.

Then $T + F$ *has a fixed point* $x \in (\overline{U}_2 \setminus U_1) \cap \Omega$.

4.1.2 Applications

Application 1

Consider the nonlinear equation

$$p(t)x^3(t) - x(t) = g(t, x(t)), \quad 0 < t < 1, \tag{4.1}$$

where

$(\mathcal{H}_1)$ $p : [0,1] \to \mathbb{R}_+$ is continuous, $g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and for each bounded function x on $[0,1]$, the superposition operator $g(\cdot, x(\cdot))$ is equicontinuous on $[0,1]$.

Let

$$p_1 : = \min_{0 \le t \le 1} p(t) \text{ and } p_2 =: \max_{0 \le t \le 1} p(t).$$

Assume that

$$(\mathcal{H}_2) \qquad 1 \le p_1 \le p_2 < 1 + 2p_1.$$

$(\mathcal{H}_3)$ There exists $R > 0$ such that

$$p(t) - 1 \le g(t,x) \le p_1 R^3 - R, \quad \forall\, (t,x) \in [0,1] \times [0, R+1] \tag{4.2}$$

and

$$3p_1 R - p_1 R^3 \ge p_2 - 1. \tag{4.3}$$

Remark 4.1.5 *(Discussion of Hypothesis $(\mathcal{H}_3)$) (a) A sufficient condition for $(\mathcal{H}_3)$ to hold is that g is uniformly bounded and*

$$0 < p_2 - 1 \le \|g\|_0 < \frac{3p_1 - 1}{2}\sqrt{\frac{3p_1 + 1}{2p_1}}, \tag{4.4}$$

where $\|g\|_0 = \sup_{0 \le t \le 1,\, x \ge 0} g(t,x)$. To see this, let the functions $\phi(R) = 3p_1 R - p_1 R^3$ and $\psi(R) = p_1 R^3 - R$. Then the function ϕ is positive on $(0, \sqrt{3})$ and assumes $2p_1$ as a maximum at the point $R = 1$. The function ψ is positive increasing function over $(\frac{1}{\sqrt{p_1}}, +\infty)$. The functions ϕ and ψ intersect at the point $R_0 = \sqrt{\frac{3p_1+1}{2p_1}}$ with $\phi(R_0) = \psi(R_0) = \frac{3p_1-1}{2}\sqrt{\frac{3p_1+1}{2p_1}}$. As a consequence, (4.2) and (4.3) hold for all $R \in (R_1, R_2)$, where $R_1 = \psi^{-1}(\|g\|_0)$ and $R_2 = \phi^{-1}(p_2 - 1)$ (actually $1 < R_1 < R_2 < \sqrt{3}$).

(b) As for the first inequality in (4.2), it suffices that it holds for $(t,x) \in [0,1] \times [0, +\infty)$.

Our main existence result is

Theorem 4.1.6 *Under Assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, Equation (4.1) has at least one solution $x \in \mathcal{C}([0,1])$ such that $x(t) \ge 1$, for $0 \le t \le 1$.*

Proof. Consider the Banach space $E = \mathcal{C}([0,1], \mathbb{R})$ with the sup-norm $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$. Let the cone

$$\mathcal{K} = \{x \in E : \; x(t) \geq 1\}$$

and the set

$$\mathcal{K}_R = \mathcal{K} \cap \mathcal{B}(1, R) = \{x \in \mathcal{K} : \; \|x - 1\|_0 < R\},$$

where R is defined in $(\mathcal{H}_3)$. In view of Proposition 3.4.7, we introduce the operators $T, F : \overline{\mathcal{K}}_R \to E$ by

$$(Tx)(t) = x(t) - p(t)x^3(t)$$

and

$$(Fx)(t) = x(t) + g(t, x(t)),$$

respectively, for $t \in [0,1]$. Then Equation (4.1) is equivalent to the abstract equation $x = Tx + Fx$.

Step 1. (a) T and F clearly map $\overline{\mathcal{K}}_R$ into E. Moreover

$$\|Tx - Ty\|_0 \geq (3p_1 - 1)\|x - y\|_0, \; \forall\, x, y \in \overline{\mathcal{K}}_R,$$

that is $T : \overline{\mathcal{K}}_R \to E$ is expansive with constant $h = 3p_1 - 1 > 1$.

(b) If $x \in \overline{\mathcal{K}}_R$, then $\|x - 1\|_0 \leq R$ and

$$\|x - Fx\|_0 \leq \sup_{0\leq t\leq 1;\, 1\leq u\leq 1+R} g(t, u) < +\infty, \tag{4.5}$$

which implies that $(I - F)(\overline{\mathcal{K}}_R)$ is uniformly bounded. $(\mathcal{H}_1)$ further implies that $(I - F)(\overline{\mathcal{K}}_R)$ is equicontinuous in E. By Arzéla-Ascoli Lemma, $(I - F)$ maps bounded sets of $\overline{\mathcal{K}}_R$ into relatively compact sets. Since g is continuous, then so is $(I - F)$. Hence $I - F : \overline{\mathcal{K}}_R \to E$ is completely continuous, i.e., is a 0-set contraction.

(c) By (4.3), for all $x \in \partial\mathcal{K}_R$ and $t \in [0,1]$, we have

$$\begin{aligned} |x - Fx(t) - T\theta(t)| &= | - g(t, x(t)) + p(t) - 1| \\ &\leq p_1 R^3 - R + p_2 - 1 \\ &\leq (3p_1 - 1)R = hR, \end{aligned}$$

i.e.,

$$\|x - Fx + T\theta\|_0 \leq h\|x - \theta\|_0, \quad \forall\, x \in \partial\mathcal{K}_R.$$

Step 2. We claim that

$$(I - F)(\overline{\mathcal{K}}_R) \subset T(\overline{\mathcal{K}}_R). \tag{4.6}$$

Let $y \in (I - F)(\overline{\mathcal{K}}_R)$ and $x \in \overline{\mathcal{K}}_R$ be such that $y = (I - F)x$.

(a) First we claim that

$$\overline{\mathcal{K}}_R \subset y + (I - T)(\overline{\mathcal{K}}_R). \tag{4.7}$$

Let $u \in \overline{\mathcal{K}}_R$ and

$$v(t) = \sqrt[3]{\frac{u(t) + g(t, x(t))}{p(t)}}, \quad t \in [0, 1].$$

Using Assumptions $(\mathcal{H}_2) - (\mathcal{H}_3)$, for all $t \in [0, 1]$, we obtain the estimates

$$1 \leq \sqrt[3]{\frac{1 + g(t, x(t))}{p(t)}} \leq v(t) \leq \sqrt[3]{\frac{1 + R + g(t, x(t))}{p(t)}} \leq \sqrt[3]{\frac{p_1 R^3 + 1}{p(t)}} \leq R + 1.$$

Thus, $v \in \overline{\mathcal{K}}_R$ and

$$u(t) = -g(t, x(t)) + p(t)v^3(t), \quad t \in [0, 1].$$

Since $y = x - Fx = -g(\cdot, x(\cdot))$, then $u = y + (I - T)(v)$ with $v \in \overline{\mathcal{K}}_R$, that is $u \in y + (I - T)(\overline{\mathcal{K}}_R)$, proving (4.7).

(b) To show (4.6), notice that the mapping $y + (I - T) : \overline{\mathcal{K}}_R \to E$ is $3p_1$-expansive. Owing to Lemma 1.3.23 with $D = \overline{\mathcal{K}}_R$ and using (4.7), we conclude that $y + (I - T)$ has a unique fixed point, i.e., there exists $w \in \overline{\mathcal{K}}_R$ such that

$$y + (I - T)(w) = w \iff y = T(w),$$

that is $y \in T(\overline{\mathcal{K}}_R)$, proving (4.6). Finally, assume that $Tx + Fx \neq x$ on $\partial\mathcal{K}_R$, otherwise we are done. Letting $U = \mathcal{K}_R$ and $\Omega = \overline{\mathcal{K}}_R$ in Proposition 3.4.7, we obtain

$$i_*\,(T + F, \mathcal{K}_R, \mathcal{K}) = 1.$$

By the existence property of the index, the mapping $T + F$ has at least one positive fixed point x in $\overline{\mathcal{K}}_R$, solution of Equation (4.1).

■

Application 2

Consider the nonlinear integral equation

$$x(t) = \int_0^{+\infty} G(t,s)\, f(s, x(s))\, ds, \quad t \geq 0, \tag{4.8}$$

where $f, G \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\lim\limits_{t \to +\infty} G(t,s) = \ell$, for all positive s. Suppose that the following conditions hold:

$(\mathcal{H}_1)$ $\quad \exists\, p > 0,\ p \neq 1,\ 0 \leq f(t,x) \leq a(t) + b(t)x^p,\ \forall\, (t,x) \in [0,+\infty) \times [0,+\infty),$

where the coefficients $a, b \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$.

$(\mathcal{H}_2)$ $\quad$ Assume that

$$\begin{cases} M_1 := \sup\limits_{t \in [0,+\infty)} \int_0^{+\infty} G(t,s) a(s)\, ds < \infty \\ M_2 := \sup\limits_{t \in [0,+\infty)} \int_0^{+\infty} G(t,s) b(s)\, ds < \infty, \end{cases}$$

and there exist $\varepsilon \in (0,1)$ and $R > \frac{1+\varepsilon}{2}$ such that

$$M_1 + M_2 R^p < \frac{1+\varepsilon}{2}.$$

Remark 4.1.7 *As example, the values* $M_1 = \frac{1}{20}$*,* $M_2 = \frac{1}{10}$*,* $p \in \mathbb{R}$*,* $\varepsilon = \frac{1}{2}$*, and* $R = 1$ *validate the inequality in Assumption* $(\mathcal{H}_2)$*.*

Theorem 4.1.8 *Under Assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*, Equation (4.8) has at least one positive solution* $x \in \mathcal{C}([0,+\infty), \mathbb{R})$ *such that* $0 < x(t) \leq R,\ \forall\, t \geq 0$*.*

Proof. Consider the Banach space

$$E = \left\{ x \in \mathcal{C}([0,+\infty), \mathbb{R}) : \lim_{t \to +\infty} x(t) \text{ exists} \right\}$$

with norm

$$\|x\| = \sup_{t \in [0,+\infty)} |x(t)|$$

and the positive cone

$$\mathcal{P} = \{x \in E : \ x(t) \geq 0,\ t \geq 0\}.$$

Let $R_1 = \dfrac{\varepsilon R + M_1 + M_2 R^p}{1+\varepsilon}$ and let $\mathcal{B}_R = \mathcal{B}(0, R)$ denote the open ball centered at the origin with radius R. Consider the open sets:

$$\begin{aligned} U &= \mathcal{B}_R \cap \{x \in E : \; x(t) \geq \frac{1+\varepsilon}{2}, \; \forall\, t \in J\}, \\ \Omega &= \mathcal{B}_{R_1} \cap \mathcal{P}, \end{aligned}$$

for some compact sub-interval $J \subset [0, +\infty)$. Since $R < \frac{1+\varepsilon}{2}$, then $U \neq \emptyset$. On E, define the operators

$$\begin{aligned} Tx(t) &= (1+\varepsilon)x(t), \\ Fx(t) &= (1-\varepsilon)x(t) - \int_0^{+\infty} G(t,s) f(s, x(s)) ds. \end{aligned}$$

Then Equation (4.8) is equivalent to the operator equation $x = Tx + Fx$. Next, we check that all assumptions of Corollary 3.4.8 are satisfied. First we have $T : \Omega \to E$ and

$$\|Tx - Ty\| = (1+\varepsilon)\|x - y\|,$$

for all $x, y \in \Omega$, i.e., $T : \Omega \to E$ is an expansive operator with a constant $h = 1 + \varepsilon$.

1. **Step 1.** We have $I - F : \overline{U} \to E$ is continuous, bounded mapping and for $x \in \overline{U}$,

$$\begin{aligned} \int_0^{+\infty} G(t,s)|f(s,x(s))| ds &\leq \int_0^{+\infty} G(t,s)(a(s) + b(s)s(s)) ds \\ &\leq M_1 + M_2 R^p < \infty. \end{aligned}$$

Hence, by the properties of the kernel G, Lebesgue's dominated convergence theorem yields

$$\begin{aligned} &\left| \int_0^{+\infty} G(t_1,s) f(s,x(s))\, ds - \int_0^{+\infty} G(t_2,s) f(s,x(s))\, ds \right| \\ \leq\; &\int_0^{\infty} |G(t_1,s) - G(t_2,s)| f(s,x(s))\, ds, \end{aligned}$$

which tends to 0, uniformly in $x \in \mathcal{B}_R$, as $|t_1 - t_2| \to 0$. Moreover

$$\begin{aligned} &\lim_{t\to+\infty} \left| \int_0^{+\infty} G(t,s) f(s,x(s))\, ds - \lim_{y\to+\infty} \int_0^{+\infty} G(y,s) f(s,x(s))\, ds \right| \\ = \;&\lim_{t\to+\infty} \left| \int_0^{+\infty} G(t,s) f(s,x(s))\, ds - l \right| = 0. \end{aligned}$$

As a consequence, Corduneanu's compactness criterion Lemma 1.2.34 assures that for all $t \in [0, +\infty)$ and every bounded subset $B \subset \overline{U}$, the set $\left\{t \mapsto \int_0^{+\infty} G(t,s)f(s,x(s))ds,\ x \in B\right\}$ is relatively compact. Furthermore, the operator $I - F$ is written as sum of a ε-contraction and a completely continuous mapping. Thus, $I - F : \overline{U} \to E$ is a ε-set contraction.

2. **Step 2.** Let $y \in \mathcal{B}_R$ be arbitrarily chosen. For $t \geq 0$, take

$$z(t) = \frac{\varepsilon y + \int_0^{+\infty} G(t,s)f(s,y(s))ds,}{1+\varepsilon}.$$

Then

$$0 \leq z(t) \leq \frac{\varepsilon R + M_1 + M_2 R^p}{1+\varepsilon} = R_1,$$

i.e., $z \in \Omega$ and

$$\varepsilon y + \int_0^{+\infty} G(t,s)f(s,y(s))\,ds = (1+\varepsilon)z(t) = Tz(t), \quad t \geq 0.$$

Therefore $(I - F)(\overline{U}) \subset T(\Omega)$.

3. **Step 3.** Assume that there exist some $x_0 \in \partial U$ and $\lambda_0 \geq 1$ such that $\lambda_0 x_0 \in \Omega$ and

$$x_0(t) - Fx_0(t) = T(\lambda_0 x_0(t)), \quad t \geq 0.$$

Then

$$\varepsilon x_0(t) + \int_0^{+\infty} G(t,s)f(s,x_0(s))ds = \lambda_0(1+\varepsilon)x_0(t), \quad t \geq 0.$$

Hence

$$\int_0^{+\infty} G(t,s)f(s,x_0(s))ds = (\lambda_0 + (\lambda_0 - 1)\varepsilon)x_0(t).$$

Let $t_1 \in J$ be such that

$$x_0(t_1) \geq \frac{1+\varepsilon}{2}.$$

Since $\lambda_0 \geq 1$, we have the estimates

$$\begin{aligned}\frac{1+\varepsilon}{2} \leq x_0(t_1) &\leq (\lambda_0 + (\lambda_0 - 1)\varepsilon)x_0(t_1) \\ &= \int_0^{+\infty} G(t_1,s)f(s,x_0(s))ds \\ &\leq M_1 + M_2 R^p < \frac{1+\varepsilon}{2},\end{aligned}$$

which is a contradiction. By Corollary 3.4.8, Equation (4.8) has a non trivial positive solution x in $\mathcal{C}([0,+\infty))$ such that $0 \leq x(t) \leq R$, for all $x \in [0,+\infty)$. This completes the proof of Theorem 4.1.8.

■

4.2 Expansion-Compression fixed point theorem of Leggett-Williams type for the sum of two operators

The results given in this section are obtained by Benslimane-Goergiev-Mebarki in [11].

Let Ψ and δ be nonnegative continuous functionals on $\mathcal{P}$; then, for positive real numbers a and b, we define the sets:

$$\mathcal{P}(\Psi, b) = \{x \in \mathcal{P} : \Psi(x) \leq b\},$$

and

$$\mathcal{P}(\Psi, \delta, a, b) = \{x \in \mathcal{P} : a \leq \Psi(x) \text{ and } \delta(x) \leq b\}.$$

Krasnosel'skii type compression-expansion fixed point theorems gives us fixed points localized in a conical shell of the form $\{x \in \mathcal{P} : a \leq \|x\| \leq b\}$, where $a, b \in (0, \infty)$, while with the Leggett-Williams type they are localized in a conical shell of the form $\mathcal{P}(\alpha, \beta, a, b)$, where α is a concave nonnegative functional, and β a convex nonnegative functional. The original Leggett-Williams fixed point theorem (see [56, Theorem 3.2]) discusses the existence of at least one fixed point in a conical shell of the form $\{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$, where $a, b \in (0, +\infty)$ and α is a nonnegative concave functional. Noting that this result has been widely extended in many directions, (see for example [4, 8, 38, 43, 56]).

In [5, Theorem 4.1], Anderson and al. have discussed the existence of at least one solution in $\mathcal{P}(\beta, \alpha, r, R)$ or in $\mathcal{P}(\alpha, \beta, r, R)$ for the nonlinear operational equation $Ax = x$, where A is a completely continuous nonlinear map acting in $\mathcal{P}$, α is a nonnegative continuous concave functional on $\mathcal{P}$ and β is a nonnegative continuous convex functional on $\mathcal{P}$. In this result, the authors have used techniques similar to those of Leggett-Williams that require only subsets of both boundaries to be mapped inward and outward, respectively, as well as in Krasnosel'skii's cone compression and expansion one. Moreover, conditions involving the norm in the original Leggett-Williams fixed point theorem are replaced by more general conditions on a convex functional.

In this context, the Leggett-Williams approach provides more general results than those obtained by using the Krasnosel'skii one. Noting that, in [5], the authors provided more general results than those obtained in [4, 9, 43, 44, 56, 76] for completely continuous mappings.

In this work, Benslimane-Goergiev-Mebarki use the fixed point index theory developed in [29] to generalize the main result of [5] for the sum $T + F$ where T is an expansive mapping with constant $h > 1$ and $I - F$ is a k-set contraction with $k < h$.

4.2.1 Main result

Let Ω be a subset of $\mathcal{P}$ such that $0 \in \Omega$. We consider the nonlinear equation

$$Tx + Fx = x, \tag{4.9}$$

where $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, and $I - F : \mathcal{P} \to E$ a k-set contraction with $0 \leq k < h$.

In what follows, we will establish an extension of [5, Theorem 4.1], which guarantees the existence of at least one non trivial nonnegative solution of Equation (4.9).

Theorem 4.2.1 *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β be a nonnegative continuous convex functional on $\mathcal{P}$ with $\beta(x) \leq \|x\|$ for all $x \in \mathcal{P}$. Assume that there exist nonnegative numbers a, b, c, d and $z_0 \in \mathcal{P}$ such that $\|T0\| < h\min(b, d)$ and $\alpha(T^{-1}z_0) > \max(a, c)$.*

Suppose that:

(A1) *if $x \in \mathcal{P}$ with $\beta(x) = b$, then $\alpha(Tx + x) \geq a$;*

(A2) *if $x \in \mathcal{P}$ with $\beta(x) = b$ and $\alpha(x) \geq a$, then $\beta(Tx + Fx) < b$ and $\beta(Tx + x) \leq b$;*

(A3) *if $x \in \mathcal{P}$ with $\beta(x) = b$ and $\alpha(Tx + Fx) < a$, then $\beta(Tx + Fx) < b$ and $\beta(Tx + x) \leq b$;*

($A4$) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$, *then* $\beta(Tx + x - z_0) \leq d$;

($A5$) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $\beta(x) \leq d$, *then* $\alpha(Tx + Fx) > c$ *and* $\alpha(Tx + x - z_0) \geq c$;

($A6$) *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $\beta(Tx+Fx) > d$, *then* $\alpha(Tx+Fx) > c$ *and* $\alpha(Tx+x-z_0) \geq c$.

Then,

1. *(Expansive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ *if*

   ($H1$) $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x)$ *and* $\alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha, c)$ *is bounded and*

$$t(I - F)(\mathcal{P}(\beta, b)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{4.10}$$

$$t(I - F)(\mathcal{P}(\alpha, c)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{4.11}$$

2. *(Compressive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\alpha, \beta, a, d) \cap \Omega$ *if*

   ($H2$) $c < a$, $d < b$, $\{x \in \mathcal{P} : a < \alpha(x)$ *and* $\beta(x) < d\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$, $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$, *and* $\mathcal{P}(\beta, d)$ *is bounded and*

$$t(I - F)(\mathcal{P}(\beta, d)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{4.12}$$

$$t(I - F)(\mathcal{P}(\alpha, a)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{4.13}$$

Proof. We will prove the expansion form. The proof of the compression form is nearly identical. If we list

$$U = \{x \in \mathcal{P} : \beta(x) < b\}, \tag{4.14}$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\}, \tag{4.15}$$

then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} : \; b < \beta(x) \text{ and } \alpha(x) < c\}.$$

Thus U, V and W are bounded (they are subsets of V which is bounded by condition ($H1$)), not empty (by condition ($H1$)) and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for the sum $T+F$ in $\mathcal{P}(\beta,\alpha,b,c)\cap\Omega$, it is enough for us to show that $i_*(T+F,W\cap\Omega,\mathcal{P})\neq 0$ since W is the interior of $\mathcal{P}(\beta,\alpha,b,c)$.

Claim 1. $Tx+Fx\neq x$ for all $x\in\partial U\cap\Omega$.

Let $x_0\in\partial U\cap\Omega$, then $\beta(x_0)=b$. Suppose that $x_0=Tx_0+Fx_0$, then $\beta(Tx_0+Fx_0)=b$. If $\alpha(x_0)\geq a$, then $\beta(Tx_0+Fx_0)<b$ by condition $(A2)$, and if $\alpha(x_0)<a$, then $\alpha(Tx_0+Fx_0)<a$, then $\beta(Tx_0+Fx_0)<b$ by condition $(A3)$.

This is a contradiction. Thus we have $Tx+Fx\neq x$ for all $x\in\partial U\cap\Omega$.

Claim 2. $Tx+Fx\neq x$ for all $x\in\partial V\cap\Omega$.

Let $x_1\in\partial V\cap\Omega$, then $\alpha(x_1)=c$. Suppose that $x_1=Tx_1+Fx_1$, then $\alpha(Tx_1+Fx_1)=c$. If $\beta(x_1)\leq d$, then $\alpha(Tx_1+Fx_1)>c$ by condition $(A5)$, and if $\beta(x_1)>d$, then $\beta(Tx_1+Fx_1)>d$, then $\alpha(Tx_1+Fx_1)>c$ by condition $(A6)$.

This is a contradiction. Thus we have $Tx+Fx\neq x$ for all $x\in\partial V\cap\Omega$.

Claim 3. $i_*(T+F,U\cap\Omega,\mathcal{P})=1$.

? Let $H_1:[0,1]\times\overline{U}\to E$ be defined by

$$H_1(t,x)=tFx+(1-t)\,x.$$

Clearly H_1 is uniformly continuous in t with respect to $x\in\overline{U}$ and $(I-H_1)$ is continuous, and from (4.10) we easily see that $(I-H_1([0,1]\times\overline{U}))\subset T(\Omega)$. Moreover $(I-H_1(t,.)):\overline{U}\to E$ is a k-set contraction for all $t\in[0,1]$ and $Tx+H_1(t,x)\neq x$ for all $(t,x)\in[0,1]\times\partial U\cap\Omega$. Otherwise, there would exists $(t_2,x_2)\in[0,1]\times\partial U\cap\Omega$ such that $Tx_2+H_1(t_2,x_2)=x_2$. Since $x_2\in\partial U$, $\beta(x_2)=b$. Either $\alpha(Tx_2+Fx_2)<a$ or $\alpha(Tx_2+Fx_2)\geq a$.

Case (1): If $\alpha(Tx_2+Fx_2)<a$, the convexity of β and the condition $(A3)$ lead

$$\begin{aligned} b=\beta(x_2) &= \beta\left(Tx_2+H_1(t_2,x_2)\right)\\ &= \beta\left(Tx_2+t_2Fx_2+(1-t_2)x_2\right)\\ &\leq t_2\beta\left(Tx_2+Fx_2\right)+(1-t_2)\,\beta(Tx_2+x_2)\\ &< b, \end{aligned}$$

which is a contradiction.

Case (2): If $\alpha(Tx_2 + Fx_2) \geq a$, from the concavity of α and the condition $(A1)$, we obtain $\alpha(x_2) \geq a$. Indeed,

$$\begin{aligned}\alpha(x_2) &= \alpha\left(Tx_2 + H_1(t_2, x_2)\right) \\ &\geq t_2\alpha\left(Tx_2 + Fx_2\right) + (1-t_2)\,\alpha(Tx_2 + x_2) \\ &\geq a,\end{aligned}$$

and thus by condition $(A2)$, we have $\beta(Tx_2 + Fx_2) < b$, which is the same contradiction we arrived at in the previous case.

Being $T^{-1}0 \in U$ (we have $h\beta(T^{-1}0) \leq h\|T^{-1}0\| \leq \|T0\| < hb$), the homotopy invariance property (iii) and the normality property (i) of the fixed point index i_* lead

$$i_*(T+F, U\cap\Omega, \mathcal{P}) = i_*(T+I, U\cap\Omega, \mathcal{P}) = 1.$$

Claim 4. $i(T+F, W\cap\Omega, \mathcal{P}) = -1$.

Let $H_2 : [0,1]\times\overline{V} \to E$ be defined by

$$H_2(t,x) = tFx + (1-t)(x - z_0).$$

Clearly H_2 is uniformly continuous in t with respect to $x \in \overline{V}$ and $(I - H_2)$ is continuous, and from (4.11) we easily see that $(I - H_2([0,1]\times\overline{V})) \subset T(\Omega)$. Moreover $I - H_2(t,.) : \overline{V} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_2(t,x) \neq x$ for all $(t,x) \in [0,1]\times\partial V\cap\Omega$. Otherwise, there would exists $(t_3, x_3) \in [0,1]\times\partial V\cap\Omega$ such that $H_2(t_3,x_3) = x_3$. Since $x_3 \in \partial V$ we have that $\alpha(x_3) = c$. Either $\beta(Tx_3 + Fx_3) \leq d$ or $\beta(Tx_3 + Fx_3) > d$.

Case (1): If $\beta(Tx_3 + Fx_3) > d$. the concavity of α and the condition $(A6)$ lead

$$\begin{aligned}c = \alpha(x_3) &= \alpha(Tx_3 + H_2(t_3, x_3)) \\ &= \alpha(Tx_3 + t_3Fx_3 + (1-t_3)(x_3 - z_0)) \\ &\geq t_3\alpha(Tx_3 + Fx_3) + t_3\alpha(Tx_3 + x_3 - z_0) \\ &> c.\end{aligned}$$

This is a contradiction.

Case (2): If $\beta(Tx_3 + Fx_3) \leq d$, from the convexity of β and the condition $(A4)$, we obtain $\beta(x_3) \leq d$. Indeed,

$$\begin{aligned}
\beta(x_3) &= \beta(Tx_3 + H_2(t_3, x_3)) \\
&\leq t_3\beta(Tx_3 + Fx_3) + (1 - t_3)\beta(Tx_3 + x_3 - z_0) \\
&\leq d,
\end{aligned}$$

and thus by condition $(A5)$, we have $\alpha(Tx_3 + Fx_3) > c$, which is the same contradiction we arrived at in the previous case.

The homotopy invariance property (iii) of the fixed index i_* yields

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + I - z_0, V \cap \Omega, \mathcal{P}),$$

and by the solvability property (iv) of the index i_* (since $T^{-1}z_0 \notin V$ the index cannot be nonzero) we have

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + I - z_0, V \cap \Omega, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and $T + F$ has no fixed point in $\overline{V} - (U \cup W)$ (by claims 1 and 2), from the additivity property (ii) of the index i_*, we deduce

$$i_*(T + F, V \cap \Omega, \mathcal{P}) = i_*(T + F, U \cap \Omega, \mathcal{P}) + i_*(T + F, W \cap \Omega, \mathcal{P}).$$

Consequently, we have

$$i(T + F, W \cap \Omega, \mathcal{P}) = -1,$$

and thus by the solvability property (iv) of the fixed point index i_*, the sum $T + F$ has a fixed point $x^* \in W \subset \mathcal{P}(\beta, \alpha, b, c) \cap \Omega$. ■

4.2.2 Applications

Application 1

In this subsection, we will investigate the three-point BVP

$$\begin{aligned}
&y'' + f(t, y) = 0, \quad t \in (0, 1), \\
&y(0) = ky(\eta), \quad y(1) = 0,
\end{aligned} \tag{4.16}$$

where

(B1) $f \in \mathcal{C}([0,1]\times\mathbb{R}_+)$, $0 < \widetilde{A} \le f(t,u) \le A$, $t \in [0,1]$, $u \in [0,\infty)$, for some positive constants $A \ge \widetilde{A}$.

(B2) $\eta \in (0,1)$, $k > 0$, $k(1-\eta) < 1$, $B = \frac{1+k\eta}{1-k(1-\eta)}$ $\epsilon \in (1,2)$, $c = 0$ and there exist $a, b, d, z_0 > 0$ so that $z_0 = a$ and

$$a < d < b, \quad 2z_0 < \epsilon d, \quad (\epsilon-1)b + 2z_0 < \tfrac{d}{2},$$
$$(\epsilon-1)b + \epsilon AB < d, \quad a < \tfrac{\epsilon AB + 2z_0}{\epsilon} \le d.$$

After the proof of the main result in this subsection, we will give an example for a function f and constants A, $\widetilde{A}$, B, η, k, a, b, d, ϵ, z_0 which satisfy $(B1)$ and $(B2)$. We will investigate the BVP (4.16) for existence of at least one non trivial nonnegative solution. Our main result is as follows.

Theorem 4.2.2 *Suppose* $(B1)$ *and* $(B2)$. *Then the BVP* (4.16) *has at least one non trivial nonnegative solution* y.

To prove our main result, we will use Theorem 4.2.1.

In [84] the BVP (4.16) is investigated when the function f satisfies the following conditions

(B3) $f(t,u)$ is nonnegative and continuous on $(0,1)\times[0,\infty)$, $f(t,u)$ is monotone increasing on u for fixed $t \in (0,1)$, there exists $q \in (0,1)$ such that

$$f(t,ru) > r^q f(t,u), \quad 0 < r < 1, \quad (t,u) \in (0,1)\times[0,\infty),$$

and in [84] it is proved that the BVP (4.16) has a unique solution $u \in \mathcal{C}([0,1])\bigcap\mathcal{C}^2((0,1))$. We will note that there are cases for the function f for which we can apply Theorem 4.2.2 and we can not apply Theorem 4.1 in [84] and the conversely. For example, if $f(t,u) = 1 + \frac{1}{1+u}$, $t, u \in [0,\infty)$, then it is bounded below and above and we can apply Theorem 4.2.2. At the same time, it is decreasing with respect to u for $t, u \in [0,\infty)$ and we can not apply Theorem 4.1 in [84]. If $f(t,u) = \sum_{j=1}^m a_j(t)u^{\alpha_j}$, where $a_j \in \mathcal{C}([0,\infty))$ are nonnegative functions and $\alpha_j \in (0,1)$, $j \in \{1,\ldots,m\}$, as it is shown in [84], it satisfies $(B3)$. On the other hand, it is unbounded above and we can not apply Theorem 4.2.2. Thus, our result Theorem 4.2.2 and Theorem 4.1 in [84] are complementary to one another.

Proof. of Theorem 4.2.2

Set

$$H(t,s)=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1,\end{cases}$$

and

$$G(t,s)=H(t,s)+\frac{k(1-t)}{1-k(1-\eta)}H(\eta,s),\quad t,s\in[0,1].$$

Note that $0\le H(t,s)\le 1$, $t,s\in[0,1]$. Hence,

$$0\le G(t,s)\le 1+\frac{k}{1-k(1-\eta)}=\frac{1-k+k\eta+k}{1-k(1-\eta)}=\frac{1+k\eta}{1-k(1-\eta)}=B,$$

$t,s\in[0,1]$. Moreover, for $t,s\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]$, we have

$$H(t,s)\ge\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)$$

and

$$G(t,s)\ge H(t,s)\ge\frac{\eta}{3}\left(1-\frac{\eta}{2}\right).$$

Next,

$$H_t(t,s)=\begin{cases} -s, & 0\le s\le t\le 1,\\ 1-s, & 0\le t\le s\le 1.\end{cases}$$

Hence, $|H_t(t,s)|\le 1$, $t,s\in[0,1]$, and

$$\begin{aligned}|G_t(t,s)| &= \left|H_t(t,s)-\frac{k}{1-k(1-\eta)}H(\eta,s)\right|\\ &\le |H_t(t,s)|+\frac{k}{1-k(1-\eta)}H(\eta,s)\\ &\le 1+\frac{k}{1-k(1-\eta)}=\frac{1+k\eta}{1-k(1-\eta)}=B,\quad t,s\in[0,1].\end{aligned}$$

Let $E=\mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|y\|=\max_{t\in[0,1]}|y(t)|.$$

On E, define

$$\alpha(y)=\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]}|y(t)|+z_0,\quad \beta(y)=\max_{t\in[0,1]}|y(t)|.$$

In [84] it is proved that the solution of the BVP (4.16) can be expressed in the following form

$$y(t)=\int_0^1 G(t,s)f(s,y(s))ds,\quad t\in[0,1].$$

Set

$$k_1=\frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right\}}{d\epsilon}.$$

Define

$$\begin{aligned}\mathcal{P} &= \{y\in E: y(t)\geq 0,\quad t\in[0,1],\quad \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t)\geq k_1\max_{t\in[0,1]} y(t)\},\\ \Omega &= \{y\in\mathcal{P}:\|y\|\leq\frac{2z_0+\epsilon AB}{\epsilon}\}.\end{aligned}$$

Note that $0\in\Omega$ and $\Omega\subset\mathcal{P}$. For $y\in\mathcal{P}$, define the operators

$$\begin{aligned}Ty(t) &= -\epsilon y(t)+2z_0,\\ Fy(t) &= y(t)-2z_0+\epsilon\int_0^1 G(t,s)f(s,y(s))ds,\quad t\in[0,1].\end{aligned}$$

Note that if $y\in\mathcal{P}$ is a fixed point of the operator $T+F$, then it is a solution to the BVP (4.16). Next, if $y\in\mathcal{P}$ and $\beta(y)\leq b$, we have

$$\begin{aligned}|Ty(t)+y(t)| &\leq (\epsilon-1)y(t)+2z_0\\ &\leq (\epsilon-1)b+2z_0\\ &< \frac{d}{2},\quad t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}|Ty(t)+Fy(t)| &= \left|-(\epsilon-1)y(t)+\epsilon\int_0^1 G(t,s)f(s,y(s))ds\right|\\ &\leq (\epsilon-1)y(t)+\epsilon\int_0^1 G(t,s)f(s,y(s))ds\\ &\leq (\epsilon-1)b+\epsilon A\int_0^1 G(t,s)ds\\ &\leq (\epsilon-1)b+\epsilon AB\\ &< d.\end{aligned}$$

Therefore, if $y\in\mathcal{P}$ and $\beta(y)\leq b$, we have

$$\beta(Ty+y)<d \tag{4.17}$$

and

$$\beta(Ty + Fy) < d. \tag{4.18}$$

For $y, z \in \mathcal{P}$, we have

$$|Ty(t) - Tz(t)| = \epsilon|y(t) - z(t)|, \quad t \in [0,1].$$

Hence,

$$\|Ty - Tz\| = \epsilon\|y - z\|.$$

Thus, $T : \mathcal{P} \to E$ is an expansive operator with constant $h = \epsilon$.

Let now, $y \in \mathcal{P}$. Then

$$\begin{aligned}
|(I - F)y(t))| &= \epsilon\left|\int_0^1 G(t,s)f(s,y(s))ds\right| \\
&\leq \epsilon A \int_0^1 G(t,s)ds \\
&\leq \epsilon AB, \quad t \in [0,1],
\end{aligned}$$

whereupon

$$\|(I - F)y\| \leq \epsilon AB$$

and $I - F : \mathcal{P} \to E$ is uniformly bounded. Moreover,

$$\begin{aligned}
\left|\frac{d}{dt}(I - F)y(t)\right| &= \left|\int_0^1 G_t(t,s)f(s,y(s))ds\right| \\
&\leq \int_0^1 |G_t(t,s)|f(s,y(s))ds \\
&\leq AB, \quad t \in [0,1].
\end{aligned}$$

Consequently, $I - F : \mathcal{P} \to E$ is completely continuous. Then $I - F : \mathcal{P} \to E$ is a 0-set contraction.

Note that

$$\|T0\| = 2z_0 < \epsilon \min\{b, d\}.$$

For $y \in E$, we have

$$T^{-1}y = -\frac{y - 2z_0}{\epsilon}.$$

Hence,

$$\alpha\left(T^{-1}z_0\right) = \alpha\left(\frac{z_0}{\epsilon}\right) = \frac{z_0}{\epsilon} + z_0 > \max\{a, c\}.$$

Suppose that $y \in \mathcal{P}$ with $\beta(y) = b$. Then

$$\alpha(Ty + y) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} |Ty(t) + y(t)| + z_0 \geq z_0 = a.$$

Consequently $(A1)$ holds.

Now, we take $y \in \mathcal{P}$ with $\beta(y) = b$, $\alpha(y) \geq a$. Then, using $d < b$, (4.17) and (4.18), we obtain

$$\beta(Ty + y) < b \quad \text{and} \quad \beta(Ty + Fy) < b.$$

Consequently $(A2)$ holds.

Observe that, if $y \in \mathcal{P}$, $\beta(y) = b$ and $\alpha(Ty + Fy) < a$, using $d < b$ and (4.17), (4.18), we find

$$\beta(Ty + Fy) < b \quad \text{and} \quad \beta(Ty + y) < b.$$

Thus, $(A3)$ holds.

Since $c = 0$ and $\alpha(y) > 0$ for any $y \in \mathcal{P}$, the case $\alpha(y) = c$ is impossible.

Let now, $a_1 \in \left(a, \frac{\epsilon AB + z_0}{\epsilon}\right)$ be arbitrarily chosen. Then

$$\alpha(a_1) = a_1 + z_0 > a$$

and

$$\beta(a_1) = a_1 < \frac{\epsilon AB + 2z_0}{\epsilon} \leq d.$$

Therefore

$$\{y \in \mathcal{P} : a < \alpha(y) \quad \text{and} \quad \beta(y) < d\} \cap \Omega \neq \emptyset.$$

Let $y \in \mathcal{P}(\alpha, a)$. Then $y \in \mathcal{P}$ and $\alpha(y) \leq a$. Hence,

$$a \geq \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) + z_0 = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) + a.$$

Therefore $\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0$ and using the definition of the cone $\mathcal{P}$, we find

$$\beta(y) = \max_{t\in[0,1]} y(t) \leq \frac{1}{k_1} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0 \leq d.$$

Thus, $y \in \mathcal{P}(\beta, d)$ and $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$.

Since $0 \in \mathcal{P}(\alpha, a)$, we have $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$.

Note that $\mathcal{P}(\beta, d)$ is bounded.

Let $\lambda \in [0, 1]$ is fixed and $u \in \mathcal{P}(\alpha, a)$ is arbitrarily chosen. Then $\beta(u) \leq d < b$. Set

$$v(t) = \frac{\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon}, \quad t \in [0,1].$$

We have that $v(t) \geq 0$, $t \in [0, 1]$, and

$$v(t) \leq \frac{\epsilon AB + z_0}{\epsilon} \leq d, \quad t \in [0,1],$$

and

$$\|v\| \leq \frac{\epsilon AB + z_0}{\epsilon} \leq d.$$

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} v(t) &\geq \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon \left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{\epsilon} \\
&= \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{d\epsilon} d \\
&\geq k_1 \max_{t\in[0,1]} v(t).
\end{aligned}$$

Thus, $v \in \Omega$. Next,

$$\begin{aligned}
\lambda(I-F)u(t) + (1-\lambda)z_0 &= 2\lambda z_0 - \lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + z_0 - \lambda z_0 \\
&= -\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1+\lambda)z_0 \\
&= -\epsilon \frac{\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} + 2z_0 \\
&= Tv(t), \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\alpha, a)) + (1-\lambda)z_0 \subset T(\Omega).$$

Let $\lambda \in [0,1]$ be fixed and $u \in \mathcal{P}(\beta, d)$ be arbitrarily chosen. Take

$$w(t) = \frac{2(1-\lambda)z_0 + \lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds}{\epsilon}, \quad t \in [0,1].$$

We have $v(t) \geq 0$, $t \in [0,1]$, and

$$w(t) \leq \frac{\epsilon AB + 2z_0}{\epsilon} \leq d, \quad t \in [0,1].$$

Moreover,

$$\begin{aligned}
\min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} w(t) &\geq \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon \left(\frac{\eta}{2} - \frac{\eta}{3}\right) \frac{\eta}{3} \left(1 - \frac{\eta}{2}\right) \widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left\{\epsilon \frac{\eta^2}{18} \left(1 - \frac{\eta}{2}\right) \widetilde{A}, z_0\right\}}{\epsilon} \\
&= \frac{\min\left\{\epsilon \frac{\eta^2}{18} \left(1 - \frac{\eta}{2}\right) \widetilde{A}, z_0\right\}}{d\epsilon} d \\
&\geq k_1 \max_{t \in [0,1]} w(t).
\end{aligned}$$

Therefore $w \in \Omega$. Also,

$$\begin{aligned}
\lambda(I-F)u(t) &= \lambda \left(2z_0 - \epsilon \int_0^1 G(t,s)f(s,u(s))ds\right) \\
&= -\epsilon \frac{\epsilon \int_0^1 G(t,s)f(s,u(s))ds + 2(1-\lambda)z_0}{\epsilon} + 2z_0 \\
&= -\epsilon w(t) + 2z_0 \\
&= Tw(t), \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\beta,d)) \subset T(\Omega).$$

By Theorem 4.2.1, it follows that the BVP (4.16) has at least one solution in $\{y \in \mathcal{P} : a < \alpha(y) \quad \text{and} \quad \beta(y) < d\} \cap \Omega \subset P(\alpha, \beta, a, d) \cap \Omega$. ∎

An Example

Consider the BVP

$$\begin{aligned}
y'' + \tfrac{1}{300(1+t^2)(1+y)} + \tfrac{1}{300} &= 0, \quad t \in (0,1), \\
y(0) = y\left(\tfrac{1}{2}\right), \quad y(1) &= 0.
\end{aligned} \tag{4.19}$$

Here

$$f(t,y)=\frac{1}{300(1+t^2)(1+y)}+\frac{1}{300},\quad t\in(0,1),\quad y\in[0,\infty),\quad k=1,\quad \eta=\frac{1}{2}.$$

Note that for the function f we can not apply Theorem 4.1 in [84] because it is a decreasing function with respect to y for $t,y\in[0,\infty)$. Take the constants

$$\begin{aligned}&\epsilon=\tfrac{41}{40},\quad B=3,\quad A=\tfrac{1}{123},\quad \widetilde{A}=\tfrac{1}{300},\quad b=1,\quad d=\tfrac{1}{2},\\ &z_0=\tfrac{1}{400},\quad a=\tfrac{1}{400},\quad q=\tfrac{1}{1000}.\end{aligned}$$

We have

$$\begin{aligned}a<d<b,\quad 2z_0=2a=\frac{1}{200}&<\frac{41}{80}=\epsilon d,\\ (\epsilon-1)b+2z_0=\frac{1}{40}+\frac{1}{200}&=\frac{3}{100}<\frac{1}{4}=\frac{d}{2},\\ (\epsilon-1)b+\epsilon AB=\frac{1}{40}+\frac{41}{40}\cdot\frac{3}{123}&=\frac{1}{40}+\frac{1}{40}=\frac{1}{20}<\frac{1}{2}=d,\\ \frac{1}{400}=a<\frac{\epsilon AB+2z_0}{\epsilon}&=\frac{40}{41}\cdot\left(\frac{41}{40}\cdot\frac{3}{123}+\frac{1}{200}\right)<\frac{1}{2}=d.\end{aligned}$$

Thus, $(B2)$ holds. Next, $f\in\mathcal{C}([0,1]\times\mathbb{R}_+)$ and

$$\frac{1}{300}\leq f(t,y)=\frac{1}{300(1+t^2)(1+y)}+\frac{1}{300}\leq\frac{1}{150}\leq\frac{1}{123}=A,$$

i.e., $(B1)$ holds. By Theorem 4.2.1, it follows that the BVP (4.19) has at least one nonnegative solution.

Application 2

In this part, we will investigate the following BVP

$$\begin{aligned}x''(t)+g(x(t))&=0,\quad t\in(0,1),\\ x(0)=0&=x'(1),\end{aligned}\tag{4.20}$$

where

(C1) $g\in\mathcal{C}(\mathbb{R}^+)$, $0<\widetilde{A}_1\leq g(x)\leq A_1$, $x\in[0,\infty)$, for some positive constants $A_1\geq\widetilde{A}_1$.

(C2) The nonnegative constants z_1, a_1, b_1, c_1, d_1, ϵ_1 satisfy

$$\epsilon_1\in(1,2),\quad(\epsilon_1-1)b_1+2z_1<\frac{d_1}{2},\quad(\epsilon_1-1)b_1+\epsilon_1A_1<d_1,$$

$$c_1 = 0, \quad 2z_1 < \epsilon_1 \min\{b_1, d_1\}, \quad \frac{z_1}{\epsilon_1} + z_1 > \max\{a_1, c_1\}, \quad z_1 = a_1,$$
$$a_1 < d_1 < b_1, \quad a_1 < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} \leq d_1.$$

Our main result in this subsection is as follows.

Theorem 4.2.3 *Suppose* $(C1)$ *and* $(C2)$. *Then the BVP* (4.20) *has at least one non trivial nonnegative solution.*

The BVP (4.20) is investigated in [5] under the following conditions

(C1.1) $\tau \in (0,1)$ is fixed, b and c are positive constants with $3b \leq c$, $g : [0,\infty) \to [0,\infty)$ is a continuous function such that

1. $g(w) > \frac{c}{\tau(1-\tau)}, \quad w \in \left[c, \frac{c}{\tau}\right]$,
2. g is decreasing on $[a, b\tau]$ with $g(b\tau) \geq g(w)$ for $w \in [b\tau, b]$.
3. $\int_0^\tau sg(s)ds \leq \frac{2b - g(b\tau)(1-\tau^2)}{2}$,

and it is proved that the BVP (4.20) has at least one nonnegative solution. Note that there are cases for the function g for which we can apply Theorem 4.2.3 and we can not apply Theorem 5.1 in [5] and the conversely. For instance, if $g(x) = \frac{x}{1+x} + 1$, $x \in [0,\infty)$, then it is bounded above and below and we can apply Theorem 4.2.3. On the other hand, g is an increasing function on $[0,\infty)$ and we can not apply Theorem 5.1 in [5]. If $g(x) = \frac{1}{\sqrt{x}} + e^{x-2}$, $x \in (0,\infty)$, as it is shown in [5], we can apply for it Theorem 5.1 in [5]. Since it is unbounded above, we can not apply Theorem 4.2.3. Therefore the main result of [11] Theorem 4.2.1 and the main result Theorem 4.1 in [5] are complementary to one another.

After the proof of Theorem 4.2.3, we will give an example for a function g and constants A_1, $\widetilde{A}_1$, z_1, a_1, b_1, c_1, d_1, ϵ_1 that satisfy $(C1)$ and $(C2)$.

Proof. of Theorem 4.2.3. Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define

$$G_1(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1] \times [0,1].$$

Note that

$$0 \leq G_1(t,s) \leq 1, \quad (t,s) \in [0,1] \times [0,1],$$

and

$$G_1(t,s) \geq \frac{1}{3}, \quad t,s \in \left[\frac{1}{3}, \frac{1}{2}\right].$$

On E, define the following functionals

$$\alpha_1(x) = \min_{t\in[0,1]} |x(t)| + z_1, \quad \beta_1(x) = \max_{t\in[0,1]} |x(t)|.$$

In [5] it is proved that the solution of the BVP (4.20) can be represented in the form

$$x(t) = \int_0^1 G_1(t,s) g(x(s)) ds, \quad t \in [0,1].$$

Set

$$k_2 = \frac{\min\left\{\frac{\epsilon_1 \widetilde{A}_1}{3}, z_1\right\}}{d_1 \epsilon_1}.$$

Define

$$\begin{aligned} \mathcal{P}_1 &= \{x \in E : x(t) \geq 0, \quad t \in [0,1], \quad \min_{t\in\left[\frac{1}{3},\frac{1}{2}\right]} x(t) \geq k_2 \max_{t\in[0,1]} x(t)\}, \\ \Omega_1 &= \left\{x \in \mathcal{P}_1 : \|x\| \leq \frac{2z_1 + \epsilon_1 A_1}{\epsilon_1}\right\}. \end{aligned}$$

Note that $0 \in \Omega_1$ and $\Omega_1 \subset \mathcal{P}_1$. For $x \in \mathcal{P}_1$, define the following operators.

$$\begin{aligned} T_1 x(t) &= -\epsilon_1 x(t) + 2z_1, \\ F_1 x(t) &= x(t) - 2z_0 + \epsilon_1 \int_0^1 G_1(t,s) g(x(s)) ds, \quad t \in [0,1]. \end{aligned}$$

Now, the proof of Theorem 4.2.3 follows similar arguments to those in the proof of ([11] Theorem4.1). ■

An example

Consider the BVP

$$\begin{aligned} x''(t) + \tfrac{x(t)}{400(1+x(t))} + \tfrac{1}{400} &= 0, \quad t \in (0,1), \\ x(0) = 0 &= x'(1). \end{aligned} \tag{4.21}$$

Here

$$g(x) = \frac{x}{400(1+x)} + \frac{1}{400}, \quad x \in [0, \infty).$$

Note that the function g is an increasing function on $[0, \infty)$ and then we cannot apply Theorem 5.1 in [5]. Take

$$\begin{aligned}
\epsilon_1 &= \frac{41}{40}, \quad A_1 = \frac{1}{123}, \quad \widetilde{A}_1 = \frac{1}{400}, \quad b_1 = 1, \quad d_1 = \frac{1}{2}, \\
z_1 &= \frac{1}{400}, \quad a_1 = \frac{1}{400}, \quad c_1 = 0.
\end{aligned}$$

Then, $\epsilon_1 > 1$ and

$$(\epsilon_1 - 1)b_1 + 2z_1 = \frac{1}{40} + \frac{1}{200} < \frac{1}{4} = \frac{d_1}{2},$$

$$(\epsilon_1 - 1)b_1 + \epsilon_1 A_1 = \frac{1}{40} + \frac{41}{40} \cdot \frac{1}{123} = \frac{1}{40} + \frac{1}{120} < \frac{1}{2} = d_1,$$

$$\epsilon_1 \min\{b_1, d_1\} = \frac{41}{40} \cdot \frac{1}{2} = \frac{41}{80} > \frac{1}{200} = 2z_1,$$

$$\frac{z_1}{\epsilon_1} + z_1 = \frac{\frac{1}{400}}{\frac{41}{40}} = \frac{1}{410} + \frac{1}{400} > \frac{1}{400} = \max\{a_1, c_1\},$$

$$a_1 < d_1 < b_1,$$

$$\begin{aligned}
a_1 = \frac{1}{400} < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} &= \frac{\frac{41}{40} \cdot \frac{1}{123} + \frac{1}{200}}{\frac{41}{40}} = \frac{\frac{1}{120} + \frac{1}{200}}{\frac{41}{40}} \\
&= \frac{\frac{1}{3} + \frac{1}{5}}{41} = \frac{8}{615} < \frac{1}{2} = d_1.
\end{aligned}$$

Thus, $(C2)$ holds. Next,

$$\frac{1}{400} \leq g(x) \leq \frac{1}{200}, \quad x \in [0, \infty).$$

So, $(C2)$ holds. Hence, applying Theorem 4.2.3, we conclude that the BVP (4.21) has at least one nonnegative solution.

Chapitre 5

Multiple nonnegative solutions for a class of fourth-order BVPs

Contents

The results of this chapter are obtained by Benslimane, Goergiev and Mebarki in [12].

5.1 Introduction

Since 1970, the interest for fourth order boundary value porblems (BVPs for short) has risen due to their important applications in pratical problems. For instance, the deformation of an elastic beam under an external force h supported at both ends is described by the linear boundary value problem

$$x^{(4)}(t) = h(t), \quad t \in (0,1),$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$

where vanishing moments at the ends of the attached beam motivate the boundary conditions (see [42] for more details). The existence of solutions for nonlinear fourth-order BVPs has gained much interest in the last years (see, e.g., [57, 58, 59, 74, 78, 81, 83]). Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary conditions as special cases.

In this work, we investigate the existence of at least two nonnegative solutions to the fourth-order nonlinear boundary value problem with integral boundary conditions:

$$\begin{aligned} x^{(4)}(t) &= w(t)f(t,x(t),x''(t)), \quad t \in (0,1), \\ x(0) &= \int_0^1 h_1(s)x(s)ds, \quad x(1) = \int_0^1 k_1(s)x(s)ds, \\ x''(0) &= \int_0^1 h_2(s)x''(s)ds, \quad x''(1) = \int_0^1 k_2(s)x''(s)ds, \end{aligned} \tag{5.1}$$

where

(H1) $w \in L^1([0,1])$ is nonnegative and may be singular at $t = 0$ and (or) $t = 1$, $f \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R})$ and satisfies the polynomial growth condition:

$$|f(t,u,v)| \leq a_1(t)|u|^{p_1} + a_2(t)|v|^{p_2} + a_3(t), \quad t \in [0,1], \quad u,v \in \mathbb{R},$$

$a_1, a_2, a_3 \in \mathcal{C}([0,1])$ are given nonnegative functions, p_1, p_2 are given nonnegative constants.

(H2) h_1, h_2, k_1, $k_2 \in L^1([0,1])$ with $m_1\nu_1 + n_1\mu_1 \neq 0$, $m_2\nu_2 + n_2\mu_2 \neq 0$, for

$$
\begin{aligned}
m_1 &= \int_0^1 sh_1(s)ds, \quad m_2 = \int_0^1 sh_2(s)ds, \\
n_1 &= 1 - \int_0^1 sk_1(s)ds, \quad n_2 = 1 - \int_0^1 sk_2(s)ds, \\
\mu_1 &= 1 - \int_0^1 h_1(s)ds, \quad \mu_2 = 1 - \int_0^1 h_2(s)ds, \\
\nu_1 &= 1 - \int_0^1 k_1(s)ds, \quad \nu_2 = 1 - \int_0^1 k_2(s)ds.
\end{aligned}
$$

In 2003 and 2004, the authors of [57, 86] studied the existence of solutions of Problem (5.1) for $h_1 = h_2 = k_1 = k_2 = 0$, by using the Krasnels'kii's fixed point theorem and fixed point index theory on cones of Banach spaces respectively.

By using the Krasnosel'skii fixed point theorem of cone expansion and compression, in [78] is proved the existence of at least two positive solution of BVP (5.1) when w may be singular at $t = 0$ and (or) $t = 1$, $w \in L^1([0,1])$, $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous, h_1, h_2, k_1, $k_2 \in L^1([0,1])$ are nonnegative with $\mu_1 > 0$, $\nu_1 > 0$, $\mu_2 > 0$, $\nu_2 > 0$.

This work complements and improves similar results obtained in [78]. In Section 5.5, we discuss and compare our result with those obtained in [78]. We end by giving an example of application with some numerical computations.

5.2 Multiple fixed points theorem

The following theorem is useful to provide existence of two fixed points in a cone. It will be used to prove the main result of [12]. We refer the reader to [40] and [29] for more details.

Theorem 5.2.1 *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exist* $\varepsilon > 0$ *small enough and* $Sx \neq (I-T)(\lambda x)$, *for all* $\lambda \geq 1+\varepsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

(iii) $Sx \neq (I-T)(x-\lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T+S$ *has at least two non-zero fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

5.3 Integral formulation of the problem

Let

$$\begin{aligned}
G(t,s) &= \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases} \\
H_1(t,s) &= G(t,s) + \frac{m_1 + \mu_1 t}{m_1\nu_1 + n_1\mu_1} \int_0^1 k_1(\nu) G(t,\nu) d\nu \\
&\quad + \frac{n_1 - \nu_1 t}{m_1\nu_1 + n_1\mu_1} \int_0^1 h_1(\nu) G(t,\nu) d\nu, \\
H_2(t,s) &= G(t,s) + \frac{m_2 + \mu_2 t}{m_2\nu_2 + n_2\mu_2} \int_0^1 k_2(\nu) G(t,\nu) d\nu \\
&\quad + \frac{n_2 - \nu_2 t}{m_2\nu_2 + n_2\mu_2} \int_0^1 h_2(\nu) G(t,\nu) d\nu, \\
H(t,s) &= \int_0^1 H_1(t,\nu) H_2(\nu,s) d\nu, \quad t,s \in [0,1], \\
\mathbb{K}_1 &= \int_0^1 |k_1(\nu)| d\nu, \quad \mathbb{K}_2 = \int_0^1 |k_2(\nu)| d\nu, \\
\mathbb{H}_1 &= \int_0^1 |h_1(\nu)| d\nu, \quad \mathbb{H}_2 = \int_0^1 |h_2(\nu)| d\nu,
\end{aligned}$$

$$\begin{aligned}
A_1 &= 1+\frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{K}_1+\frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{H}_1,\\
A_2 &= 1+\frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{K}_2+\frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{H}_2,\\
A_3 &= \int_0^1 w(s)a_1(s)ds,\\
A_4 &= \int_0^1 w(s)a_2(s)ds,\\
A_5 &= \int_0^1 w(s)a_3(s)ds.
\end{aligned}$$

Then

$$0\le G(t,s)\le 1,\quad t,s\in[0,1],$$

and

$$\begin{aligned}
|H_1(t,s)| &\le G(t,s)+\frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\int_0^1|k_1(\nu)|G(t,\nu)d\nu\\
&\quad+\frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\int_0^1|h_1(\nu)|G(t,\nu)d\nu\\
&\le 1+\frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{K}_1+\frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{H}_1\\
&= A_1,\\
|H_2(t,s)| &\le G(t,s)+\frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\int_0^1|k_2(\nu)|G(t,\nu)d\nu\\
&\quad+\frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\int_0^1|h_2(\nu)|G(t,\nu)d\nu\\
&\le 1+\frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{K}_2+\frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{H}_2\\
&= A_2,\\
|H(t,s)| &= \left|\int_0^1 H_1(t,\nu)H_2(\nu,s)d\nu\right|\\
&\le \int_0^1|H_1(t,\nu)||H_2(t,\nu)|d\nu\\
&\le A_1A_2,\quad t,s\in[0,1].
\end{aligned}$$

In [78, Lemma 5], it is proved that if $x\in\mathcal{C}^2([0,1])$ is a solution to the integral equation

$$x(t)=\int_0^1 H(t,s)w(s)f(s,x(s),x''(s))ds,$$

then $x\in\mathcal{C}^2([0,1])\cap\mathcal{C}^4((0,1))$ and it satisfies the BVP (5.1).

In addition of above conditions, we assume the following.

(H3) Let $m > 0$ be large enough and A, r_1, L_1 and R_1 be positive constants satisfy the following conditions

$$r_1 < L_1 < R_1, \quad R_1 > \left(\frac{2}{5m} + 1\right) L_1,$$

$$A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) < \frac{L_1}{5},$$

(H4) There exists a nonnegative function $g \in \mathcal{C}([0,1])$ with $g \not\equiv 0$ so that

$$\int_0^1 \left((1-s)^2 + 2(1-s) + 2\right) g(s)ds \leq A.$$

In the last section, we will give an example for the constants p_1, p_2, A, m, A_1, A_2, A_3, A_4, A_5, r_1, L_1, R_1 and the function g that satisfy $(H3)$ and $(H4)$. For $x \in \mathcal{C}^2([0,1])$, define the operator

$$Fx(t) = \int_0^t (t-s)^2 g(s) \left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f\left(s_1, x(s_1), x''(s_1)\right) ds_1\right) ds, \quad t \in [0,1].$$

Lemma 5.3.1 *Suppose $(H1), (H2)$ and $(H4)$. If $x \in \mathcal{C}^2([0,1])$ is a solution to the equation*

$$0 = \frac{L_1}{5} + Fx(t), \quad t \in [0,1], \tag{5.2}$$

then $x \in \mathcal{C}^2([0,1]) \cap \mathcal{C}^4((0,1))$ is a solution to the BVP (5.1).

Proof. Let $x \in \mathcal{C}^2([0,1])$ is a solution to Equation (5.2). We differentiate three times with respect to t Equation (5.2) and we get

$$0 = g(t)\left(-x(t) + \int_0^1 H(t,s_1)w(s_1)f\left(s_1, x(s_1), x''(s_1)\right) ds_1\right), \quad t \in [0,1],$$

whereupon

$$x(t) = \int_0^1 H(t,s_1)w(s_1)f\left(s_1, x(s_1), x''(s_1)\right) ds_1, \quad t \in [0,1].$$

Then $x \in \mathcal{C}^2([0,1]) \cap \mathcal{C}^4((0,1))$ is a solution to the BVP (5.1). This completes the proof. ■

Lemma 5.3.2 *Assume $(H1), (H2)$ and $(H4)$. If $x \in \mathcal{C}^2([0,1])$ and $\|x\| \leq c$ for some positive constant c, then*

$$\|Fx\| \leq A\left(c + A_1 A_2\left(A_3 c^{p_1} + A_4 c^{p_2} + A_5\right)\right).$$

Proof. Let $x \in \mathcal{C}^2([0,1])$ and $\|x\| \leq c$. Then

$$\begin{aligned}
|Fx(t)| &= \left| \int_0^t (t-s)^2 g(s) \left(-x(s) + \int_0^1 H(s,s_1) w(s_1) f(s_1, x(s_1), x''(s_1)) ds_1\right) ds \right| \\
&\leq \int_0^t (t-s)^2 g(s) \left(|x(s)| + \int_0^1 |H(s,s_1)| w(s_1) |f(s_1, x(s_1), x''(s_1))| ds_1\right) ds \\
&\leq \int_0^1 (1-s)^2 g(s) \left(c + A_1 A_2 \int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1} + a_2(s_1)|x''(s_1)|^{p_2} + a_3(s_1)\right) ds_1\right) ds \\
&\leq \int_0^1 (1-s)^2 g(s) \Bigg(c + A_1 A_2 \Bigg(c^{p_1} \int_0^1 w(s_1) a_1(s_1) ds_1 + c^{p_2} \int_0^1 w(s_1) a_2(s_1) ds_1 \\
&\qquad + \int_0^1 w(s_1) a_3(s_1) ds_1 \Bigg)\Bigg) ds \\
&\leq \left(c + A_1 A_2\left(c^{p_1} A_3 + c^{p_2} A_4 + A_5\right)\right) \int_0^1 (1-s)^2 g(s) ds \\
&\leq A\left(c + A_1 A_2\left(c^{p_1} A_3 + c^{p_2} A_4 + A_5\right)\right), \quad t \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
& |(Fx)'(t)| \\
= & \left| 2\int_0^t (t-s) g(s) \left(-x(s) + \int_0^1 H(s,s_1) w(s_1) f(s_1, x(s_1), x''(s_1)) ds_1\right) ds \right| \\
\leq & \ 2\int_0^t (t-s) g(s) \left(|x(s)| + \int_0^1 |H(s,s_1)| w(s_1) |f(s_1, x(s_1), x''(s_1))| ds_1\right) ds \\
\leq & \ 2\int_0^1 (1-s) g(s) \left(c + A_1 A_2 \int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1} + a_2(s_1)|x''(s_1)|^{p_2} + a_3(s_1)\right) ds_1\right) ds \\
\leq & \ 2\int_0^1 (1-s) g(s) \Bigg(c + A_1 A_2 \Bigg(c^{p_1} \int_0^1 w(s_1) a_1(s_1) ds_1 + c^{p_2} \int_0^1 w(s_1) a_2(s_1) ds_1 \\
& \qquad + \int_0^1 w(s_1) a_3(s_1) ds_1 \Bigg)\Bigg) ds \\
\leq & \ 2\left(c + A_1 A_2\left(c^{p_1} A_3 + c^{p_2} A_4 + A_5\right)\right) \int_0^1 (1-s) g(s) ds \\
\leq & \ A\left(c + A_1 A_2\left(c^{p_1} A_3 + c^{p_2} A_4 + A_5\right)\right), \quad t \in [0,1],
\end{aligned}$$

and

$$
\begin{aligned}
& |(Fx)''(t)| \\
= & \left|2\int_0^t g(s)\left(-x(s)+\int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\right| \\
\leq & 2\int_0^t g(s)\left(|x(s)|+\int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds \\
\leq & 2\int_0^1 g(s)\left(c+A_1A_2\int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1}+a_2(s_1)|x''(s_1)|^{p_2}+a_3(s_1)\right)ds_1\right)ds \\
\leq & 2\int_0^1 g(s)\Bigg(c+A_1A_2\Bigg(c^{p_1}\int_0^1 w(s_1)a_1(s_1)ds_1+c^{p_2}\int_0^1 w(s_1)a_2(s_1)ds_1 \\
& +\int_0^1 w(s_1)a_3(s_1)ds_1\Bigg)\Bigg)ds \\
\leq & 2\left(c+A_1A_2\left(c^{p_1}A_3+c^{p_2}A_4+A_5\right)\right)\int_0^1 g(s)ds \\
\leq & A\left(c+A_1A_2\left(c^{p_1}A_3+c^{p_2}A_4+A_5\right)\right), \quad t\in[0,1].
\end{aligned}
$$

Consequently

$$
\|Fx\| \leq A\left(c+A_1A_2\left(c^{p_1}A_3+c^{p_2}A_4+A_5\right)\right).
$$

This completes the proof. ■

5.4 Main Result

Theorem 5.4.1 *Under the assumptions* $(H1)$-$(H4)$, *the BVP* (5.1) *has at least two non trivial nonnegative classical solutions in* $\mathcal{C}^2([0,1])\cap\mathcal{C}^4((0,1))$.

Consider the Banach space $E=\mathcal{C}^2([0,1])$ endowed with the norm

$$
\|x\|=\max\{\max_{t\in[0,1]}|x(t)|,\ \max_{t\in[0,1]}|x'(t)|,\ \max_{t\in[0,1]}|x''(t)|\},
$$

and the positive cone

$$
\mathcal{P}=\{x\in E: x\geq 0 \quad \text{on} \quad [0,1]\}.
$$

Let ϵ be positive constant. For $x\in E$, define the operators

$$
\begin{aligned}
Tx(t) &= (1+m\epsilon)x(t)-\epsilon\frac{L_1}{10}, \\
Sx(t) &= -\epsilon Fx(t)-m\epsilon x(t)-\epsilon\frac{L_1}{10},\ t\in[0,1].
\end{aligned}
$$

Note that any fixed point $x \in E$ of the operator $T + S$ is a solution to the IVP (5.1).

Define

$$
\begin{aligned}
\mathcal{P}_{r_1} &= \{v \in \mathcal{P} : \|v\| < r_1\}, \\
\mathcal{P}_{L_1} &= \{v \in \mathcal{P} : \|v\| < L_1\}, \\
\mathcal{P}_{R_1} &= \{v \in \mathcal{P} : \|v\| < R_1\}, \\
R_2 &= R_1 + \frac{A}{m}\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) + \frac{L_1}{5m}, \\
\Omega &= \mathcal{P}_{R_2} = \{v \in \mathcal{P} : \|v\| < R_2\}.
\end{aligned}
$$

1. For $x_1, x_2 \in \Omega$, we have

$$\|Tx_1 - Tx_2\| = (1 + m\epsilon)\|x_1 - x_2\|,$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\epsilon > 1$.

2. We prove that S is 0-set contraction.

(a) S is continuous. Indeed, let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in E. We have

$$|Sx_n(t) - Sx(t)| \leq \epsilon|Fx_n(t) - Fx(t)| + m\epsilon|x_n(t) - x(t)|, \quad \forall t \in [0, 1]. \tag{5.3}$$

We know that $|x_n(t) - x(t)| \to 0$, as $n \to \infty$ and

$$
\begin{aligned}
& |Fx_n(t) - Fx(t)| \\
\leq & \int_0^t (t - s)^2 g(s) \left(|x_n(s) - x(s)|\right. \\
+ & \left.\int_0^1 H(s, s_1) w(s_1) |f(s_1, x_n(s_1), x_n''(s_1)) - f(s_1, x(s_1), x''(s_1))| ds_1\right) ds, \quad t \in [0, 1].
\end{aligned}
$$

By continuity of f

$$\lim_{n \to +\infty} f(s_1, x_n(s_1), x_n''(s_1)) = f(s_1, x(s_1), x''(s_1)),$$

Then, Lemma 5.3.2 and the Lebesgue Dominated Convergence Theorem imply that

$$\int_0^1 H(s, s_1) w(s_1) |f(s_1, x_n(s_1), x_n''(s_1)) - f(s_1, x(s_1), x''(s_1))| ds_1 \to 0, \text{ as } n \to \infty.$$

So $|Fx_n(t) - Fx(t)| \to 0$, as $n \to \infty$. Thus $|Sx_n(t) - Sx(t)| \to 0$, as $n \to \infty$.

In the same way, we prove that $|(Sx_n)'(t) - (Sx)'(t)| \to 0$ and $|(Sx_n)''(t) - (Sx)''(t)| \to 0$, as $n \to \infty$, and then conclude that $Sx_n \to Sx$, as $n \to \infty$ in E, which ends the proof.

(b) $S(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Indeed, for $x \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned}\|Sx\| &\leq \epsilon\|Fx\| + m\epsilon\|x\| + \epsilon\frac{L_1}{10}\\ &\leq \epsilon\Big(A\left(R_1 + A_1A_2\left(R_1^{p_1}A_3 + R_1^{p_2}A_4 + A_5\right)\right) + mR_1 + \frac{L_1}{10}\Big).\end{aligned}$$

(c) $S(\overline{\mathcal{P}}_{R_1})$ is equicontinuous in E. Indeed, let $t_1, t_2 \in [0,1], t_1 < t_2$ and $x \in \overline{\mathcal{P}}_{R_1}$. From Lemma 5.3.2, we deduce

$$\begin{aligned}&|Fx(t_1) - Fx(t_2)|\\ =& \Big|\int_0^{t_1}(t_1 - s)^2 g(s)\left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f(s_1, x(s_1), x''(s_1))ds_1\right)ds\\ &- \int_0^{t_2}(t_2 - s)^2 g(s)\left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f(s_1, x(s_1), x''(s_1))ds_1\right)ds\Big|\\ \leq& \int_0^{t_1}\left((t_1 - s)^2 - (t_2 - s)^2\right)g(s)\left(|x(s)| + \int_0^1 |H(s,s_1)|w(s_1)|f(s_1, x(s_1), x''(s_1))|ds_1\right)ds\\ &+ \int_{t_1}^{t_2}(t_2 - s)^2 g(s)\left(|x(s)| + \int_0^1 |H(s,s_1)|w(s_1)|f(s_1, x(s_1), x''(s_1))|ds_1\right)ds\\ \leq& \int_0^1\left((t_1 - s)^2 - (t_2 - s)^2\right)g(s)\left(|x(s)| + \int_0^1 |H(s,s_1)|w(s_1)|f(s_1, x(s_1), x''(s_1))|ds_1\right)ds\\ &+ \int_{t_1}^{t_2}(1 - s)^2 g(s)\left(|x(s)| + \int_0^1 |H(s,s_1)|w(s_1)|f(s_1, x(s_1), x''(s_1))|ds_1\right)ds\\ &\to 0, \quad \text{as } t_1 \to t_2.\end{aligned}$$

Similarly,

$$|(Fx)'(t_2) - (Fx)'(t_1)| \to 0, \quad \text{as } t_1 \to t_2,$$

and

$$|(Fx)''(t_2) - (Fx)''(t_1)| \to 0, \quad \text{as } t_1 \to t_2.$$

Consequently,

$$\begin{aligned}|Sx(t_2) - Sx(t_1)| &\leq \epsilon|Fx(t_2) - Fx(t_1)| + \epsilon m|x(t_2) - x(t_1)| \to 0,\\ |(Sx)'(t_2) - (Sx)'(t_1)| &\leq \epsilon|(Fx)'(t_2) - (Fx)'(t_1)| + \epsilon m|x'(t_2) - x'(t_1)| \to 0,\\ |(Sx)''(t_2) - (Sx)''(t_1)| &\leq \epsilon|(Fx)''(t_2) - (Fx)''(t_1)| + \epsilon m|x''(t_2) - x''(t_1)| \to 0,\\ &\text{as } t_1 \to t_2.\end{aligned}$$

Therefore, $S(\overline{\mathcal{P}}_{R_1})$ is equicontinuous.

According to the Arzela-Ascoli compactness criteria, we conclude that $S : \overline{\mathcal{P}}_{R_1} \to E$ is completely continuous. So, it is 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}.$$

Note that by the second inequality of $(H3)$ and by Lemma 5.3.2, it follows that $\epsilon Fv_1 + \epsilon\frac{L_1}{5} \geq 0$ on $[t_0, \infty)$. We have $v_2 \geq 0$ on $[t_0, \infty)$ and

$$\begin{aligned} \|v_2\| &\leq \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m} \\ &\leq R_1 + \frac{A}{m}\left(R_1 + A_1A_2\left(R_1^{p_1}A_3 + R_1^{p_2}A_4 + A_5\right)\right) + \frac{L_1}{5m} \\ &= R_2. \end{aligned}$$

Therefore, $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon F v_1 - \epsilon\frac{L_1}{10} - \epsilon\frac{L_1}{10}$$

or

$$\begin{aligned} (I-T)v_2 &= -\epsilon m v_2 + \epsilon\frac{L_1}{10} \\ &= Sv_1. \end{aligned}$$

Consequently, $S(\overline{\mathcal{P}}_{R_1}) \subset (I-T)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda_0 > 0$ and $x_0 \in \partial\mathcal{P}_{r_1} \cap (\Omega + \lambda_0 u_0)$ or $x_0 \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda_0 u_0)$ such that

$$Sx_0 = (I-T)(x_0 - \lambda_0 u_0).$$

Then

$$-\epsilon F x_0(t) - \epsilon m x_0(t) - \epsilon\frac{L_1}{10} = -\epsilon m(x_0(t) - \lambda_0 u_0) + \epsilon\frac{L_1}{10},\ t \in [0,1].$$

Whereupon,

$$Fx_0(t) = -\lambda_0 m u_0 - \frac{L_1}{5},\ t \in [0,1].$$

So,

$$\|Fx_0\| = \|\lambda_0 m u_0 + \frac{L_1}{5}\| > \frac{L_1}{5},$$

which contradicts Lemma 5.3.2 and the second inequality of (H3).

5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $\lambda_1 \geq \varepsilon_1 + 1$ and $x_1 \in \partial\mathcal{P}_{L_1}$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ such that

$$Sx_1 = (I - T)(\lambda_1 x_1).$$

Note that $x_1 \in \partial\mathcal{P}_{L_1}$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ imply

$$\left(\frac{2}{5m} + 1\right) L_1 < \lambda_1 L_1 = \lambda_1 \|x_1\| \leq R_1.$$

Then

$$-\epsilon F x_1 - m\epsilon x_1 - \epsilon \frac{L_1}{10} = -\lambda_1 m \epsilon x_1 + \epsilon \frac{L_1}{10},$$

or

$$Fx_1 + \frac{L_1}{5} = (\lambda_1 - 1) m x_1.$$

Hence,

$$2\frac{L_1}{5} > \|Fx_1 + \frac{L_1}{5}\| = (\lambda_1 - 1)m\|x_1\| = (\lambda_1 - 1)mL_1,$$

or

$$\lambda_1 < \frac{2}{5m} + 1,$$

which is a contradiction.

Therefore all conditions of Theorem 5.2.1 hold. Hence, the BVP (5.1) has at least two solutions x_1 and x_2 such that

$$r_1 \leq \|x_1\| < L_1 < \|x_2\| \leq R_1.$$

5.5 Concluding remarks

1. The conclusion of the main result of [12] remains true if we replace the condition $w \in L^1([0,1])$ by the following one:

$$w : [0,1] \to \mathbb{R} \quad \text{is a nonnegative function such that } w a_i \in L^1([0,1]),\ i = 1, 2.$$

2. In [78], the BVP (5.1) is investigated in the case when

   (A1) w may be singular at $t=0$ and (or) $t=1$, $w \in L^1([0,1])$, $f:[0,1]\times[0,\infty)\times(-\infty,0]\to[0,\infty)$ is continuous, $h_1,\ h_2,\ k_1,\ k_2 \in L^1([0,1])$ are nonnegative with $\mu_1>0,\ \nu_1>0,\ \mu_2>0,\ \nu_2>0$.

   If $(A1)$ holds and $Nf_0>1$, $Nf_\infty>1$, and there exists $b>0$ such that $\max\limits_{t\in[0,1],0<|x|+|y|\le b} f(t,x,y)<\frac{b}{L}$, where

$$f_\beta=\liminf_{|x|+|y|\to\beta}\min_{t\in[0,1]}\frac{f(t,x,y)}{|x|+|y|},\quad \beta=0,\quad \beta=\infty,$$

   and

$$\begin{aligned}
L &= \left(\frac{\eta_1\eta_2}{16}+\frac{\eta_2}{4}\right)\int_0^1 w(s)ds,\\
N &= \left(\frac{\rho_1\rho_2}{120}+\frac{\rho_2}{4}\right)\delta^2\int_\delta^{1-\delta} e(s)w(s)ds,\\
\eta_1 &= \frac{m_1+n_1+\mu_1(1-\nu_1)}{m_1\nu_1+n_1\mu_1},\quad \eta_2=\frac{m_2+n_2+\mu_2(1-\nu_2)}{m_2\nu_2+n_2\mu_2},\\
\rho_1 &= \frac{1}{m_1\nu_1+n_1\mu_1}\left(\mu_1\int_0^1 e(\tau)k_1(\tau)d\tau+\nu_1\int_0^1 e(\tau)h_1(\tau)d\tau\right),\\
\rho_2 &= \frac{1}{m_2\nu_2+\nu_2\mu_2}\left(\mu_2\int_0^1 e(\tau)k_2(\tau)d\tau+\nu_2\int_0^1 e(\tau)h_2(\tau)d\tau\right),\\
e(t) &= t(1-t),\quad t\in[0,1],
\end{aligned}$$

   in [78], it is proved that the BVP (5.1) has at least two positive solutions.

   Moreover, if $(A1)$ holds and $Lf^0<1$, $Lf^\infty<1$, and there exist $\delta\in\left(0,\frac12\right)$ and $B>0$ such that $f(t,x,y)>\frac{\delta^2B}{N}$ for all $t\in J_\delta$, $x\in[\delta^2B,B]$, $y\in[-B,-\delta^2B]$, where $J_\delta=[\delta,1-\delta]$,

$$f^\beta=\limsup_{|x|+|y|\to\beta}\max_{t\in[0,1]}\frac{f(t,x,y)}{|x|+|y|},\quad \beta=0,\quad \beta=\infty,$$

   in [78], it is proved that the BVP (5.1) has at least two positive solutions.

   When $\mu_1<0$ or $\nu_1<0$, or $\mu_2<0$, or $\nu_2<0$, then we can not apply the results in [78] and we can apply the main result of [12]. Thus, the main result of [12] and the results in [78] are complementary.

5.6 Example

Let

$$\begin{aligned} r_1 &= 1, \quad L_1 = 10, \quad R_1 = 20, \\ p_1 &= p_2 = 0, \quad m = 1000, \quad A = \frac{1}{10^2}. \end{aligned}$$

Let also,

$$h_1(s) = h_2(s) = k_1(s) = k_2(s) = s, \quad a_1(s) = a_2(s) = a_3(s) = \frac{1}{3}, \quad w(s) = \frac{1}{\sqrt{s}}, \quad s \in [0,1].$$

Then

$$\begin{aligned} m_1 &= m_2 = \int_0^1 s^2 ds = \frac{1}{3}, \\ n_1 &= n_2 = 1 - \frac{1}{3} = \frac{2}{3}, \\ \mu_1 &= \mu_2 = \nu_1 = \nu_2 = 1 - \int_0^1 s ds = \frac{1}{2}, \\ \mathbb{K}_1 &= \mathbb{K}_2 = \mathbb{H}_1 = \mathbb{H}_2 = \int_0^1 s ds = \frac{1}{2}, \\ A_1 &= A_2 = 1 + \frac{\frac{1}{3} + \frac{1}{2}}{\frac{1}{6} + \frac{1}{3}} \cdot \frac{1}{2} + \frac{\frac{2}{3} + \frac{1}{2}}{\frac{1}{6} + \frac{1}{3}} \cdot \frac{1}{2} \\ &= 1 + \frac{\frac{5}{6}}{\frac{3}{6}} \cdot \frac{1}{2} + \frac{\frac{7}{6}}{\frac{3}{6}} \cdot \frac{1}{2} \\ &= 1 + \frac{5}{6} + \frac{7}{6} = 1 + 2 = 3, \\ A_3 &= A_4 = A_5 = \frac{1}{3} \int_0^1 \frac{ds}{\sqrt{s}} = \frac{2}{3} \sqrt{s} \Big|_{s=0}^{s=1} = \frac{2}{3}. \end{aligned}$$

Then

$$\begin{aligned} A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) &= \frac{1}{10^2}(20 + 9 \cdot 2) \\ &< 2 = \frac{L_1}{5}. \end{aligned}$$

$$\frac{R_1}{L_1} = 2 > \frac{2}{5000} + 1 = \frac{2}{5m} + 1.$$

Thus, $(H3)$ holds. Let $g(s) = \frac{1}{10^3}$, $s \in [0,1]$. Then

$$\int_0^1 ((1-s)^2 + 2(1-s) + 2) g(s) ds = \frac{1}{10^3} \int_0^1 (s^2 - 4s + 5) ds = \frac{1}{3 \cdot 10^2} < A.$$

Therefore $(H4)$ holds. Consequently the BVP

$$\begin{aligned} x^{(4)}(t) &= \frac{1}{\sqrt{t}}\left(\frac{1}{1+(x''(t))^2}\right), \quad t \in (0,1), \\ x(0) &= x(1) = \int_0^1 s x(s) ds, \quad x''(0) = x''(1) = x'(1), \end{aligned}$$

has at least two nonnegative solutions.

Chapitre 6

Appendix: Green's functions for some boundary value problems in ODEs

The Green's function plays an important role in solving boundary value problems of ordinary differential equations. The solutions of some boundary value problems for linear ordinary differential equations can be expressed by their respective Green's functions, in what follows, we give some examples. The interest of Green's function resides mainly in the resolution of non-homogeneous differential equations, it is necessary on the one hand to determine the general solution of the homogeneous equation associated, and on the other hand, to find a particular solution of the complete equation, then add both solutions to determine the integration constants with indispensable additional data. Green's function makes it possible to find precisely this particular solution. Some boundary value problems for nonlinear differential equations can be transformed into nonlinear integral equations whose kernel are the Green's functions of corresponding linear differential equations. Such integral equations can be studied using the properties of Green's functions. The concept, the significance and the development of Green's functions can be seen in [21, 22, 31, 68].

Second-order differential equation with linear boundary conditions

Consider the following linear second order differential equation

$$(\mathcal{E})\ \ p(x)y'' + q(x)y' + r(x)y = f(x), \quad x \in [a, b],$$

where p, q, r and f are continuous functions in $[a, b]$, associated to non separated linear boundary conditions :

$$(\mathcal{F})\ \begin{cases} U_1(y) = \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) = \gamma, \\ U_2(y) = \beta_1 y(a) + \beta_2 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = \delta, \end{cases}$$

where α_i, β_i, $i = 1, 4$ and γ, δ are real constants.

We call associated homogeneous boundary value problem to $(\mathcal{E}) + (\mathcal{F})$ the problem $(\mathcal{E}_H) + (\mathcal{F}_H)$ such that :

$$(\mathcal{E}_H)\ \ p(x)y'' + q(x)y' + r(x)y = 0, \quad a < x < b$$

and

$$(\mathcal{F}_H)\ \begin{cases} U_1(y) = 0, \\ U_2(y) = 0. \end{cases}$$

If ($f \neq 0$ and $\gamma = \delta = 0$) or ($f = 0$ and ($\gamma \neq 0$ or $\delta \neq 0$)), we say that the problem $(\mathcal{E}) + (\mathcal{F})$ is semi homogeneous.

Remark 6.0.1 *1. The boundary value problem $(\mathcal{E}) + (\mathcal{F})$ is said to be regular if a and b are finite numbers, p, q, r are bounded functions on $[a, b]$ and $p(x) \neq 0\ \forall x \in [a, b]$, otherwise we say that it is singular.*

2. The linear boundary conditions $(\mathcal{F})$ are general, in particular they include :

a) The Dirichlet's conditions : $y(a) = \gamma$, $y(b) = \delta$;

b) Neuman's conditions : $y'(a) = \gamma$, $y'(b) = \delta$;

c) The mixed conditions: $y(a) = \gamma$, $y'(b) = \delta$ or $y'(a) = \gamma$, $y(b) = \delta$;

d) The linear separated boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = \gamma \\ \beta_1 y(b) + \beta_2 y'(b) = \delta, \end{cases}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ *and* $\beta_1^2 + \beta_2^2 \neq 0$*;*

e) The linear periodic boundary conditions

$$\begin{cases} y(a) = y(b) \\ y'(a) = y'(b). \end{cases}$$

In what follows, we present a result, called Fredholm alternative, that assures the existence and the unicity of the solution of the problem $(\mathcal{E}) + (\mathcal{F})$.

Theorem 6.0.2 ***(Fredholm alternative)*** ([1], *page* 236)
The nonhomogeneous problem $(\mathcal{E}) + (\mathcal{F})$ *admits a unique solution if and only the homogeneous problem* $(\mathcal{E}_H) + (\mathcal{F}_H)$ *admits only the trivial solution* $y \equiv 0$*.*

Definition 6.0.3 $G : [a, b] \times [a, b] \to \mathbb{R}$ *is called Green's function of the problem* $(\mathcal{E}_H) + (\mathcal{F}_H)$ *if it verifies the following properties :*

1. *G is continuous in $[a, b] \times [a, b]$;*
2. *$\frac{\partial G}{\partial x}$ is continuous in each point $(x, t) \in [a, b] \times [a, b]$ such that $x \neq t$;*
3. *$\frac{\partial G}{\partial x}(x, x^-) - \frac{\partial G}{\partial x}(x, x^+) = \frac{1}{p(x)}$ $\forall x \in [a, b]$, where*
$$\frac{\partial G}{\partial x}(x, x^-) = \lim_{t \to x^-} \frac{\partial G}{\partial x}(x, t) \;\; and \;\; \frac{\partial G}{\partial x}(x, x^+) = \lim_{t \to x^+} \frac{\partial G}{\partial x}(x, t);$$
4. *$\forall t \in (a, b)$ the function $x \mapsto G(x, t)$ verifies the homogeneous equation $(\mathcal{E}_H)$ in each of the intervals $[a, t)$ and $(t, b]$;*
5. *$\forall t \in (a, b)$ the function $x \mapsto G(x, t)$ verifies the homogeneous conditions $(\mathcal{F}_H)$.*

Theorem 6.0.4 *([1], pp 240-244) Suppose that the homogeneous problem $(\mathcal{E}_H) + (\mathcal{F}_H)$ has only the trivial solution. Then, there exists a unique function G, called Green's function, such* *that for every continuous function f the solution y of the semi homogeneous problem $(\mathcal{E})+(\mathcal{F}_H)$ is uniquely written like the following:*

$$y(x) = \int_a^b G(x,t)f(t)dt.$$

Proof. Existence, uniqueness and construction of the function G

Let φ_1, φ_2 two independent solutions of $(\mathcal{E}_H)$. By defintion, the partial function $x \mapsto G(x,t)$ is solution of the equation $(\mathcal{E}_H)$ in each interval $[a,t[$ and $]t,b]$, there exist four functions on t such that :

$$G(x,t) = \begin{cases} \lambda_1(t)\varphi_1(x) + \lambda_2(t)\varphi_2(x) \text{ if } \ a \le x \le t \\ \mu_1(t)\varphi_1(x) + \mu_2(t)\varphi_2(x) \text{ if } \ t \le x \le b, \end{cases} \tag{6.1}$$

Next, the Properties 1 and 3 give the system :

$$\begin{cases} \lambda_1(t)\varphi_1(t) + \lambda_2(t)\varphi_2(t) = \mu_1(t)\varphi_1(t) + \mu_2(t)\varphi_2(t) \\ \mu_1(t)\varphi'(t) + \mu_2(t)\varphi'(t) - \lambda_1(t)\varphi'(t) - \lambda_2(t)\varphi'(t) = \frac{1}{p(t)}. \end{cases} \tag{6.2}$$

Posing $v_1(t) = \mu_1(t) - \lambda_1(t)$ et $v_2(t) = \mu_2(t) - \lambda_2(t)$, the System (6.2) becomes

$$\begin{cases} v_1(t)\varphi_1(t) + v_2(t)\varphi_2(t) = 0 \\ v_1(t)\varphi'(t) + v_2(t)\varphi'(t) = \frac{1}{p(t)}. \end{cases} \tag{6.3}$$

Since the Wronksian $W(\varphi_1, \varphi_2)(x) \neq 0$ for all $t \in [a,b]$ the system (6.3) admits a unique solution $(v_1(t),\, v_2(t))$. Using the relations $\mu_1(t) = \lambda_1(t) + v_1(t)$ et $\mu_2(t) = \lambda_2(t) + v_2(t)$, the Green's function G becomes *:*

$$G(x,t) = \begin{cases} \lambda_1(t)\varphi_1(x) + \lambda_2(t)\varphi_2(x), \quad if \ \ a \le x \le t \le b \\ \lambda_1(t)\varphi_1(x) + \lambda_2(t)\varphi_2(x) + v_1(t)\varphi_1(x) + v_2(t)\varphi_2(x), \quad if \ \ a \le t \le x \le b. \end{cases}$$

Next, the Property 5 gives the system

$$\begin{cases} U_1(\varphi_1)\lambda_1(t) + U_1(\varphi_2)\lambda_2(t) = k_1(t) \\ U_2(\varphi_1)\lambda_1(t) + U_2(\varphi_2)\lambda_2(t) = k_2(t), \end{cases} \tag{6.4}$$

where

$$\begin{cases} k_1(t) = -v_1(t)[\alpha_3\varphi_1(b) + \alpha_4\varphi'(b)] - v_2(t)[\alpha_3\varphi_2(b) + \alpha_4\varphi'(b)], \\ k_2(t) = -v_1(t)[\beta_3\varphi_1(b) + \beta_4\varphi'(b)] - v_2(t)[\beta_3\varphi_2(b) + \beta_4\varphi'(b)]. \end{cases}$$

Indeed, we have

$$\begin{aligned} G(a,t) &= \lambda_1(t)\varphi_1(a) + \lambda_2(t)\varphi_2(a), \quad (a \le t) \\ \tfrac{\partial G}{\partial x}(a,t) &= \lambda_1(t)\varphi'(a) + \lambda_2(t)\varphi'(a), \\ G(b,t) &= \lambda_1(t)\varphi_1(b) + \lambda_2(t)\varphi_2(b) + v_1(t)\varphi_1(b) + v_2(t)\varphi_2(b), \ (t \le b) \\ \tfrac{\partial G}{\partial x}(b,t) &= \lambda_1(t)\varphi'(b) + \lambda_2(t)\varphi'(b) + v_1(t)\varphi'(b) + v_2(t)\varphi'(b). \end{aligned}$$

Since the function $x \mapsto G(x,t)$ verifies the boundary conditions $(\mathcal{F}_H)$ for all $t \in [a,b]$, then

$$\alpha_1 G(a,t) + \alpha_2 \tfrac{\partial G}{\partial x}(a,t) + \alpha_3 G(b,t) + \alpha_4 \tfrac{\partial G}{\partial x}(b,t) = 0,$$

which gives the equation

$$\lambda_1(t)[\alpha_1\varphi_1(a) + \alpha_2\varphi'(a) + \alpha_3\varphi_1(b) + \alpha_4\varphi'(b)] + \lambda_2(t)[\alpha_1\varphi_2(a) + \alpha_2\varphi'(a) +$$
$$\alpha_3\varphi_2(b) + \alpha_4\varphi'(b)] + \nu_1(t)[\alpha_3\varphi_1(b) + \alpha_4\varphi'(b)] + v_2(t)[\alpha_3\varphi_2(b) + \alpha_4\varphi'(b)] = 0,$$

which is equivalent to

$$\lambda_1(t)[\alpha_1\varphi_1(a) + \alpha_2\varphi'(a) + \alpha_3\varphi_1(b) + \alpha_4\varphi'(b)] + \lambda_2(t)[\alpha_1\varphi_2(a) + \alpha_2\varphi'(a) +$$
$$\alpha_3\varphi_2(b) + \alpha_4\varphi'(b)]$$
$$= -v_1(t)[\alpha_3\varphi_1(b) + \alpha_4\varphi'(b)] - v_2(t)[\alpha_3\varphi_2(b) + \alpha_4\varphi'(b)] = k_1(t).$$

In the same way, we get

$$\beta_1 G(a,t) + \beta_2 \tfrac{\partial G}{\partial x}(a,t) + \beta_3 G(b,t) + \beta_4 \tfrac{\partial G}{\partial x}(b,t) = 0$$

which gives the second equation of the system(6.4).

By hypothesis, the homogeneous problem $(\mathcal{E}_H)$+$(\mathcal{F}_H)$ admits only the trivial solution, the determinant of the system (6.4) is non-zero. Thus, this system admits a unique solution $(\lambda_1(t), \lambda_2(t))$.

■

Example 6.0.5 *Let us consider the following periodic boundary problem :*

$$(P) \begin{cases} y''(x) + k^2 y(x) = 0, \quad 0 < x < a, \quad k > 0 \\ y(0) = y(a), \\ y'(0) = y'(a), \quad a > 0. \end{cases}$$

Let $\varphi_1(x) = \cos kx$ *and* $\varphi_2(x) = \sin kx$ *two linearly independent solutions of the equation* $y''(x) + k^2 y(x) = 0$. *The homogeneous problem associated to the problem* (P) *having only one solution* $y \equiv 0$ *if and only if* $\Delta = 4k \sin^2 \dfrac{ka}{2} \neq 0$.

Let $a \in]0, \frac{2\pi}{k}[$. *The Green's function* G *associated to the problem* (P) *is written like the following*

$$G(x,t) = \begin{cases} \lambda_1(t)\cos kx + \lambda_2(t)\sin kx & if \quad 0 \leq x < t \\ \mu_1(t)\cos kx + \mu_2(t)\sin kx & if \quad t < x \leq a. \end{cases}$$

Let $v_1(t) = \mu_1(t) - \lambda_1(t)$ *and* $v_2(t) = \mu_2(t) - \lambda_2(t)$. *Then,* $v_1(t)$ *et* $v_2(t)$ *verify the system*

$$\begin{cases} \cos(kt)v_1(t) + \sin(kt)v_2(t) = 0, \\ -k\sin(kt)v_1(t) + k\cos(kt)v_2(t) = 1, \end{cases}$$

which gives

$$v_1(t) = -\frac{1}{k}\sin kt \quad et \quad v_2(t) = \frac{1}{k}\cos kt.$$

Then, $\lambda_1(t)$ *and* $\lambda_2(t)$ *verify the system*

$$\begin{cases} (1-\cos ka)\lambda_1(t) - \sin ka\lambda_2(t) = \dfrac{1}{k}\sin k(a-t) \\ \sin ka\lambda_1(t) + (1-\cos ka)\lambda_2(t) = \dfrac{1}{k}\cos k(a-t). \end{cases}$$

The determinant Δ *of this system in* $(\lambda_1(t),\ \lambda_2(t))$ *is nonzero, then* $\lambda_1(t)$ *and* $\lambda_2(t)$ *are well defined and unique such that*

$$\lambda_1(t) = \frac{1}{2k\sin\frac{k}{2}}\cos k(t-\frac{a}{2}) \quad et \quad \lambda_2(t) = \frac{1}{2k\sin\frac{k}{2}}\sin k(t-\frac{a}{2}).$$

We replace these functions in the expression of the Green's function and we get

$$G(x,t) = \frac{1}{2k\sin\frac{k}{2}}\begin{cases} \cos k(x-t+\dfrac{a}{2}) & 0 \leq x \leq t \\ \cos k(t-x+\dfrac{a}{2}) & t \leq x \leq a \end{cases}.$$

Particular case: separate linear boundary conditions

Let consider the following linear differential second order equation

$$(\mathcal{E}) \quad p(x)y'' + q(x)y' + r(x)y = f(x), \quad x \in [a,b],$$

where p, q, r et f are regular functions associated to separated linear boundary conditions :

$$(\mathfrak{F}) \quad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = \gamma \\ \beta_1 y(b) + \beta_2 y'(b) = \delta, \end{cases}$$

where $\alpha_1^2+\alpha_2^2\neq 0$ et $\beta_1^2+\beta_2^2\neq 0$. In this case, The Green's function associated to the problem $(\mathcal{E}_H)+(\mathfrak{F}_H)$ can be determined with an easier way like the following :

$$G(x,t)=\frac{1}{p(t)W(t)}\begin{cases}\phi_1(x)\phi_2(t), \ a\leq x\leq t,\\ \phi_1(t)\phi_2(x), \ t\leq x\leq b,\end{cases}$$

where ϕ_1 and ϕ_2 are the solutions of the initial conditions problems respectively

$$(\mathcal{E}_H)+\begin{cases}\phi_1(a)=\alpha_2\\ \phi_1'(a)=-\alpha_1\end{cases} \quad \text{and} \quad (\mathcal{E}_H)+\begin{cases}\phi_2(b)=\beta_2\\ \phi_2'(b)=-\beta_1,\end{cases}$$

$W(t)=\phi_1(t)\phi_2'(t)-\phi_1'(t)\phi_2(t)\neq 0$ is their Wronskian and $p(t)=\exp\left(\int^t\frac{q(s)}{p(s)}ds\right)$.

Note that the product pW is constant in $[a,b]$.

Example 6.0.6 *Consider the Dirichlet's problem posed in* $[a,b]$

$$(\mathcal{P})\begin{cases}y''=f(x), \ \ a<x<b\\ y(a)=y(b)=0.\end{cases}$$

Let build the functions ϕ_1 and ϕ_2 solutions of Cauchy's problems :

$$\begin{cases}\phi_1''=0\\ \phi_1(a)=0\\ \phi'(a)=-1.\end{cases} \quad \textit{and} \quad \begin{cases}\phi_2''=0\\ \phi_2(b)=0\\ \phi'(b)=-1.\end{cases}$$

We find $\phi_1(x)=a-x$, $\phi_2(x)=b-x$, $W(\phi_1,\phi_2)=b-a\neq 0$ *et* $p(t)=1,\ t\in[a,b]$.

Hence the Green's function

$$G(x,t)=\begin{cases}\frac{(x-a)(t-b)}{b-a}, \ \ \textit{if} \ \ a\leq x\leq t\leq b\\ \frac{(t-a)(x-b)}{b-a}, \ \ \textit{if} \ \ a\leq t\leq x\leq b.\end{cases} \tag{6.5}$$

Second-order differential equation with three point boundary conditions

In this paragraph, we consider the Green's functions for a second-order linear ordinary differential equation with some three-point boundary conditions. The results presented here are developed by Zhao in [85].

We consider the second-order linear differential equation

$$y'' + f(t) = 0,\ t \in [a, b], \tag{6.6}$$

satisfying the boundary conditions

$$y(a) = ky(\eta), \quad y(b) = 0, \tag{6.7}$$

where k is a given real number and $\eta \in (a, b)$ is a given point.

The Green's function of Equation (6.6) with the boundary condition (6.7)

Theorem 6.0.7 *Assume $k(b - \eta) \neq b - a$. Then, the Green's function for the (6.6)-(6.7), is given by*

$$G(t, s) = K(t, s) + \frac{k(b-t)}{b-a-k(b-\eta)} K(\eta, s),$$

where

$$K(t, s) \begin{cases} \frac{(s-a)(b-t)}{b-a}, & a \leq s \leq t \leq b \\ \frac{(t-a)(b-s)}{b-a}, & a \leq t \leq s \leq b. \end{cases} \tag{6.8}$$

Proof. It is well known that the Green's function is $K(t, s)$ as in (6.8) for the second order two-point linear boundary value problem

$$\begin{cases} u'' + f(t) = 0, \quad t \in (a, b), \\ \quad u(a), \quad u(b) = 0, \end{cases} \tag{6.9}$$

and the solution of (6.9) is given by

$$w(t) = \int_a^b K(t, s) f(s) ds, \tag{6.10}$$

and

$$w(a) = 0, \quad w(b) = 0, \quad w(\eta) = \int_a^b K(\eta, s) f(s) ds. \tag{6.11}$$

The three-point boundary value problem (6.6) − (6.7) can be obtained from replacing $u(a) = 0$ by $u(a) = ku(\eta)$ in (6.9). Thus, we suppose that the solution of the three-point boundary value problem (6.6) − (6.7) can be expressed by

$$u(t) = w(t) + (c + d\,t) w(\eta); \tag{6.12}$$

where c and d are constants that will be determined. From (6.11),(6.12), we know that

$$u(a) = (c + da)w(\eta),$$

$$u(b) = (c + db)w(\eta),$$

$$u(\eta) = (c + d\eta + 1)w(\eta).$$

Putting these into (6.7) yields

$$\begin{cases} c + da = k(c + d\eta + 1), \\ c + db = 0. \end{cases}$$

Since $k(b - \eta) \neq b - a$, by solving the system of linear equations on the unknown numbers c, d, we obtain

$$\begin{cases} c = \frac{kb}{b-a-k(b-\eta)}, \\ d = \frac{-k}{b-a-k(b-\eta)}, \end{cases}$$

hence, $c + dt = \frac{k(b-t)}{b-a-k(b-\eta)}$. By substitution in (6.12), we get

$$u(t) = w(t) + \frac{k(b - t)}{b - a - k(b - \eta)}.$$

This together with (6.10) implies that

$$u(t) = \int_a^b K(t, s)f(s) + \frac{k(b - t)}{b - a - k(b - \eta)} \int_a^b K(\eta, s)f(s)ds.$$

Consequently, the Green's function $G_1(t, s)$ for the boundary value problem $(6.6) - (6.7)$ is as described in Theorem 6.0.7. ■

From Theorem 6.0.7 we obtain the following corollary.

Corollary 6.0.8 *If $k(b - \eta) \neq b - a$, then the second-order three-point linear boundary value problem*

$$\begin{cases} u'' + f(t) = 0, \quad t \in [a, b], \\ u(a) = ku(\eta), \quad u(b) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_a^b G(t, s)f(s)ds.$$

Consequetly, for $a = 0$ and $b = 1$, we have the following result.

Corollary 6.0.9 *If $k(1-\eta) \neq 1$, then the Green's function for the second-order three-point boundary value problem*

$$\begin{cases} u'' + f(t) = 0, \quad t \in [0,1], \\ u(0) = ku(\eta), \quad u(1) = 0 \end{cases} \tag{6.13}$$

is

$$G(t,s) = H(t,s) + \frac{k(1-t)}{1-k(1-\eta)} H(\eta, s), \tag{6.14}$$

where

$$H(t,s) \begin{cases} s(1-t), \quad 0 \leq s \leq t \leq 1 \\ t(1-s), \quad 0 \leq t \leq s \leq 1. \end{cases} \tag{6.15}$$

Hence the problem (6.13) has a unique solution

$$u(t) = \int_0^1 G(t,s) f(s) ds.$$

If $g(t,u)$ is continuous in $[0,1] \times \mathbb{R}$, then the nonlinear boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, \quad t \in [0,1], \\ u(0) = ku(\eta), \quad u(1) = 0 \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) g(s, u(s)) ds.$$

Fourth-order differential equation with integral boundary conditions

We consider the following fourth-order boundary value problem with integral conditions

$$\begin{cases} x^{(4)}(t) = w(t) f\left(t, x(t), x''(t)\right), \quad t \in (0,1), \\ x(0) = \int_0^1 h_1(s) x(s) ds, \quad x(1) = \int_0^1 k_1(s) x(s) ds, \\ x''(0) = \int_0^1 h_2(s) x''(s) ds, \quad x''(1) = \int_0^1 k_2(s) x''(s) ds, \end{cases} \tag{6.16}$$

where

(A1) w is nonnegative, and $w \in L^1[0,1]$ may have singularities at $t=0$ and(or) $t=1$;

(A2) $f \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R}$;

(A3) $h_1, h_2, k_1, k_2 \in L^1[0,1]$ are nonnegative and

$$\mu_1 = 1 - \int_0^1 h_1(s)ds > 0,$$
$$v_1 = 1 - \int_0^1 k_1(s)ds,$$
$$\mu_2 = 1 - \int_0^1 h_2(s)ds > 0,$$
$$v_2 = 1 - \int_0^1 k_2(s)ds,$$

In order to get the Green's function of problem (6.16) we need the following Lemma.

Lemma 6.0.10 *If $h, k \in L^1[0,1]$ are nonnegative, and $\mu = 1 - \int_0^1 h(s)ds > 0$, $v = 1 - \int_0^1 k(s)ds > 0$, then for any $y \in \mathcal{C}(0,1)$, the BVP*

$$\begin{cases} -x''(t) = y(t), \quad t \in (0,1), \\ x(0) = \int_0^1 h(s)x(s)ds, \quad x(1) = \int_0^1 k(s)x(s)ds, \end{cases} \tag{6.17}$$

has a unique solution x which is given by

$$x(t) = \int_0^1 \tilde{H}(t,s)y(s)ds, \tag{6.18}$$

where

$$\tilde{H}(t,s) = G(t,s) + \frac{m+\mu t}{mv+n\mu}\int_0^1 k(\tau)G(s,\tau)d\tau + \frac{n-vt}{mv+n\mu}\int_0^1 h(\tau)G(s,\tau)d\tau, \tag{6.19}$$

$$G(t,s) = \begin{cases} s(1-t), \quad 0 \le s \le t \le 1, \\ t(1-s), \quad 0 \le t \le s \le 1, \end{cases} \tag{6.20}$$

and

$$m = \int_0^1 sh(s)ds, \quad n = 1 - \int_0^1 sk(s)ds. \tag{6.21}$$

Proof. The general solution $x''(t) = y(t)$ can be written as

$$x(t) = -\int_0^1 (t-s)y(s)ds + At + B. \tag{6.22}$$

Now, we solve for A, B by $x(0) = \int_0^1 h(s)x(s)ds$ and $x(1) = \int_0^1 k(s)x(s)ds$, it follows that

$$\begin{aligned} B &= -\int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau + A\int_0^1 \tau h(\tau)d\tau + B\int_0^1 h(\tau)d\tau, \\ &- \int_0^1(1-s)y(s)ds + A + B \\ &= -\int_0^1 k(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau + A\int_0^1 \tau k(\tau)d\tau + B\int_0^1 k(\tau)d\tau, \end{aligned} \tag{6.23}$$

that is,

$$\begin{aligned} A\int_0^1 \tau h(\tau)d\tau - B\left(1 - \int_0^1 h(\tau)d\tau\right) &= \int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau, \\ A\left(1 - \int_0^1 \tau k(\tau)d\tau\right) &+ B\left(1 - \int_0^1 k(\tau)d\tau\right) \\ &= \int_0^1(1-s)y(s)ds - \int_0^1 k(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau. \end{aligned} \tag{6.24}$$

Solving the above equations, we get

$$\begin{aligned} A &= \tfrac{1}{mv+n\mu}(v\int_0^1 h(\tau)\int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau + \mu(\int_0^1(1-s)y(s)ds \\ &- \int_0^1 k(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau)), \\ B &= \tfrac{1}{mv+n\mu}(m(\int_0^1(1-s)y(s)ds - \int_0^1(k(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau) \\ &-n\int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau). \end{aligned} \tag{6.25}$$

Therefore, (6.17) has a unique solution

$$\begin{aligned} &x(t) = \\ &-\int_0^t(t-s)y(s)ds + \tfrac{1}{mv+n\mu} \times \Big[vt\int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)dsd\tau \\ &+\mu t\left(\int_0^1(1-s)y(s)ds - \int_0^1 k(\tau)\ \int_0^\tau(\tau - s)y(s)dsd\tau\right) \\ &+m\left(\int_0^1(1-s)y(s)ds - \int_0^1 k(\tau)\ \int_0^\tau(\tau - s)y(s)dsd\tau\right) \\ &- n\int_0^1 h(\tau)\int_0^\tau(\tau - s)y(s)d\tau\Big]. \end{aligned}$$

The unique solution of (6.17) is expressed as the following

$$
\begin{aligned}
x(t) &= \int_0^t s(1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds \\
&+\tfrac{1}{mv+n\mu}\Big[\mu t\int_0^1 \tau k(\tau)\int_0^1(1-s)y(s)ds - \mu t\int_0^1(1-s)y(s)ds \\
&-vt\int_0^1 \tau h(\tau)d\tau\int_0^1(1-s)y(s)ds + vt\int_0^1 h(\tau)\int_0^\tau(\tau-s)y(s)dsd\tau \\
&+\mu t\int_0^1(1-s)y(s)ds - \mu t\int_0^1 k(\tau)\int_0^\tau(\tau-s)y(s)dsd\tau \\
&+\int_0^1 \tau h(\tau)d\tau\int_0^1(1-s)y(s)ds - \int_0^1 \tau h(\tau)d\tau\int_0^1 k(\tau)\int_0^\tau(\tau-s)y(s)dsd\tau \\
&-\int_0^1 h(\tau)\int_0^\tau(\tau-s)y(s)dsd\tau + \int_0^1 \tau k(\tau)d\tau\int_0^1 h(\tau)\int_0^\tau(\tau-s)y(s)dsd\tau \\
&-\int_0^1 \tau k(\tau)d\tau\int_0^1 \tau h(\tau)d\tau\int_0^1(1-s)y(s)ds \\
&+\int_0^1 \tau h(\tau)\int_0^1 \tau k(\tau)d\tau\int_0^1(1-s)y(s)ds\Big] \\
&= \int_0^t s(1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds \\
&+\tfrac{1}{mv+n\mu}\Big[\mu t\Big(k(\tau)\int_0^\tau s(1-\tau)y(s)dsd\tau + \int_0^1 k(\tau)\int_\tau^1 \tau(1-s)y(s)dsd\tau\Big) \\
&-vt\Big(\int_0^1 h(\tau)\int_0^\tau s(1-\tau)y(s)dsd\tau + \int_0^1 h(\tau)\int_\tau^1 \tau(1-s)dsd\tau\Big) \\
&+\Big(\int_0^1 h(\tau)\int_0^\tau s(1-\tau)y(s)dsd\tau + \int_0^1 g(\tau)\int_0^1 \tau(1-s)dsd\tau\Big) - \int_0^1 \tau k(\tau)d\tau \\
&\times\Big(\int_0^1 h(\tau\int_0^\tau s(1-\tau)y(s)dsd\tau \\
&+\int_0^1 h(\tau)\int_\tau^1 \tau(1-s)y(s)dsd\tau\Big) \\
&+\int_0^1 \tau h(\tau)d\tau \\
&\times\Big(\int_0^1 k(\tau)\int_0^\tau s(1-\tau)y(s)dsd\tau \\
&+\int_0^1 k(\tau)\int_\tau^1 \tau(1-s)y(s)dsd\tau\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)y(s)ds + \tfrac{1}{mv+n\mu}\times\Big[\mu t\int_0^1 k(\tau)\int_0^1 G(s,\tau)y(s)dsd\tau \\
&-vt\int_0^1 h(\tau)\int_0^1 G(s,\tau)y(s)dsd\tau \\
&+\int_0^1 h(\tau)\int_0^1 G(s,t)y(s)dsd\tau \\
&-\int_0^1 \tau k(\tau)d\tau\int_0^1 h(\tau) \\
&\times\int_0^1 G(s,\tau)y(s)dsd\tau \\
&+\int_0^1 \tau h(\tau)d\tau\int_0^1 k(\tau) \\
&\times G(s,\tau)y(s)dsd\tau] \\
&= \int_0^1 G(t,s)y(s)ds + \tfrac{m+\mu t}{mv+n\mu}\int_0^1 y(s)\int_0^1 k(\tau)G(s,\tau)d\tau ds \\
&+\tfrac{n-vt}{mv+n\mu}\int_0^1 y(s)\int_0^1 h(\tau)G(s,\tau)d\tau ds.
\end{aligned}
\tag{6.26}
$$

Therefore, the unique solution of (6.17) is $x(t) = \int_0^1 \tilde{H}(t,s)y(s)ds$. ■

Theorem 6.0.11 *Assume that (A1)-(A3) hold. If $x(t) \in \mathcal{C}^2[0,1]$ is a solution of the following integral equation*

$$x(t) = \int_0^1 H(t,s)w(s)f(s,x(s),x''(s))ds, \tag{6.27}$$

then $x(t) \in \mathcal{C}^2[0,1] \cup \mathcal{C}^4(0,1)$ is a solution of BVP (6.16), where

$$H(t,s) = \int_0^1 H_1(t,\tau)H_2(\tau,s)d\tau, \tag{6.28}$$

$$\begin{aligned} H_1(t,\tau) = G(t,\tau) + \tfrac{m_1+\mu_1 t}{m_1 v_1 + n_1\mu_1} \int_0^1 k_1(v)G(\tau,v)dv \\ + \tfrac{n_1 - v_1 t}{m_1 v_1 + n_1 \mu_1} \int_0^1 h_1(v)G(\tau,v)dv, \end{aligned} \tag{6.29}$$

$$\begin{aligned} H_2(\tau,s) = G(\tau,s) + \tfrac{m_2+\mu_2 \tau}{m_2 v_2 + n_2\mu_2} \int_0^1 k_2(v)G(s,v)dv \\ + \tfrac{n_2 - v_2 \tau}{m_2 v_2 + n_2 \mu_2} \int_0^1 h_2(v)G(s,v)dv, \end{aligned} \tag{6.30}$$

$$\begin{aligned} m_1 = \textstyle\int_0^1 sh_1(s)ds, \quad n_1 - \int_0^1 sk_1(s)ds, \\ m_2 = \textstyle\int_0^1 sh_2(s)ds, \quad n_2 = 1 - \int_0^1 sk_2(s)ds. \end{aligned} \tag{6.31}$$

Proof. By using Lemma (6.0.10), the conclusion is abvious. ■

Example 6.0.12 *Let consider the following fourth-order boundary value problem*

$$\begin{cases} x^{(4)}(t) = \frac{1}{\sqrt{s}}\left(\frac{1}{1+(x''(t))^2}\right), & t \in (0,1), \\ x(0) = x(1) = \int_0^1 sx(s)ds, \\ x''(0) = x''(1) = \int_0^1 s(s)x''(s)ds. \end{cases} \tag{6.32}$$

Then

$$m_1 = m_2 = \int_0^1 s^2 ds = \frac{1}{3},$$
$$n_1 = n_2 = \frac{2}{3}.$$
$$\mu_1 = \mu_2 = v_1 = v_2 = \frac{1}{2}$$

and we have

$$H_1(t,\tau) = \begin{cases} \tau(\frac{1}{3} - t) + \frac{2}{3}, & 0 \le \tau \le t \le 1 \\ \tau(\frac{1}{3} - t) + t, & 0 \le t \le \tau \le 1, \end{cases}$$

with

$$H_2(\tau, s) = \begin{cases} s(\frac{1}{3} - \tau) + \frac{2}{3}, & 0 \leq s \leq \tau \leq 1 \\ s(\frac{1}{3} - \tau) + \tau, & 0 \leq \tau \leq s \leq 1, \end{cases}$$

and finaly we get

$$H(t, s) = \begin{cases} \frac{1}{18}(3st + 10 - 3s - 6t) & 0 \leq s \leq t \leq 1 \\ \frac{1}{18}(3t - s + 2) & 0 \leq t \leq s \leq 1. \end{cases}$$

Conclusion

This work is a contribution to the fixed point theory on cones of Banach spaces for the sum of two operators. The motivation for this study stems from the fact that many problems emanating from other fields of science are modeled as a sum of two operators. More precisely, the purpose of this thesis work is twofold, firstly, we construct a generalized fixed point index for operators that are sums of the form $T+F$, where T is an expansive operator and $I-F$ is a k-set contraction. For this, we appeal to the fixed point index theory for strict set contractions. After computing this new index, several fixed point theorems and recent results are derived, including Krasnosel'skii type theorems and Leggett-Williams type ones. Secondly, we use some of our obtained results to investigate the existence, nonnegativity, localization and multiplicity of solutions for two-point BVPs and for three-point BVPs as well as to study a class of fourth-order boundary value problems with integral boundary conditions. The study of these types of problems is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics, and the life sciences can be modeled in this way.

Fixed point theory is a flourishing area of research for many mathematicians with an enormous number or a wide range of applications in various fields of mathematics. The subject has become so vast that no single work can cover all its theoretical and applied parts and this theory still the object of intense research activity.

This work is a contribution to both theoretical and applied parts of the fixed point theory. We suggest the following topics to study later:

- Discrete Fixed Point Theory (Tarski's Fixed Point Theorem).
- Application to Navier-Stokes equations.

- Fixed point theory under weak topology.
- Application to fractional differential equations (FDEs).
- Random fixed point theory.

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Abstract

This thesis consists on the study of the fixed point index theory for the sum T+F on ordered Banach spaces and applications to some problems emanating from other fields. First, we present the necessary elements for the elaboration of this thesis such as the Kuratowski'smeasure of noncompactness, the topological degree theory as well as the fixed point index theory in cones. Secondly, we develop a new fixed point index for the same sum in the case where T is an h-expansive mapping with $h > 1$ and I-F is a k-set contraction with $0 \le k < h$. Finally, we use this fixed point index to develop fixed point theorems for this class of operators which will allow us to prove existence of nonnegative solutions for some boundary value problems.

Résumé

Cette thèse consiste en l'étude de la théorie de l'indice du point fixe pour la somme T+F dans des espaces de Banach ordonnés et ses applications à certains problèmes émanant d'autres domaines de la science. Dans un premier temps nous présentons les éléments nécessaires à l'élaboration de cette thèse tels que la mesure de non compacité de Kuratowski, la théorie du degré topologique ainsi que la théorie de l'indice du point fixe sur les cônes. Ensuite, nous développons un indice du point fixe pour cette somme dans le cas où T est un opérateur expansif avec la constante $h>1$ et I - F est un opérateur k-contractant d'ensembles avec $0 \le k < h$.Finalement, en utilisant cet indice, nous développons des théorèmes du point fixequi nous permettent de trouver des solutions positives à des problèmes aux limites associés à des équations différentielles d'ordre deux et d'ordre quatre.

ملخّص

نهتم في هذه الأطروحة بدراسة مؤشر النقطة الصامدة الخاصة بالمؤثرات التي تكتب على شكل مجموع مؤثرين ذات خصائص مختلفة والمعرّفة على مجموعات محدبة ومغلقة من فضاء بناخي. هذا المؤشر سوف يمكّننا فيما بعد، من جهة، من تطوير نظريات النقطة الصامدة على المخروطات بالنسبة لهذا الصنف من المؤثرات، ومن جهة أخرى سوف يمكّننا من إثبات وجود حلول موجبة لبعض المعادلات التكاملية وكذا لبعض المعادلات التفاضلية الغير خطية المرفقة بأنماط مختلفة من الشروط الحدية. للإشارة فإن نظرية النقطة الصامدة تأثرت بشكل كبير بالتقدم الموازي للأعمال البحثية المنجزة على الدرجة الطوبولوجية لأصناف مختلفة من المؤثرات. يعود التطور الهائل الذي عرفته نظرية المؤشر الطوبولوجي (الدرجة الطوبولوجية ومؤشر النقطة الصامدة) إلى أنه أداة قوية ومرنة من أجل دراسة وجود الحلول للكثير من المسائل الغير خطية.